Asymptotic behavior of the partial derivatives of Laguerre kernels and some applications

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Abstract

The aim of this paper is to present some new results about the asymptotic behavior of the partial derivatives of the kernel polynomials associated with the Gamma distribution. We also show how these results can be used in order to obtain the inner relative asymptotics for certain Laguerre-Sobolev type polynomials.

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1. Introduction

Let μ be a finite positive Borel measure supported on an infinite subset of $\mathbb{R}$. It is well-known that the polynomial kernels (also called reproducing, Christoffel-Darboux or Dirichlet kernels) associated with the sequences of orthogonal polynomials corresponding to μ are frequently used as a basic tool in spectral analysis, convergence of orthogonal expansions [2; 23; 27], and other aspects of mathematical analysis (see [26] and the references therein.)

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In the setting of orthogonal polynomial theory these kernels have been especially used by Freud and Nevai [4; 21; 22] and, more recently, the remarkable Lubinsky's works [9; 10] have caused heightened interest in this topic. Also, other interesting and related results corresponding to Fourier-Sobolev expansions may be found in [11–15; 17–19; 25].

Our goal here will be to analyze the asymptotic behavior of the partial derivatives of the diagonal Christoffel-Darboux kernels corresponding to classical Laguerre orthogonal polynomials (in short, the diagonal Laguerre kernels), i.e., we will consider the n-th Christoffel-Darboux kernel $K_n(x,y)$, given by

$$K_n(x,y)=\sum_{k=0}^{n}\frac{\widehat{L}_k^{\alpha}(x)\widehat{L}_k^{\alpha}(y)}{\left\langle \widehat{L}_k^{\alpha},\widehat{L}_k^{\alpha}\right\rangle_{\alpha}},$$

and its partial derivatives

$$K_n^{(j,k)}(x,y):=\frac{\partial^{j+k}K_n(x,y)}{\partial x^j\partial y^k},\quad 0\leq i,j\leq n,$$

where, as it is usual, $\{\widehat{L}_n^{\alpha}(x)\}_{n\geq 0}$ is the sequence of monic polynomials orthogonal with respect to the inner product

$$\langle f,g\rangle_{\alpha}=\int_0^{\infty}f(x)g(x)\,x^{\alpha}e^{-x}dx,\quad \alpha>-1,\quad f,g\in\mathbb{P},$$

and $\mathbb{P}$ denotes the linear space of polynomials with real coefficients. Then, for $c>0$ we will study the asymptotic behavior of $K_n^{(j,k)}(c,c),\quad 0\leq j,k\leq n$. From this starting point, we will focus our attention on the study of asymptotic properties of the sequences of polynomials orthogonal with respect to the following Sobolev-type inner product on the linear space of polynomials with real coefficients $\mathbb{P}$:

$$\langle f,g\rangle_S=\langle f,g\rangle_{\alpha}+\sum_{k=0}^{N}M_kf^{(k)}(c)g^{(k)}(c), \tag{1}$$

where $c>0$, $M_k\geq 0$, for $k=0,\ldots,N-1$, and $M_N>0$.

To the best of our knowledge, asymptotic properties of the diagonal Laguerre kernels $K_n^{(j,k)}(c,c)$, $0\leq j,k\leq n$, are not available in the literature up to those cases where the following situations have been considered.

- *Case 1:* $c \geq 0$ and either $j = k = 0$ or $0 \leq j, k \leq 1$ (cf. [6; 8]).
- *Case 2:* $c = 0$ and either $0 \leq j, k \leq 1$ or $0 \leq j, k \leq n$ (cf. [3; 24]).

The outline of the paper is as follows. Section 2 provides some basic background about structural and asymptotic properties of the classical Laguerre polynomials. The estimates of the partial derivatives of the diagonal Christoffel-Darboux kernels $K_n^{(j,k)}(c,c)$ (Theorem 2) are deduced. In Section 3 we prove our main result (Theorem 3), where an estimate in the Laguerre weighted L^2-norm for the difference between Laguerre orthonormal polynomials and the Laguerre-Sobolev type polynomials $\widetilde{L}_n^{\alpha,\underline{M}}(x)$, orthogonal with respect to (1) with $c > 0$, is obtained.

Let consider the multi-indexes $\underline{M} = (M_0, \ldots, M_N)$ of nonnegative real numbers, $M_N > 0$. The notation $u_n \sim_n v_n$ will always mean that the sequence u_n/v_n converges to 1 when n tends to infinity. Positive constants will be denoted by C, $C_1, C_{i,j}, \ldots$ and they may vary at every occurrence. Any other standard notation will be properly introduced whenever needed.

2. Asymptotics for the partial derivatives of the diagonal Laguerre kernels

For $\alpha > -1$, let $\{\widehat{L}_n^{\alpha}(x)\}_{n\geq 0}$, $\{\tilde{L}_n^{\alpha}(x)\}_{n\geq 0}$, and $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ be the sequences of monic, orthonormal and normalized Laguerre polynomials with leading coefficient equal to $\frac{(-1)^n}{n!}$, respectively. The following proposition summarizes some structural and asymptotic properties of the classical Laguerre polynomials (see [6; 7; 19] and the references therein.)

Proposition 2.1. *Let $\{\widehat{L}_n^{\alpha}(x)\}_{n\geq 0}$ be the sequence of monic Laguerre orthogonal polynomials. Then the following statements hold.*

1. *For every $n \in \mathbb{N}$,*

$$\left\langle \widehat{L}_n^{\alpha}, \widehat{L}_n^{\alpha} \right\rangle_{\alpha} = ||\widehat{L}_n^{\alpha}||_{\alpha}^2 = \Gamma(n+1)\Gamma(n+\alpha+1). \tag{2}$$

2. *Hahn's condition. For every $n \in \mathbb{N}$,*

$$\frac{d}{dx}\widehat{L}_n^{\alpha}(x) = n\widehat{L}_{n-1}^{\alpha+1}(x). \tag{3}$$

