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AN ELECTROSTATIC MODEL FOR ZEROS OF PERTURBED LAGUERRE POLYNOMIALS

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Abstract. In this paper we consider the sequences of polynomials $\{Q_n^{(\alpha)}\}_{n\geq 0}$, orthogonal with respect to the inner product

$$\langle f,g\rangle_\nu = \int_0^{+\infty} f(x)g(x)d\mu(x) + \sum_{j=1}^{m} a_j\, f(c_j)g(c_j),$$

where $d\mu(x) = x^\alpha e^{-x}$ is the Laguerre measure on $\mathbb{R}_+$, $\alpha > -1$, $c_j < 0$, $a_j > 0$ and f, g are polynomials with real coefficients. We first focus our attention in the representation of these polynomials in terms of the standard Laguerre polynomials. Next we find the explicit formula for their outer relative asymptotics, as well as the holonomic equation that such polynomials satisfy. Finally, an electrostatic interpretation of their zeros in terms of a logarithmic potential is presented.

1. Introduction

One of the most important topics in the theory of orthogonal polynomials is the location of their zeros and critical points, as well as their asymptotic behavior. This interest is motivated because both the zeros and critical points play a key role in several applications in many areas of engineering and physics, such as interpolation, quadrature formulas, rational approximation, and electrostatics, among others.

The Laguerre monic orthogonal polynomials $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$, $\alpha > -1$, (see for instance [19, Ch. 5], [11], [2], [17] among others), are a family of *classical orthogonal polynomials* defined by the orthogonality relations

$$\int_0^{+\infty} \widehat{L}_n^\alpha(x) x^k d\mu(x) = 0, \quad \text{for } k = 0,1,\ldots,n-1,$$

where $\widehat{L}_n^\alpha(x) = x^n +$ *lower degree terms* and $d\mu(x) = x^\alpha e^{-x}dx$, $\alpha > -1$. Equivalently, they are the family of monic polynomials orthogonal with respect to the inner product

$$\langle f,g\rangle_\alpha = \int_0^{+\infty} f(x)g(x)d\mu(x), \tag{1.1}$$

where f, $g \in \mathbb{P}$, the linear space of polynomials with real coefficients.

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The monic Laguerre polynomials are the polynomial solutions of the second order linear differential equation with polynomial coefficients (see [19, 5.1, Formula (5.1.3)])

$$x\,y'' + (\alpha+1-x)\,y' + n\,y = 0, \quad n \geq 0. \tag{1.2}$$

Furthermore, they satisfy (see [19, 5.1, Formula (5.1.14)])

$$x[\widehat{L}_n^{\alpha}(x)]'(x) = n\widehat{L}_n^{\alpha}(x) + n(n+\alpha)\widehat{L}_{n-1}^{\alpha}(x), \tag{1.3}$$

$$x[\widehat{L}_{n-1}^{\alpha}(x)]'(x) = -\widehat{L}_n^{\alpha}(x) + (x-(n+\alpha))\widehat{L}_{n-1}^{\alpha}(x). \tag{1.4}$$

The differential equation (1.2) can be deduced by different and alternative ways which are closely connected with the calculation of the discriminant of Laguerre polynomials and can be interpreted in terms of a problem of electrostatic equilibrium (see [19, Th. 6.7.2 and Prob. 38]).

Using the Laguerre measure $d\mu$ we introduce another measure

$$d\nu = d\nu_{\mathbf{c}} = d\mu + \sum_{j=1}^{m} a_j\,\delta_{c_j},$$

where δ_{c_j} are unit masses located at the points $\mathbf{c} = \{c_1, c_2, \ldots, c_m\} \subset \mathbb{R}\setminus\mathbb{R}_+$, such that if $i \neq j$ then $c_i \neq c_j$, a_j are positive real numbers and m is a positive integer. Let us introduce the following inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients

$$\langle f, g\rangle_\nu = \int_0^{+\infty} f(x)g(x)d\nu(x) = \int_0^{+\infty} f(x)g(x)d\mu(x) + \sum_{j=1}^{m} a_j\,f(c_j)g(c_j). \tag{1.5}$$

In recent years, there has been an increasing interest in the so called spectral transformations of measures. They have been analyzed from different points of view by several authors. This work is focused on the application of Stieltjes' ideas (see [7] and [20]) to study of the electrostatic interpretation of zeros of orthogonal polynomial sequences associated with Uvarov transformations of the measures.

A particular case of such perturbations appears in the pioneer work of T. H. Koornwinder [13], who analyzed a general situation for Jacobi weights when two mass points are added at the end points of the support of the Jacobi measure. In [12] analytic properties of orthogonal polynomials with respect to a perturbation of the Laguerre weight when a mass is added at $x = 0$ are considered. In [18], the holonomic equation for such perturbations when the mass point is located in the negative real semi-axis is deduced.

When $m = 1$ and $c_1 = 0$ in (1.5), an electrostatic interpretation of the zeros as equilibrium points with respect to a logarithmic potential, under the action of an external field, has been obtained in [3] and [9]. Analytic properties, as well as the outer relative asymptotics, Mehler-Heine, and Plancherel-Rotach formulas for these polynomials have been obtained in [1].

When $c_1 \in \mathbb{R}_-$, (i.e. the mass point is located outside the support of the standard Laguerre measure), a first approach was done in [4] and [6], where a representation of these polynomials in terms of standard Laguerre polynomials is given. From it, in [5] the authors deduce a hypergeometric representation as well as a second order linear differential equation that they satisfy. In [8] an electrostatic interpretation of the zeros of such perturbed orthogonal polynomials is given, following a different approach based in the work of Ismail (see [9], [10]). There, the authors provide a complete description of the case $m = 1$ for two different locations of the mass points, namely when either $c_1 = 0$ or $c_1 < 0$, among other results. About the current state of art on electrostatic models for zeros of orthogonal polynomials we recommend to read the survey papers ([9], [10] and [16]).

In this contribution we focus our attention on some analytic properties of orthogonal polynomials with respect to the inner product (1.5), and we deal with a natural generalization of the problems considered in [5] and [8]. In Section 2 we introduce the notion of Laguerre-type orthogonal polynomials and give a representation of these polynomials in terms of Laguerre polynomials and the kernel polynomial associated with them. The main result of Section 3 is the outer relative asymptotics of these polynomials, that is a new contribution obtained in an alternative way. In Section 4, the holonomic equation satisfied by these polynomials is given. This is a new result in the literature. As a consequence, in Section 5, an electrostatic interpretation of zeros of Laguerre type orthogonal polynomials is presented. The last section contains a numerical experiment on the location of zeros of Laguerre type orthogonal polynomials outside th e interval $[0, +\infty)$ and the position of the source-charges associated with the above interpretation.

2. Perturbed Laguerre polynomials

Firstly, we have summarized some basic properties of Laguerre orthogonal polynomials to be used in the sequel.

Proposition 2.1. *Let $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ be the sequence of standard Laguerre orthogonal polynomials, i.e the sequence of orthogonal polynomials with respect to the inner product (1.1) and leading coefficient $\frac{(-1)^n}{n!}$. Then the following statements hold (see* [19, §5.1]*).*

(1)
$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^{\alpha}(x).$$

(2) *Three term recurrence relation.*

$$nL_n^{(\alpha)}(x) = (-x + 2n + \alpha - 1)L_{n-1}^{(\alpha)}(x) - (n + \alpha - 1)L_{n-2}^{(\alpha)}(x), \quad n \geq 2, \tag{2.1}$$

with $L_0^{(\alpha)}(x) = 1$, $L_1^{(\alpha)}(x) = -x + \alpha + 1$.

(3) *Structure relation. For every $n \in \mathbb{N}$,*
$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x).$$

(4) *Norm. For every $n \in \mathbb{N}$,*
$$||L_n^{(\alpha)}||_\alpha^2 = \langle L_n^{(\alpha)}, L_n^{(\alpha)}\rangle_\alpha = \Gamma(\alpha+1)\binom{n+\alpha}{n}.$$

(5) *Hahn's condition. For every $n \in \mathbb{N}$,*
$$[L_n^{(\alpha)}]'(x) = -L_{n-1}^{(\alpha+1)}(x).$$

Furthermore, we denote the n-th degree kernel polynomial associated with Laguerre polynomials by

$$K_n(x,y) = \sum_{j=0}^{n} \frac{L_j^{(\alpha)}(x)L_j^{(\alpha)}(y)}{||L_j^{(\alpha)}||_\alpha^2}. \tag{2.2}$$