3. *The n-th Laguerre kernel $K_n(x,y)$ satisfies the Christoffel-Darboux formula (cf. [27, Theorem 3.2.2]):*

$$K_n(x,y) = \frac{1}{||\widehat{L}_n^{\alpha}||_{\alpha}^2}\left(\frac{\widehat{L}_{n+1}^{\alpha}(x)\widehat{L}_n^{\alpha}(y) - \widehat{L}_n^{\alpha}(x)\widehat{L}_{n+1}^{\alpha}(y)}{x-y}\right), \quad x \neq y, \quad n \geq 0. \tag{4}$$

4. *The so-called confluent form of the above kernel is*

$$K_n(x,x) = \frac{1}{||\widehat{L}_n^{\alpha}||_{\alpha}^2}\left\{[\widehat{L}_{n+1}^{\alpha}]'(x)\widehat{L}_n^{\alpha}(x) - [\widehat{L}_n^{\alpha}]'(x)\widehat{L}_{n+1}^{\alpha}(x)\right\}, \quad n \geq 0.$$

5. *[27, Theorem 8.22.2] Perron generalization of Fejér formula on $\mathbb{R}_+$. Let $\alpha \in \mathbb{R}$. Then for $x > 0$ we have*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}\cos\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \\ &\qquad \cdot \left\{\sum_{k=0}^{p-1} A_k(x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\} \\ &\quad + \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}\sin\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\} \\ &\qquad \cdot \left\{\sum_{k=0}^{p-1} B_k(x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\}, \end{aligned} \tag{5}$$

where $A_k(x)$ and $B_k(x)$ are certain functions of x independent of n and regular for $x > 0$. The bound for the remainder holds uniformly in $[\epsilon, \omega]$. For $k = 0$ we have $A_0(x) = 1$ and $B_0(x) = 0$.

In the next result, we show a confluent form for the partial derivatives of the kernel polynomial $K_{n-1}(x,y)$ for $x = y = c$.

Proposition 2.2. *For every $n \in \mathbb{N}$ and $0 \leq j, k \leq n-1$, we have*

$$\begin{aligned} K_{n-1}^{(k,j)}(c,c) = \frac{j!k!}{(j+k+1)!||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}\Big[\sum_{l=0}^{j}\binom{j+k+1}{l} \\ \Big([\widehat{L}_{n-1}^{\alpha}]^{(l)}(c)\,[\widehat{L}_n^{\alpha}]^{(j+k+1-l)}(c) - [\widehat{L}_n^{\alpha}]^{(l)}(c)\,[\widehat{L}_{n-1}^{\alpha}]^{(j+k+1-l)}(c)\Big)\Big]. \end{aligned} \tag{6}$$

PROOF. For $k = 0$ and $0 \leq j \leq n-1$, it is enough to follow a standard technique in literature (see, for instance [1, p. 269]) by taking derivatives in

(4) with respect to the variable y and then to evaluate at $y = c$. Thus we obtain

$$K_{n-1}^{(0,j)}(x,c) = \frac{j!}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}(x-c)^{j+1}} \left(T_j(x,c;\widehat{L}_{n-1}^{\alpha})\widehat{L}_{n}^{\alpha}(x) - T_j(x,c;\widehat{L}_{n}^{\alpha})\widehat{L}_{n-1}^{\alpha}(x) \right), \tag{7}$$

where $T_j(x,c;f)$ is the j-th Taylor polynomial of f around $y = c$.

Using the Taylor expansion of $\widehat{L}_{n}^{\alpha}(x)$ and $\widehat{L}_{n-1}^{\alpha}(x)$ in (7), we only need to look for the coefficients of $(x-c)^{j+k+1}$ therein in order to find (6). □

Taking $p = 1$ in (5), it is difficult to analyze the behavior of $\widehat{L}_{n}^{(\alpha)}(x)$, $x \in \mathbb{R}_+$, for n large enough, i.e.

$$\widehat{L}_{n}^{\alpha}(x) = (-1)^n \Gamma(n+1)\pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4}n^{\alpha/2-1/4}$$
$$\cdot \cos\{2\,(nx)^{1/2} - \alpha\pi/2 - \pi/4\}(1 + \mathcal{O}(n^{-1/2})).$$

So, we can rewrite the above expression as follows

$$\widehat{L}_{n}^{\alpha}(x) = (-1)^n \Gamma(n+1) n^{\frac{\alpha}{2}-\frac{1}{4}} \sigma^{\alpha}(x) \cos\varphi_{n}^{\alpha}(x)(1 + \mathcal{O}(n^{-1/2})), \tag{8}$$

where

$$\varphi_{n}^{\alpha}(x) = 2(nx)^{1/2} - \frac{\alpha\pi}{2} - \frac{\pi}{4},$$

and

$$\sigma^{\alpha}(x) = \pi^{-1/2}e^{x/2}x^{-\alpha/2-1/4},$$

is a function independent of n.

Now, our task is to find the asymptotic behavior of the diagonal Laguerre kernels. In order to do this we must estimate expressions like

$$\cos\varphi_{n-n_1}^{\alpha}(c)\cos\varphi_{n-n_2}^{\alpha+n_0}(c) - \cos\varphi_{n-n_3}^{\alpha+n_0}(c)\cos\varphi_{n}^{\alpha}(c).$$

Under some conditions on the parameters n_0, n_1, n_2, and n_3, we will prove that the above expression goes to zero when n tends to infinity and, moreover, we can compute its speed of convergence. The result reads as follows.

Lemma 1. *Let $n, n_0 \in \mathbb{N}$, $\alpha > -1$, and $c \in \mathbb{R}_+$. For fixed $n_1, n_2, n_3 \in \mathbb{N}$ with $n_3 = n_1 + n_2$ and $n \geq n_3$, let us consider*

$$F^{\alpha,c}(n) := \cos\varphi_{n-n_1}^{\alpha}(c)\cos\varphi_{n-n_2}^{\alpha+n_0}(c) - \cos\varphi_{n-n_3}^{\alpha+n_0}(c)\cos\varphi_{n}^{\alpha}(c).$$