For every $n \in \mathbb{N}$, we have the so called Christoffel-Darboux formula (see [19, Th. 3.2.2])

$$K_n(x,y) = \frac{(n+1)}{||L_n^{(\alpha)}||_\alpha^2} \cdot \frac{L_n^{(\alpha)}(x)L_{n+1}^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(x)L_n^{(\alpha)}(y)}{x-y}. \tag{2.3}$$

The confluent form of the n-th degree kernel is

$$K_n(x,x) = \frac{(n+1)}{||L_n^{(\alpha)}||_\alpha^2} \cdot \left\{ [L_n^{(\alpha)}(x)]' L_{n+1}^{(\alpha)}(x) - [L_{n+1}^{(\alpha)}(x)]' L_n^{(\alpha)}(x) \right\}. \tag{2.4}$$

Let $\widehat{Q}_n(x) = \widehat{Q}_n^{(\alpha,\mathbf{c})}(x)$ be the n-th monic orthogonal polynomial with respect to $d\nu$, $n \in \mathbb{Z}_+$, i.e.

$$\int_0^{+\infty} \widehat{Q}_n(x)\, x^k\, d\mu(x) + \sum_{j=1}^{m} a_j\, \widehat{Q}_n(c_j)\, c_j^k = 0, \quad k = 0,1,2,\ldots,n-1. \tag{2.5}$$

$\widehat{Q}_n(x)$ is said to be the n-th Laguerre perturbed monic polynomial or generalized Krall-Laguerre type orthogonal polynomial. If

$$R_m(z) = \prod_{j=1}^{m}(z-c_j), \tag{2.6}$$

then is straightforward to see that $\widehat{Q}_n(x)$ is quasi-orthogonal of order m with respect to $R_m(x)d\mu$, i.e.

$$\int_0^{+\infty} x^k \widehat{Q}_n(x) R_m(x) d\mu(x) = 0, \quad k = 0,1,2,\ldots,n-m-1. \tag{2.7}$$

As a well known consequence (see [19, §3.3]), the polynomial $\widehat{Q}_n(x)$ has at least $n-m$ changes of sign on $[0,+\infty)$. Hence, $\widehat{Q}_n(x)$ has at least $n-m$ zeros of odd multiplicity on $(0,+\infty)$. Furthermore, there is at most one zero of $\widehat{Q}_n$ in each gap between c_k's, assuming $c_0 = 0$. This can be proved by contradiction. Suppose that the polynomial $\widehat{Q}_n(x)$, orthogonal with respect to the inner product (1.5), has two different simple zeros x_1 and x_2 both inside the interval (c_k, c_{k-1}). We can write $\widehat{Q}_n(x)$ in the form

$$\widehat{Q}_n(x) = (x-x_1)(x-x_2)q_{n-2}(x), \tag{2.8}$$

where $q_{n-2}(x)$ is certain polynomial of degree $n-2$. Obviously,

$$\int_0^{+\infty} \widehat{Q}_n(x) q_{n-2}(x) d\nu(x) = 0 \tag{2.9}$$

because the orthogonality of $\widehat{Q}_n(x)$ to polynomials of lower degree. Note that $(x-x_1)(x-x_2) > 0$ if $x \notin (c_k, c_{k-1})$.

On the other hand, from (2.8)

$$\begin{aligned} &\int_0^{+\infty} \widehat{Q}_n(x) q_{n-2}(x) d\nu(x) \\ = &\int_0^{+\infty} (x-x_1)(x-x_2) q_{n-2}^2(x) d\nu(x) > 0, \end{aligned}$$

contrary to (2.9). This completes the proof.

From the Fourier expansion of the polynomials $\{\widehat{Q}_n(x)\}_{n\geq 0}$ in terms of the monic polynomials $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ and the definition (2.2) of kernel polynomial associated with Laguerre polynomials, it is straightforward that $\{\widehat{Q}_n\}_{n\geq 0}$ and $\{\widehat{L}_n^\alpha\}_{n\geq 0}$ are related by

$$\widehat{Q}_n(x) = \widehat{L}_n^\alpha(x) - \sum_{j=1}^{m} a_j\, \widehat{Q}_n(c_j)\, K_{n-1}(x, c_j). \tag{2.10}$$

Evaluating (2.10) in $x = c_k$, with $k = 1, 2, \ldots, m$, we obtain the following system of m linear equations ($1 \leq k \leq m$) with m unknowns $\widehat{Q}_n(c_j)$ ($1 \leq j \leq m$)

$$\widehat{L}_n^\alpha(c_k) = (1 + a_k K_{n-1}(c_k, c_k))\, \widehat{Q}_n(c_k) + \sum_{\substack{j=1\\ j\neq k}}^{m} a_j K_{n-1}(c_j, c_k)\, \widehat{Q}_n(c_j)$$

or, equivalently, $\mathbb{A}_n \mathbb{X}_n = \mathbb{L}_n$, where

$$\mathbb{A}_n = \begin{pmatrix} 1 + a_1 K_{n-1}(c_1, c_1) & a_2 K_{n-1}(c_2, c_1) & \cdots & a_m K_{n-1}(c_m, c_1) \\ a_1 K_{n-1}(c_1, c_2) & 1 + a_2 K_{n-1}(c_2, c_2) & \cdots & a_m K_{n-1}(c_m, c_2) \\ \vdots & \vdots & \ddots & \vdots \\ a_1 K_{n-1}(c_1, c_m) & a_2 K_{n-1}(c_2, c_m) & \cdots & 1 + a_m K_{n-1}(c_m, c_m) \end{pmatrix},$$

$$\mathbb{X}_n = \begin{pmatrix} \widehat{Q}_n(c_1) \\ \widehat{Q}_n(c_2) \\ \vdots \\ \widehat{Q}_n(c_m) \end{pmatrix} \quad \text{and} \quad \mathbb{L}_n = \begin{pmatrix} \widehat{L}_n^\alpha(c_1) \\ \widehat{L}_n^\alpha(c_2) \\ \vdots \\ \widehat{L}_n^\alpha(c_m) \end{pmatrix}.$$

Additionally, we denote by $\det \mathbb{M}$ the determinant of the square matrix $\mathbb{M}$ and by $\mathbb{A}_{n,j}$ the square matrix of order m obtained by replacing the j-th column of the matrix $\mathbb{A}_n$ by $\mathbb{L}_n$. A linear system as above has a unique solution if and only if the determinant of the matrix $\mathbb{A}_n$ does not vanish for each fixed $n \in \mathbb{N}$. It is important to point out that the last one is a necessary and sufficient condition for the existence of every perturbed polynomial $\widehat{Q}_n(x)$ of n-th degree. Hence, the values $\widehat{Q}_n(c_j)$ can be calculated from the values of $\widehat{L}_n^\alpha(c_j)$ as

$$\widehat{Q}_n(c_j) = \frac{\det \mathbb{A}_{n,j}}{\det \mathbb{A}_n}, \quad \text{with } j = 1, 2, \ldots, m.$$

3. Outer Relative Asymptotics

In this section we need two useful lemmas concerning the rate of convergence of the ratio of two classical Laguerre polynomials of different parameter and degree (see Lemmas 1 and 2 in [5]) outside the support of the measure.

Lemma 3.1. *Given two Laguerre polynomials as in Proposition 2.1, of the same parameter α and different degree, the following statement holds. For $x \in \mathbb{C} \setminus \mathbb{R}_+$*

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}} j + \left[\left(\frac{\alpha}{2} - \frac{1}{4}\right) j - \frac{x}{2} j^2\right] \frac{1}{n} + \mathcal{O}(n^{-3/2})$$

where $\sqrt{-x}$ must be taken real and positive if $x < 0$.

Lemma 3.2. *For any $\alpha > -1$ and $x \in \mathbb{C} \setminus \mathbb{R}_+$, locally uniform it holds that*

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+1)}(x)} = \frac{\sqrt{-x}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2} + \frac{1}{4}\right) + \frac{x}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$
$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+2)}(x)} = \frac{-x}{n} + \mathcal{O}(n^{-3/2}),$$

where $\sqrt{-x}$ must be taken real and positive if $x < 0$.