Then,

$$F^{\alpha,c}(n) \sim_n \begin{cases} \frac{-1}{4}(n_2-n_1-n_3)(n_2-n_1+n_3)c\,n^{-1} & \text{if } n_0 \equiv 0 \mod 4, \\ \frac{-1}{2}(n_2-n_1-n_3)\sqrt{c}\,n^{-1/2} & \text{if } n_0 \equiv 1 \mod 4, \\ \frac{1}{4}(n_2-n_1-n_3)(n_2-n_1+n_3)c\,n^{-1} & \text{if } n_0 \equiv 2 \mod 4, \\ \frac{1}{2}(n_2-n_1-n_3)\sqrt{c}\,n^{-1/2} & \text{if } n_0 \equiv 3 \mod 4. \end{cases}$$

PROOF. By using trigonometric identities, a straightforward computation yields

$$F^{\alpha,c}(n) = f^{\alpha,c}(n) + g^c(n),$$

where

$$f^{\alpha,c}(n) = -\sin\left(\sqrt{cn} + \sqrt{c(n-n_1)} + \sqrt{c(n-n_2)} + \sqrt{c(n-n_3)} - \alpha\pi - \frac{(n_0+1)\pi}{2}\right) \times \sin\left(\sqrt{c(n-n_2)} + \sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} - \sqrt{cn}\right),$$

and

$$g^c(n) = -\sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_2)} + \sqrt{c(n-n_3)} - \sqrt{cn}\right) \times \sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} + \sqrt{cn} - \sqrt{c(n-n_2)} - \frac{n_0\pi}{2}\right).$$

Our first technical step will be to show that

$$\lim_{n\to\infty} n^{3/2}\sin\left(\sqrt{c(n-n_2)} + \sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} - \sqrt{cn}\right) = \frac{n_1 n_2\sqrt{c}}{4}. \tag{9}$$

Since $h(n) = (\sqrt{n} - \sqrt{n-n_2}) - (\sqrt{n-n_1} - \sqrt{n-n_3})$ can be written as $h(n) = k(n) - k(n-n_1)$, with $k(n) = \sqrt{n} - \sqrt{n-n_2}$, then, according to the mean value theorem, we obtain

$$h(n) = n_1 k'(\xi_n) = \frac{n_1}{2}\left(\frac{1}{\sqrt{\xi_n}} - \frac{1}{\sqrt{\xi_n - n_2}}\right), \quad \text{where } n - n_1 \le \xi_n \le n.$$

Let $l(n) = \frac{1}{\sqrt{n}}$. By using the mean value theorem we get

$$h(n) = \frac{n_1}{2}\left(l(\xi_n) - l(\xi_n - n_2)\right) = \frac{n_1}{2} n_2 l'(\delta_n) = \frac{-n_1 n_2}{2}\delta_n^{-3/2},$$

where $n-n_1-n_2 \leq \xi_n - n_2 \leq \delta_n \leq \xi_n \leq n$. Taking into account that $\lim_{n\to\infty} \frac{\delta_n}{n} = 1$, we get (9). Since the first factor in $f^{\alpha,c}(n)$ is bounded, we obtain

$$\lim_{n\to\infty} n^{1/2} f^{\alpha,c}(n) = 0 = \lim_{n\to\infty} n f^{\alpha,c}(n). \tag{10}$$

In our second technical step will show that the speed of convergence of the first factor in $g^c(n)$ is $n^{-1/2}$. In a similar way to the previous situation, we have

$$\lim_{n\to\infty} \sqrt{n} \sin\left(\sqrt{c(n-n_1)} - \sqrt{c(n-n_3)} + \sqrt{cn} - \sqrt{c(n-n_2)}\right)$$
$$= \frac{1}{2}\sqrt{c}\,(n_3 - n_1 + n_2)\,.$$

Then, the speed of convergence of $g^c(n)$ can be determined by discussing the following four cases

(i) If $n_0 \equiv 0 \mod 4$, using that $\sin(x-2\pi) = \sin(x)$, then

$$\lim_{n\to\infty} n g^c(n) = \frac{-1}{4} c(n_2 - n_1 - n_3)(n_2 - n_1 + n_3) \neq 0.$$

(ii) If $n_0 \equiv 1 \mod 4$, using that $\sin(x - \frac{\pi}{2}) = \cos(x)$, then

$$\lim_{n\to\infty} n^{1/2} g^c(n) = \frac{-1}{2}\sqrt{c}(n_2 - n_1 - n_3) \neq 0.$$

(iii) If $n_0 \equiv 2 \mod 4$, using that $\sin(x-\pi) = -\sin(x)$, then

$$\lim_{n\to\infty} n g^c(n) = \frac{1}{4} c(n_2 - n_1 - n_3)(n_2 - n_1 + n_3) \neq 0.$$

(iv) If $n_0 \equiv 3 \mod 4$, using that $\sin(x - \frac{3\pi}{2}) = -\cos(x)$, then

$$\lim_{n\to\infty} n^{1/2} g^c(n) = \frac{1}{2}\sqrt{c}(n_2 - n_1 - n_3) \neq 0.$$

Finally, from the above analysis and (10) the statement of Lemma follows.

$\square$

Theorem 2. *For $c > 0$, we get the behavior of the partial derivatives of the diagonal Laguerre kernels*

$$K_{n-1}^{(k,j)}(c,c) \sim_n \begin{cases} C_{0,k,j}\, n^{\frac{j+k+1}{2}} & \text{if } j+k \equiv 0 \mod 2, \\ \\ C_{1,k,j}\, n^{\frac{j+k}{2}} & \text{if } j+k \equiv 1 \mod 2, \end{cases}$$

where $0 \le j, k \le n-1$ and

$$C_{0,k,j} = \frac{(-1)^{\frac{j+k}{2}+j}}{k+j+1} \sigma^{\alpha}(c) \sigma^{\alpha+j+k+1}(c) \sqrt{c},$$
$$C_{1,k,j} = (-1)^{\frac{j+k+1}{2}+j} \frac{k-j}{k+j} \sigma^{\alpha}(c) \sigma^{\alpha+j+k+1}(c)\, c.$$

PROOF. Without loss of generality, we can assume that $j \le k$. From (3) and (6), we obtain

$$K_{n-1}^{(k,j)}(c,c) = \frac{j!k!\, n^{j+k+1}}{(j+k+1)!\, ||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}} \times$$
$$\sum_{l=0}^{j} \binom{j+k+1}{l} \left(\widehat{L}_{n-1-l}^{\alpha+l}(c) \widehat{L}_{n-j-k-1+l}^{\alpha+j+k+1-l}(c) - \widehat{L}_{n-l}^{\alpha+l}(c) \widehat{L}_{n-j-k-2+l}^{\alpha+j+k+1-l}(c) \right). \quad (11)$$