Using these two lemmas, we deduce

$$\begin{aligned} \frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(c_j)}{L_n^{(\alpha)}(c_j)} &= \frac{\sqrt{|c_j|} - \sqrt{|c_i|}}{\sqrt{n}} + \frac{|c_i| - |c_j|}{2n} + \mathcal{O}(n^{-3/2}), \\ \frac{L_{n-1}^{(\alpha+1)}(c_i)}{L_{n-2}^{(\alpha+1)}(c_i)} - \frac{L_n^{(\alpha)}(c_i)}{L_{n-1}^{(\alpha)}(c_i)} &= \frac{1}{2n} + \mathcal{O}\left(n^{-3/2}\right). \end{aligned} \tag{3.1}$$

On the other hand, from (2.10) we have

$$Q_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) - \sum_{j=1}^{m} a_j Q_n^{(\alpha)}(c_j) K_{n-1}(c_j, x), \tag{3.2}$$

where $Q_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\widehat{Q}_n(x)$. Dividing by $L_n^{(\alpha)}(x)$ in both sides of (3.2), we get

$$\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = 1 - \sum_{j=1}^{m} a_j Q_n^{(\alpha)}(c_j) \frac{K_{n-1}(c_j, x)}{L_n^{(\alpha)}(x)}. \tag{3.3}$$

Next, we will analyze

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \tag{3.4}$$

when $x \in \mathbb{C}\setminus\mathbb{R}_+$. In order to prove the existence of such a limit, we will find the solutions of the following linear system

$$\left[1 + a_k K_{n-1}(c_k, c_k)\right] Q_n^{(\alpha)}(c_k) + \sum_{\substack{j=1 \\ j\neq k}}^{m} a_j K_{n-1}(c_j, c_k) Q_n^{(\alpha)}(c_j) = L_n^{(\alpha)}(c_k) \tag{3.5}$$

with $k = 1, 2, \ldots, m$, obtained from (3.2) where x is evaluated at $c_1, c_2, \ldots, c_m$. Let us define

$$P_n^{(\alpha)}(c_j, x) = -a_j Q_n^{(\alpha)}(c_j) \frac{K_{n-1}(c_j, x)}{L_n^{(\alpha)}(x)} \tag{3.6}$$

and

$$\lim_{n\to\infty} P_n^{(\alpha)}(c_j, x) = \bar{p}^{(\alpha)}(c_j, x). \tag{3.7}$$

From (3.3) and (3.4) we need to figure out the values of $\bar{p}^{(\alpha)}(c_1, x), \ldots, \bar{p}^{(\alpha)}(c_m, x)$ to obtain the outer strong asymptotic for $Q_n^{(\alpha)}(x)$. From (3.6) we have

$$Q_n^{(\alpha)}(c_j) = \frac{-L_n^{(\alpha)}(x) P_n^{(\alpha)}(c_j, x)}{a_j K_{n-1}(c_j, x)}$$

and then, for $j = 1, \dots, m$, we replace these expressions in (3.5) to obtain the next linear system in the unknowns $P_n^{(\alpha)}(c_1, x), \dots, P_n^{(\alpha)}(c_m, x)$

$$
\left.\begin{array}{ccccccc}
\Phi_n(1,x)P_n^{(\alpha)}(c_1,x) & + & \cdots & + & \Psi_n(1,m,x)P_n^{(\alpha)}(c_m,x) & = & -1 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\Psi_n(m,1,x)P_n^{(\alpha)}(c_1,x) & + & \cdots & + & \Phi_n(m,x)P_n^{(\alpha)}(c_m,x) & = & -1
\end{array}\right\}, \tag{3.8}
$$

where

$$
\Phi_n(i,x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(c_i)} \frac{1/a_i + K_{n-1}(c_i,c_i)}{K_{n-1}(c_i,x)} \tag{3.9}
$$

$$
\Psi_n(i,j,x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(c_j)} \frac{K_{n-1}(c_i,c_j)}{K_{n-1}(c_i,x)}. \tag{3.10}
$$

No matter the number of equations of the previous system, in each of the m previous equations we will have always only two different quantities. Only one (in each equation) of the type $\Phi_n(i,x)$ and $m-1$ of the type $\Psi_n(i,j,x)$.

Next we estimate the rate of convergence of (3.9) and (3.10) as $n \to \infty$. Taking into account

$$
u - v = (\sqrt{|v|} + \sqrt{|u|})(\sqrt{|v|} - \sqrt{|u|}), \quad \forall u, v \in \mathbb{R}_-,
$$

in (3.10) we obtain for $x \in \mathbb{C} \setminus \mathbb{R}_+$

$$
\Psi_n(i,j,x) = \frac{(\sqrt{-x} + \sqrt{|c_i|})}{(\sqrt{|c_j|} + \sqrt{|c_i|})} \frac{(\sqrt{-x} - \sqrt{|c_i|})}{(\sqrt{|c_j|} - \sqrt{|c_i|})} \frac{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(c_j)}{L_n^{(\alpha)}(c_j)} \right)}{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \right)}, \quad i \neq j,
$$

where $\sqrt{c_j} > 0$. From (3.1) we deduce $\Psi_n(i,j,x) = \dfrac{(\sqrt{-x} + \sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}{(\sqrt{|c_j|} + \sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}$, and, as a consequence,

$$
\lim_{n \to \infty} \Psi_n(i,j,x) = \frac{\sqrt{-x} + \sqrt{|c_i|}}{\sqrt{|c_j|} + \sqrt{|c_i|}}. \tag{3.11}
$$

On the other hand,

$$
\Phi_n(i,x) = (c_i - x) \frac{\frac{||L_n^{(\alpha)}||_\alpha^2}{n \cdot a_i \left(L_n^{(\alpha)}(c_i)\right)^2} + \frac{L_{n-2}^{(\alpha+1)}(c_i)}{L_n^{(\alpha)}(c_i)} \frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} \left(\frac{L_{n-1}^{(\alpha+1)}(c_i)}{L_{n-2}^{(\alpha+1)}(c_i)} - \frac{L_n^{(\alpha)}(c_i)}{L_{n-1}^{(\alpha)}(c_i)} \right)}{\left(\frac{L_{n-1}^{(\alpha)}(c_i)}{L_n^{(\alpha)}(c_i)} - \frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \right)}
$$

and, combining (3.1), (3.2), and (3.1) we get

$$
\Phi_n(i,x) = \frac{(\sqrt{-x} + \sqrt{|c_i|})(\sqrt{-x} - \sqrt{|c_i|}) \frac{1}{2\sqrt{|c_i|}} + \mathcal{O}(n^{-1/2})}{(\sqrt{-x} - \sqrt{|c_i|}) + \mathcal{O}(n^{-1/2})}.
$$

Thus,

$$
\lim_{n \to \infty} \Phi_n(i,x) = \frac{\sqrt{-x} + \sqrt{|c_i|}}{2\sqrt{|c_i|}}. \tag{3.12}
$$

Next, from (3.7), (3.11), and (3.12), and taking limits, when $n \to \infty$, in both hand sides of (3.8)

$$\left.\begin{array}{rcl} \frac{\sqrt{-x}+\sqrt{|c_1|}}{2\sqrt{|c_1|}}\bar{p}^{(\alpha)}(c_1,x)+\cdots+\frac{\sqrt{-x}+\sqrt{|c_m|}}{\sqrt{|c_1|}+\sqrt{|c_m|}}\bar{p}^{(\alpha)}(c_m,x) & = & -1 \\ \vdots & \vdots & \vdots \\ \frac{\sqrt{-x}+\sqrt{|c_1|}}{\sqrt{|c_m|}+\sqrt{|c_1|}}\bar{p}^{(\alpha)}(c_1,x)+\cdots+\frac{\sqrt{-x}+\sqrt{|c_m|}}{2\sqrt{|c_m|}}\bar{p}^{(\alpha)}(c_m,x) & = & -1 \end{array}\right\}$$

It is not difficult to prove that the m solutions of the above linear system are

$$\bar{p}^{(\alpha)}(c_i,x)=\frac{-2\sqrt{|c_i|}}{\sqrt{-x}+\sqrt{|c_i|}}\prod_{\substack{j=1\\ j\neq i}}^{m}\left(\frac{\sqrt{|c_i|}+\sqrt{|c_j|}}{\sqrt{|c_i|}-\sqrt{|c_j|}}\right),\quad \forall i=1,\ldots,m.$$

Now, from (3.6) and (3.7) we conclude that, if $x\in\mathbb{C}\setminus\mathbb{R}_+$, then

$$\begin{aligned} \lim_{n\to\infty}\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} &= 1+\sum_{i=1}^{m}\lim_{n\to\infty}\left(-a_iQ_n^{(\alpha)}(c_i)\frac{K_{n-1}(c_i,x)}{L_n^{(\alpha)}(x)}\right)\\ &= 1+\sum_{i=1}^{m}\bar{p}^{(\alpha)}(c_i,x)\\ &= 1+\sum_{i=1}^{m}\left(\frac{-2\sqrt{|c_i|}}{\sqrt{-x}+\sqrt{|c_i|}}\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{\sqrt{|c_i|}+\sqrt{|c_j|}}{\sqrt{|c_i|}-\sqrt{|c_j|}}\right). \end{aligned}$$