Now, using (2) and (8), we get

$$K_{n-1}^{(k,j)}(c,c) \sim_n \sum_{l=0}^{j} \binom{j+k+1}{l} (-1)^{j+k}$$
$$\frac{j!k!}{(j+k+1)!} \sigma^{\alpha}(c) \sigma^{\alpha+j+k+1}(c) n^{\frac{j+k}{2}+1} F^{\alpha+l,c}(n-l).$$

On the other hand, for all $l = 0, \ldots, j$, Lemma 1 with $n_0 = j+k+1-2l$, $n_1 = 1$, $n_2 = j+k+1-2l$, and $n_3 = j+k+2-2l$ implies that

$$F^{\alpha+l,c}(n-l) \sim_n \begin{cases} (-1)^{l}(j+k+1-2l)c\, n^{-1} & \text{if } j+k+1 \equiv 0 \mod 4, \\ (-1)^{l}\sqrt{c}\, n^{-1/2} & \text{if } j+k+1 \equiv 1 \mod 4, \\ (-1)^{l+1}(j+k+1-2l)c\, n^{-1} & \text{if } j+k+1 \equiv 2 \mod 4, \\ (-1)^{l+1}\sqrt{c}\, n^{-1/2} & \text{if } j+k+1 \equiv 3 \mod 4, \end{cases}$$

Since the above relation can be reduced to

$$F^{\alpha+l,c}(n-l) \sim_n \begin{cases} (-1)^{l+\frac{j+k}{2}}\sqrt{c}\, n^{-1/2} & \text{if } j+k \equiv 0 \mod 2, \\ (-1)^{l+\frac{j+k-1}{2}}(j+k+1-2l)c\, n^{-1} & \text{if } j+k \equiv 1 \mod 2, \end{cases}$$

we obtain the statement of the Theorem with

$$C_{0,k,j} = (-1)^{\frac{j+k}{2}} \frac{j!k!}{(k+j+1)!}\sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\sqrt{c}\sum_{l=0}^{j}\binom{j+k+1}{l}(-1)^{l},$$

$$C_{1,k,j} = (-1)^{\frac{j+k-1}{2}} \frac{j!k!}{(k+j+1)!}\sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\, c\sum_{l=0}^{j}\binom{j+k+1}{l}(j+k+1-2l)(-1)^{l+1}.$$

Using the identities

$$\sum_{l=0}^{j}\binom{j+k+1}{l}(-1)^{l} = (-1)^{j}\binom{j+k}{j},$$

$$\sum_{l=0}^{j}\binom{j+k+1}{l}(j+k+1-2l)(-1)^{l+1} = (-1)^{j+1}(j+k+1)\frac{(j+k-1)!}{j!k!}(k-j),$$

the above expressions read

$$C_{0,k,j} = \frac{(-1)^{\frac{j+k}{2}+j}}{k+j+1}\sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\sqrt{c},$$

$$C_{1,k,j} = (-1)^{\frac{j+k+1}{2}+j}\frac{k-j}{k+j}\sigma^{\alpha}(c)\sigma^{\alpha+j+k+1}(c)\, c.$$

Thus, we conclude the proof of the Theorem. □

Remark 1. Notice that Theorem 2 generalizes the asymptotic behavior of the diagonal Laguerre kernels given in [6] (where only the case $0 \leq j,k \leq 1$ has been analyzed.) The interested reader can find an analogous result of Theorem 2 when $c=0$, $0 \leq j,k \leq 1$, and $c=0$, $0 \leq j,k \leq n-1$, in [3; 24], respectively. Also, it is worthwhile to point out that with a different approach the authors of [8] obtained a lower bound for the Christoffel functions when $c \geq 0$.

3. Inner relative asymptotics

As an application of Theorem 2, we will study the inner relative asymptotics for a certain family of Laguerre-Sobolev type orthogonal polynomials. More precisely, we compare the behavior of the Sobolev and standard Laguerre polynomials on $(0,\infty)$ for n large enough. The main result in this section guarantees the norm convergence of the Laguerre-Sobolev polynomials to the Laguerre ones in the Laguerre L^2-norm. Before to deal with the general case, we are going to analyze a more simple framework. For example, let us consider the Sobolev type inner product

$$\langle f,g\rangle_S = \langle f,g\rangle_\alpha + Mf'(c)g'(c), \tag{12}$$

where $\alpha > -1$, $c > 0$ and $M > 0$. Notice that this is just a particular case of the family of inner products defined in [16]. Let $\{\widehat{L}_n^{M,\alpha}(x)\}_{n\geq 0}$ be the monic Laguerre-Sobolev polynomials orthogonal with respect to (12). We also consider the normalization

$$\widetilde{L}_n^{M,\alpha}(x) = \frac{\widehat{L}_n^{M,\alpha}(x)}{||\widehat{L}_n^{\alpha}||_\alpha},$$

i.e., the normalized Laguerre-Sobolev type orthogonal polynomials with the same leading coefficient as the classical orthonormal Laguerre polynomial of degree n. Then, (see [16, equation (2.8)])

$$\widetilde{L}_n^{M,\alpha}(x) - \tilde{L}_n^{\alpha}(x) = \frac{M\left(\tilde{L}_n^{\alpha}\right)'(c)}{1+MK_{n-1}^{(1,1)}(c,c)}K_{n-1}^{(0,1)}(x,c).$$

Let consider the standard L^2-Laguerre norm of the previous expression, i.e.

$$||\widetilde{L}_n^{M,\alpha} - \tilde{L}_n^{\alpha}||_\alpha^2 = \frac{M^2\left[\left(\tilde{L}_n^{\alpha}\right)'(c)\right]^2}{\left(1+MK_{n-1}^{(1,1)}(c,c)\right)^2}K_{n-1}^{(1,1)}(c,c) \leq \frac{\left[\left(\tilde{L}_n^{\alpha}\right)'(c)\right]^2}{K_{n-1}^{(1,1)}(c,c)}.$$