From the above expression, we obtain

Theorem 3.3.

$$\lim_{n\to\infty}\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}=\prod_{k=1}^{m}\left(\frac{\sqrt{-x}-\sqrt{|c_k|}}{\sqrt{-x}+\sqrt{|c_k|}}\right) \tag{3.13}$$

uniformly on compact subsets of $\mathbb{C}\setminus\mathbb{R}_+$.

Proof. The proof is based on the partial fraction decomposition and the Residue Theorem. To simplify the notation, we write $t_i=\sqrt{|c_i|}$, $z=\sqrt{-x}$. Thus (3.13) becomes a rational function

$$\lim_{n\to\infty}\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)}=r(z)=\frac{q_m(z)}{p_m(z)},$$

where $q_m(z)$ and $p_m(z)$ are monic polynomials of degree m, i.e.

$$q_m(z)=\prod_{j=1}^{m}(z-t_j),\quad p_m(z)=\prod_{j=1}^{m}(z+t_j).$$

Notice that

$$\frac{q_m(z)}{p_m(z)}=1+\frac{[q_m(z)-p_m(z)]}{p_m(z)} \tag{3.14}$$

and the numerator in the above expression is a polynomial of degree at most $m-1$. In these conditions, $r(z)-1$ is a *proper rational function*, i.e. a ratio between two polynomials such that the degree of the numerator is less than the degree of the denominator. Under the above assumptions, when $-t_i$ are simple zeros of the polynomial $p_m(z)$, it is well known that always exists a decomposition in partial fractions of (3.14) as

$$\frac{[q_m(z)-p_m(z)]}{p_m(z)}=\sum_{i=1}^{m}\frac{A_i}{z+t_i},\quad \text{where}\quad A_i=\lim_{z\to -t_i}(z+t_i)\frac{[q_m(z)-p_m(z)]}{p_m(z)}.$$

Applying l'Hôpital's rule we have

$$A_i = \frac{[q_m(-t_i)-p_m(-t_i)]}{p_m'(-t_i)}=\frac{\displaystyle\prod_{j=1}^{m}(-t_j-t_i)-\prod_{j=1}^{m}(t_j-t_i)}{\displaystyle\prod_{\substack{j=1\\ j\neq i}}^{m}(t_j-t_i)}=-2t_i\prod_{\substack{j=1\\ j\neq i}}^{m}\frac{t_i+t_j}{t_i-t_j},$$

for all $i=1,\dots,m$; which completes the proof. □

Remark 3.4. Notice that outer relative asymptotics for orthogonal polynomials with respect to perturbations of measures supported on $\mathbb{R}_+$ or $\mathbb{R}$ have been studied in connection with rational approximation (see [14] and [15]). Our Theorem 3.3 provides an independent proof of the results contained therein. We use a different technique taking into account the polynomials considered in [14] and [15] are not exactly the same as studied here. Indeed, Theorem 3.3 deals with the particular case of the Laguerre weight. An interesting problem would be to extend our result to other measures supported on $(0,+\infty]$ as those analyzed in [14] and [15].

Remark 3.5. Notice that according to Hurwitz's theorem, each c_k attracts exactly one zero of the polynomial $Q_n^{(\alpha)}(x)$ for n large enough. In other words, we have exactly one zero in each gap.

4. Holonomic Equation

We begin by proving a lemma, concerning two connection formulas that will be needed later.

Lemma 4.1. *For the sequences of polynomials* $\{\widehat{Q}_n\}_{n\geq 0}$ *and* $\{\widehat{L}_n^{\alpha}\}_{n\geq 0}$ *we get*

$$R_m(x)\widehat{Q}_n(x) = A_1(x;n)\widehat{L}_n^{\alpha}(x)+B_1(x;n)\widehat{L}_{n-1}^{\alpha}(x), \tag{4.1}$$

$$x\left(R_m(x)\widehat{Q}_n(x)\right)' = C_1(x;n)\widehat{L}_n^{\alpha}(x)+D_1(x;n)\widehat{L}_{n-1}^{\alpha}(x), \tag{4.2}$$