Now, from Proposition 2 we obtain

$$K_{n-1}^{(1,1)}(c,c) \sim_n C\, n^{\frac{3}{2}},$$

and, on the other hand,

$$\left(\tilde{L}_n^{\alpha}\right)'(c) = \frac{n\widehat{L}_{n-1}^{\alpha+1}(c)}{||\widehat{L}_n^{\alpha}||_{\alpha}} = \frac{n!(-1)^{n-1}}{\left(\Gamma(n+1)\Gamma(n+\alpha+1)\right)^{1/2}} L_{n-1}^{(\alpha+1)}(c),$$

from which it follows that

$$\left[\left(\tilde{L}_n^{\alpha}\right)'(c)\right]^2 = \frac{n!}{\Gamma(n+\alpha+1)} |L_{n-1}^{(\alpha+1)}(c)|^2 \leq C n^{1/2}.$$

As a consequence,

$$||\widetilde{L}_n^{M,\alpha} - \tilde{L}_n^{\alpha}||_{\alpha}^2 \leq C n^{-1},$$

so, we have proved the norm convergence of the n-th Laguerre-Sobolev type orthogonal polynomial to the n-th Laguerre one:

$$\lim_{n\to\infty} ||\widetilde{L}_n^{M,\alpha} - \tilde{L}_n^{\alpha}||_{\alpha} = 0.$$

3.1. The multi-index case

Let us consider the Sobolev type inner product (1) and $\widehat{L}_n^{\alpha,\underline{M}}(x)$ the corresponding monic orthogonal polynomial of degree n. Also, we consider the normalization

$$\widetilde{L}_n^{\alpha,\underline{M}}(x) = \frac{\widehat{L}_n^{\alpha,\underline{M}}(x)}{||\widehat{L}_n^{\alpha}||_{\alpha}},$$

i.e., the Laguerre-Sobolev polynomials with the same leading coefficient as the orthonormal Laguerre ones.

From now on, we will denote by $j_1 < \cdots < j_q$ the indexes such that $M_{j_1-1} = \cdots = M_{j_q-1} = 0$.

Theorem 3. *With the above notation, the inner relative asymptotics for the Laguerre-Sobolev polynomials orthogonal with respect to (1) reads*

$$\lim_{n\to\infty} ||\tilde{L}_n^{\alpha,\underline{M}} - \tilde{L}_n^{\alpha}||_{\alpha} = 0. \tag{13}$$

PROOF. Following a standard technique we can expand the Laguerre-Sobolev type orthogonal polynomials in terms of the Laguerre classical ones to obtain

$$\begin{aligned}\widetilde{L}_n^{\alpha,\underline{M}}(x) =&\tilde{L}_n^{\alpha}(x) - \sum_{k=0}^{n-1}\sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)\left(\tilde{L}_k^{\alpha}\right)^{(j)}(c)\tilde{L}_k^{\alpha}(x)\\ =&\tilde{L}_n^{\alpha}(x) - \sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)K_{n-1}^{(j,0)}(c,x). \qquad (14)\end{aligned}$$

At this point, estimations for $\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)$ when $j = 0,\dots,N$, $j \neq j_1 - 1,\dots,j_q - 1$, are needed.

In order to do that, we can write (14) evaluated at $x = c$ in a matrix form as follows,

$$A\mathbf{L}^{\alpha,\underline{M}} = \mathbf{L}^{\alpha},$$

where

$$A = \begin{pmatrix} 1 + M_0K_{n-1}(c,c) & M_1K_{n-1}^{(1,0)}(c,c) & M_2K_{n-1}^{(2,0)}(c,c) & \dots & M_NK_{n-1}^{(N,0)}(c,c)\\ M_0K_{n-1}^{(0,1)}(c,c) & 1 + M_1K_{n-1}^{(1,1)}(c,c) & M_2K_{n-1}^{(2,1)}(c,c) & \dots & M_NK_{n-1}^{(N,1)}(c,c)\\ M_0K_{n-1}^{(0,2)}(c,c) & M_1K_{n-1}^{(1,2)}(c,c) & 1 + M_2K_{n-1}^{(2,2)}(c,c) & \dots & M_NK_{n-1}^{(N,2)}(c,c)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ M_0K_{n-1}^{(0,N)}(c,c) & M_1K_{n-1}^{(1,N)}(c,c) & M_2K_{n-1}^{(2,N)}(c,c) & \dots & 1 + M_NK_{n-1}^{(N,N)}(c,c)\end{pmatrix},$$

$$\mathbf{L}^{\alpha} = \left(\tilde{L}_n^{\alpha}(c), \left(\tilde{L}_n^{\alpha}\right)'(c), \dots, \left(\tilde{L}_n^{\alpha}\right)^{(N)}(c)\right)^T,$$

and

$$\mathbf{L}^{\alpha,\underline{M}} = \left(\widetilde{L}_n^{\alpha,\underline{M}}(c), \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)'(c), \dots, \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(N)}(c)\right)^T.$$

Here, v^T denotes the transpose of the vector v. Then, applying Cramer's rule we get

$$\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(m-1)}(c) = \frac{\det(\mathbf{A_m})}{\det(\mathbf{A})}, \quad \text{for } m = 1,\dots,N+1,$$

where $\mathbf{A_m}$ is the matrix obtained by replacing the m-th column in the matrix $\mathbf{A}$ by the column vector $\mathbf{L}^{\alpha}$.

Thus, by using Lemmas 5 and 6, for n large enough we obtain

$$\left|\left(\tilde{L}_n^{\alpha,\underline{M}}\right)^{(m-1)}(c)\right| \leq Cn^{\frac{-2m-1}{4}}, \tag{15}$$

where C is a positive constant which does not depend on n.

Finally, in order to obtain (13) we take norm in (14). Thus

$$\begin{aligned} ||\widetilde{L}_n^{\alpha,\underline{M}} - \tilde{L}_n^{\alpha}||_\alpha^2 \leq & || \sum_{j=0}^{N} M_j \left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c) K_{n-1}^{(j,0)}(c,x)||_\alpha^2 \\ \leq & (N+1)\sum_{j=0}^{N} M_j^2 \left[\left(\widetilde{L}_n^{\alpha,\underline{M}}\right)^{(j)}(c)\right]^2 K_{n-1}^{(j,j)}(c,c). \end{aligned}$$

From Theorem 2 and (15) we get

$$\begin{aligned} ||\widetilde{L}_n^{\alpha,\underline{M}} - \tilde{L}_n^{\alpha}||_\alpha^2 \leq & (N+1)\sum_{j=0}^{N} C_j M_j^2 n^{\frac{-4(j+1)-2}{4}} n^{\frac{2j+1}{2}} \\ \leq & Cn^{-1}. \end{aligned}$$

□

Remark 2. Notice that in [5] estimates in the weighted L^2-norm for the difference between continuous Sobolev orthogonal polynomials associated with a vector of measures $(\psi W, W)$ and standard orthogonal polynomials associated with W, where W is an exponential weight $W(x) = e^{-2Q(x)}$ and ψ is a measurable and positive function on a set of positive measure, such that the moments of the Sobolev product are finite, have been obtained in terms of the Mhaskar-Rakhmanov-Saff number. The authors assume that Q is an even and convex function on the real line such that Q'' is continuous in $(0,\infty)$ and $Q' > 0$ in $(0,\infty)$, as well as for some $0 < \alpha < \beta$, $\alpha \leq \frac{xQ''(x)}{Q'(x)} \leq \beta$, $x \in (0,\infty)$ holds. The study of analogue estimates as above for general exponential weights constitutes an interesting problem in which we are working.

AppendixA. Estimates for $\det(\mathbf{A})$ and $\det(\mathbf{A_m})$

First of all we will need the following well-known result, see for instance, [20, vol. III, p. 311].

Lemma 4 (Cauchy's double alternant). *Let $x_1, \dots, x_n, y_1, \dots, y_n$ be real numbers. Then,*

$$\det\left[\frac{1}{x_i + y_j}\right]_{1\le i,j\le n} = \frac{\displaystyle\prod_{1\le i<j\le n}(x_i - x_j)(y_i - y_j)}{\displaystyle\prod_{1\le i,j\le n}(x_i + y_j)}.$$

Let us denote

$$M := \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} M_{l-1}, \qquad Q := \sum_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} l.$$

Lemma 5. *With the notation introduced in Section 3, we have*

$$\det(\boldsymbol{A}) \sim_n C_1 n^{\frac{2Q-(N+1)+q}{2}}, \tag{A.1}$$

where C_1 is a positive constant independent of n. In particular, there exists a constant $C_2 > 0$ such that

$$\det(\boldsymbol{A}) > C_2 n^{\frac{2Q-(N+1)+q}{2}}$$

for n large enough.

Proof. We denote by a_{ij}, $1 \le i, j \le N+1$, the (i,j) entry of the matrix $\mathbf{A}$. Notice that these entries verify

$$a_{ij} \sim_n \begin{cases} M_{j-1} K_{n-1}^{(j-1,i-1)}(c,c) & \text{for } j \text{ such that } M_{j-1} > 0,\\ 1 & \text{if } i = j \text{ and } M_{j-1} = 0,\\ 0 & \text{if } i \neq j \text{ and } M_{j-1} = 0. \end{cases}$$

Then, from Theorem 2, we obtain

$$a_{ij} \sim_n \begin{cases} M_{j-1} C_{0,j-1,i-1} n^{\frac{i+j-1}{2}}, & \text{if } i+j \equiv 0 \mod 2 \text{ and } M_{j-1} > 0,\\ M_{j-1} C_{1,j-1,i-1} n^{\frac{i+j-2}{2}}, & \text{if } i+j \equiv 1 \mod 2 \text{ and } M_{j-1} > 0,\\ 1 & \text{if } i = j \text{ and } M_{j-1} = 0,\\ 0 & \text{if } i \neq j \text{ and } M_{j-1} = 0. \end{cases} \tag{A.2}$$

Using the definition of determinant and (A.2), we get

$$\begin{aligned}\det(\mathbf{A}) &= \sum_{\delta\in\mathcal{S}_{N+1}} \operatorname{sgn}(\delta) a_{1,\delta(1)}\cdots a_{N+1,\delta(N+1)} \\ &\sim_n \sum_{\delta\in\mathcal{S}_{N+1}} \operatorname{sgn}(\delta) C_\delta n^{\frac{-p_1(\delta)}{2}} n^{-p_2(\delta)} \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} n^{\frac{l+\delta(l)}{2}},\end{aligned} \tag{A.3}$$

where $\mathcal{S}_{N+1}$ is the group of permutations of the set $\{1,\dots,N+1\}$, $p_1(\delta)$ (resp. $p_2(\delta)$) is the number of indexes l in $\{1,\dots,N+1\}\backslash\{j_1,\dots,j_q\}$ such that $l+\sigma(l)$ is even (resp. odd) and

$$C_\delta = \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} M_{\delta(l)-1} C_{0,\delta(l)-1,l-1}.$$

Let us define the set

$$\Delta = \left\{\delta\in\mathcal{S}_{N+1} \quad : \quad \begin{matrix} l+\delta(l) \text{ is even for all } l=1,\dots,N+1, \\ \text{and } \delta(l)=l, \text{ for } l=j_1,\dots,j_q \end{matrix}\right\}.$$

Notice that $\frac{p_1(\delta)}{2}+p_2(\delta)$ attains a minimum when $p_2(\delta)=0$. Then, the asymptotic behavior of (A.3) will be given by the terms corresponding to permutations in Δ, if they do not vanish. Thus, we have to check that $\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta$ is not zero.

$$\begin{aligned}\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta &= \sum_{\delta\in\Delta}\operatorname{sgn}(\delta) \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} M_{\delta(l)-1} C_{0,\delta(l)-1,l-1} \\ &= \sum_{\delta\in\Delta}\operatorname{sgn}(\delta) \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} M_{\delta(l)-1}(-1)^{\frac{l+\sigma(l)}{2}+l}\sigma^{\alpha}(c)\sigma^{\alpha+l+\sigma(l)-1}(c)\sqrt{c}\frac{1}{l+\delta(l)-1}.\end{aligned} \tag{A.4}$$

Recalling that $\sigma^{\alpha+l+\delta(l)-1}(c) = \pi^{-1/2}e^{c/2}c^{\frac{-\alpha+l+\delta(l)-1}{2}}c^{-1/4}$, we get

$$\prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} \sigma^{\alpha+l+\delta(l)-1}(c) = \pi^{-\frac{N+1-q}{2}} e^{\frac{(N+1-q)c}{2}} c^{-\frac{\alpha(N+1-q)+2Q-(N+1-q)}{2}}$$
$$= \left(\sigma^{\alpha-1}(c)\right)^{N+1-q} c^{-Q} = \left(\sigma^{\alpha}(c)\right)^{N+1-q} c^{\frac{N+1-q}{2}} c^{-Q}.$$

After some computations, (A.4) becomes

$$\sum_{\delta\in\Delta} \operatorname{sgn}(\delta) C_\delta = M(-1)^{2Q} \left(\sigma^{\alpha}(c)\right)^{2(N+1-q)} c^{N+1-q-Q} \sum_{\delta\in\Delta} \operatorname{sgn}(\delta) \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} \frac{1}{l+\delta(l)-1},$$

Now, let consider

$$\{1,2,\dots,N+1\}\backslash\{j_1,j_2,\dots,j_q\} = \{r_1,r_2,\dots,r_{K_1}\} \cup \{s_1,s_2,\dots,s_{K_2}\}$$

where r_i is odd for $i = 1,2,\dots,K_1$ and s_i is even for $i = 1,2,\dots,K_2$. Notice that $K_1 + K_2 = N + 1 - q$. Then, we have

$$\sum_{\delta\in\Sigma} \operatorname{sgn}(\delta) \prod_{\substack{l=1\\ l\neq j_1,\dots,j_q}}^{N+1} \frac{1}{l+\delta(l)-1} =$$
$$= \sum_{\delta\in S_{K_1}} \sum_{\xi\in S_{K_2}} \operatorname{sgn}(\delta)\operatorname{sgn}(\xi) \prod_{i=1}^{K_1} \frac{1}{r_i + r_{\delta(i)} - 1} \prod_{j=1}^{K_2} \frac{1}{s_j + s_{\xi(j)} - 1}$$
$$= \sum_{\delta\in S_{K_1}} \operatorname{sgn}(\delta) \prod_{i=1}^{K_1} \frac{1}{r_i + r_{\delta(i)} - 1} \sum_{\xi\in S_{K_2}} \operatorname{sgn}(\xi) \prod_{j=1}^{K_2} \frac{1}{s_j + s_{\xi(j)} - 1}$$
$$= \frac{\prod_{1\le i<j\le K_1} (r_i - r_j)^2}{\prod_{1\le i<j\le K_1} (r_i + r_j - 1)} \frac{\prod_{1\le i<j\le K_2} (s_i - s_j)^2}{\prod_{1\le i<j\le K_2} (s_i + s_j - 1)},$$

where we have used Lemma 4 in the sense

$$\det\left[\frac{1}{r_i - \frac{1}{2} + r_j - \frac{1}{2}}\right]_{1\le i,j\le K_1} = \sum_{\delta\in S_{K_1}} \operatorname{sgn}(\delta) \prod_{i=1}^{K_1} \frac{1}{r_i + r_{\delta(i)} - 1}.$$

Finally, (A.4) becomes

$$\sum_{\delta\in\Delta}\operatorname{sgn}(\delta)C_\delta = M\,(\sigma^\alpha(c))^{2(N+1-q)}\,c^{N+1-q-Q}\,\frac{\prod\limits_{1\le i<j\le K_1}(r_i-r_j)^2}{\prod\limits_{1\le i<j\le K_1}(r_i+r_j-1)}\,\frac{\prod\limits_{1\le i<j\le K_2}(s_i-s_j)^2}{\prod\limits_{1\le i<j\le K_2}(s_i+s_j-1)}$$

which is, as desired, different from zero. Then, we can state that

$$\det(\mathbf{A}) \sim_n Cn^{\frac{2Q-(N+1)+q}{2}}, \tag{A.5}$$

where C is a positive constant independent of n. This concludes the proof.

□

Lemma 6. *For n large enough, there exists a constant $C>0$ such that*

$$|\det(\boldsymbol{A_m})| \le Cn^{\frac{2Q-m-N+q}{2}-\frac{3}{4}}.$$

Proof. Notice that for $i\neq m$, the entries of the matrix $\mathbf{A_m}$ are the same as those of the matrix $\mathbf{A}$. Their asymptotic behavior was given in (A.2). Let us denote by $\hat{a}_{im}$ the (i,m) entry of the matrix $\mathbf{A_m}$.

According to (8), we have

$$\hat{a}_{im} = (\tilde{L}_n^\alpha)^{(i-1)}(c) \sim_n (-1)^{n-i+1}\sigma^{\alpha+i-1}(c)n^{\frac{i}{2}-\frac{3}{4}}\cos\varphi_{n-i+1}^{n+i-1}(c), \tag{A.6}$$

for $i=1,\dots,N+1$.

We expand $\det(\mathbf{A_m})$ along the m-th column:

$$\det(\mathbf{A_m}) = \sum_{i=1}^{N+1}(-1)^{i+m}\hat{a}_{im}\det\mathbf{B_{im}}, \tag{A.7}$$

where $\mathbf{B_{im}}$ is the $N\times N$ matrix obtained by deleting of $\mathbf{A}$ the i-th row and the m-th column.

Using (A.6) in (A.7), we obtain

$$\det(\mathbf{A_m}) \sim_n \sum_{i=1}^{N+1}(-1)^{n+m+1}\sigma^{\alpha+i-1}(c)n^{\frac{i}{2}-\frac{3}{4}}\cos\varphi_{n-i+1}^{n+i-1}(c)\det\mathbf{B_{im}},$$

where $\det \mathbf{B_{im}}$ can be computed as

$$\det \mathbf{B_{im}} = \sum_{\sigma \in S_N} \operatorname{sgn}(\sigma) \prod_{l=1}^{N} b_{l,\sigma(l)} = \sum_{\psi \in \Psi} \operatorname{sgn}(\psi) \prod_{\substack{l=1 \\ l \neq i, j_1, \ldots, j_q}}^{N+1} a_{l,\psi(l)}, \tag{A.8}$$

with

$$\Psi = \left\{ \psi \in S_{N+1} \quad : \quad \begin{array}{c} \psi(l) = l, \text{ for } l = j_1, \ldots, j_q \\ \text{and } \psi(i) = m \end{array} \right\}.$$

Now, we will discuss two cases:

1. **Case $i + m$ even.**
   The highest power of n that can be reached in the sum (A.8) appears when $l + \psi(l)$ is even for all $l = 1, \ldots, N+1, l \neq i, j_1, \ldots, j_q$. This means that

   $$\det \mathbf{B_{im}} \sim_n C \left(\sum_{\gamma \in \Gamma} \operatorname{sgn}(\gamma) C'_\gamma \right) n^{\frac{2Q-i-m-N+q}{2}},$$

   with

   $$\Gamma = \left\{ \gamma \in S_{N+1} \quad : \quad \begin{array}{c} l + \gamma(l) \text{ is even for all } l = 1, \ldots, N+1, \\ \gamma(l) = l, \text{ for } l = j_1, \ldots, j_q \\ \text{and } \gamma(i) = m \end{array} \right\},$$

   whenever

   $$\sum_{\gamma \in \Gamma} \operatorname{sgn}(\gamma) C'_\gamma \neq 0. \tag{A.9}$$

2. **Case $i + m$ odd.**
   In this case, the highest power of n in the sum (A.8) could be at most $\frac{2Q-i-m-N+q-1}{2} - 1$, when the permutation ψ satisfies that $l + \psi(l)$ is odd for one $l \in \{1, \ldots, N+1\} \backslash \{i, j_1, \ldots, j_q\}$, and it is even for the remainder indexes.

We obtain the highest power of n for the first case, and after checking (A.9), we conclude

$$\det(\mathbf{A_m}) \sim_n \sum_{i=1}^{N+1}(-1)^{n+m+1}\sigma^{\alpha+i-1}(c)n^{\frac{2Q-m-N+q}{2}-\frac{3}{4}}\cos\varphi_{n-i+1}^{n+i-1}(c).$$

Then, for n large enough, there exists a constant $C>0$ such that

$$|\det(\mathbf{A_m})| \leq Cn^{\frac{2Q-m-N+q}{2}-\frac{3}{4}}.$$

In order to conclude the proof we must check that (A.9) holds. Indeed,

$$\begin{aligned}&\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)\prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1} M_{\gamma(l)-1}C_{0,\gamma(l)-1,l-1}\\ &=\frac{M}{M_{m-1}}(-1)^{Q-i}\left(\sigma^{\alpha}(c)\right)^{2(N-q)}c^{\frac{2N+i+m-2q-2Q}{2}}\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)\prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1}\frac{1}{l+\gamma(l)-1}.\end{aligned}$$

Let suppose now that m is even. Let

$$\{1,2,\dots,N+1\}\backslash\{i,j_1,j_2,\dots,j_q\}=\{r_1,r_2,\dots,r_{K_1}\}\cup\{m,s_1,s_2,\dots,s_{K_2}\},$$

where r_i is odd for $i=1,2,\dots,K_1$, and s_i is even for $i=1,2,\dots,K_2$. Notice that $K_1+K_2=N-q$. We can write

$$\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)\prod_{\substack{l=1\\ l\neq i,j_1,\dots,j_q}}^{N+1}\frac{1}{l+\gamma(l)-1}=\Big(\sum_{\delta\in S_{K_1}}\operatorname{sgn}(\delta)\prod_{i=1}^{K_1}\frac{1}{r_i+r_{\delta(i)}-1}\Big)\det\mathbf{B},$$

where

$$\mathbf{B}=\begin{pmatrix}\frac{1}{m+i-1} & \frac{1}{m+s_1-1} & \cdots & \frac{1}{m+s_{K_2}-1}\\ \frac{1}{s_1+i-1} & \frac{1}{s_1+s_1-1} & \cdots & \frac{1}{s_1+s_{K_2}-1}\\ \vdots & \vdots & \ddots & \vdots\\ \frac{1}{s_{K_2}+i-1} & \frac{1}{s_{K_2}+s_1-1} & \cdots & \frac{1}{s_{K_2}+s_{K_2}-1}.\end{pmatrix},$$

Using Lemma 4,

$$\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)\prod_{\substack{l=1\\ l\neq i,j_1,\ldots,j_q}}^{N+1}\frac{1}{l+\gamma(l)-1}$$
$$=\frac{\prod_{l=1}^{K_2}(m-s_l)(i-s_l)\prod_{1\le i<j\le K_2}(s_i-s_j)^2}{(m+i-1)\prod_{1\le i<j\le K_2}(s_i+s_j-1)}\frac{\prod_{1\le i<j\le K_1}(r_i-r_j)^2}{\prod_{1\le i<j\le K_1}(r_i+r_j-1)}.$$

In an analogue way, if m is odd, let

$$\{1,2,\ldots,N+1\}\backslash\{i,j_1,j_2,\ldots,j_q\}=\{m,r_1,r_2,\ldots,r_{K_1}\}\cup\{s_1,s_2,\ldots,s_{K_2}\},$$

where r_i is odd for $i=1,2,\ldots,K_1$, and s_i is even for $i=1,2,\ldots,K_2$, and

$$\sum_{\gamma\in\Gamma}\operatorname{sgn}(\gamma)\prod_{\substack{l=1\\ l\neq i,j_1,\ldots,j_q}}^{N+1}\frac{1}{l+\gamma(l)-1}$$
$$=\frac{\prod_{l=1}^{K_1}(m-r_l)(i-r_l)\prod_{1\le i<j\le K_1}(r_i-r_j)^2}{(m+i-1)\prod_{1\le i<j\le K_1}(r_i+r_j-1)}\frac{\prod_{1\le i<j\le K_2}(s_i-s_j)^2}{\prod_{1\le i<j\le K_2}(s_i+s_j-1)}.$$

This is different from zero and we get our statement.

□

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