where $R_m(x)$ *is given in (2.6),*

$$A_1(x;n) = R_m(x)-\sum_{j=1}^{m}\left(\frac{a_j\,\widehat{L}_{n-1}^{\alpha}(c_j)\,\widehat{Q}_n(c_j)}{(n-1)!\Gamma(n+\alpha)}\right)R_{m,j}(x),$$

$$B_1(x;n) = \sum_{j=1}^{m}\left(\frac{a_j\,\widehat{L}_{n}(c_j)\,\widehat{Q}_n(c_j)}{(n-1)!\Gamma(n+\alpha)}\right)R_{m,j}(x), \tag{4.3}$$

$$R_{m,k}(x) = \prod_{\substack{j=1\\ j\neq k}}^{m}(x-c_j),$$

$$C_1(x;n) = nA_1(x;n)-B_1(x;n)+xA_1'(x;n),$$

$$D_1(x;n) = n(n+\alpha)A_1(x;n)+(x-(n+\alpha))B_1(x;n)+xB_1'(x;n).$$

Proof. Since $K_{n-1}(x,y)$ is a polynomial of degree $n-1$ in the variable y, we have

$$\begin{aligned} &\langle K_{n-1}(x,y), \widehat{Q}_n(y)\rangle_\nu = 0, \\ &\langle K_{n-1}(x,y), \widehat{Q}_n(y)\rangle_\alpha = -\sum_{j=1}^{m} a_j \, K_{n-1}(x,c_j)\, \widehat{Q}_n(c_j). \end{aligned} \tag{4.4}$$

Using in (4.4) the Christoffel-Darboux formula, we have

$$\begin{aligned} \langle K_{n-1}(x,y), \widehat{Q}_n(y)\rangle_\alpha \;&=\; -\left(\sum_{j=1}^{m} \frac{a_j\, \widehat{L}^\alpha_{n-1}(c_j)\, \widehat{Q}_n(c_j)}{(n-1)!\Gamma(n+\alpha)(x-c_j)}\right) \widehat{L}^\alpha_n(x) \\ &\quad -\left(\sum_{j=1}^{m} \frac{a_j\, \widehat{L}^\alpha_{n}(c_j)\, \widehat{Q}_n(c_j)}{(n-1)!\Gamma(n+\alpha)(x-c_j)}\right) \widehat{L}^\alpha_{n-1}(x). \end{aligned} \tag{4.5}$$

Replacing (4.5) in (4.4) and multiplying by $R_m(x)$, we deduce (4.1) for $x \in \mathbb{C} \setminus \{\mathbb{R}_+ \cup \{c_1, \ldots, c_m\}\}$. To prove (4.2), we can take derivatives in both sides of (4.1)

$$\begin{aligned} \left(R_m(x)\widehat{Q}_n(x)\right)' \;&=\; A_1'(x;n)\widehat{L}^\alpha_n(x) + A_1(x;n)[\widehat{L}^\alpha_n]'(x) + \\ &\quad +B_1'(x;n)\widehat{L}^\alpha_{n-1}(x) + B_1\,(x;n)\,[\widehat{L}^\alpha_{n-1}]'(x). \end{aligned} \tag{4.6}$$

Now, multiplying (4.6) by x and using (1.3)–(1.4), we obtain (4.2). □

Lemma 4.2. *The sequences of monic polynomials* $\{\widehat{Q}_n\}_{n\geq 0}$ *and* $\{\widehat{L}^\alpha_n\}_{n\geq 0}$ *are also related by*

$$R_m(x)\widehat{Q}_{n-1}(x) \;=\; A_2(x;n)\widehat{L}^\alpha_n(x) + B_2(x;n)\widehat{L}^\alpha_{n-1}(x), \tag{4.7}$$

$$x\left(R_m(x)\widehat{Q}_{n-1}(x)\right)' \;=\; C_2(x;n)\widehat{L}^\alpha_n(x) + D_2(x;n)\widehat{L}^\alpha_{n-1}(x), \tag{4.8}$$

where

$$\begin{aligned} A_2(x;n) &= \frac{-1}{(n-1+\alpha)(n-1)} B_1(x;n-1), \\ B_2(x;n) &= A_1(x;n-1) + \frac{(x+1-2n-\alpha)}{(n-1+\alpha)(n-1)} B_1(x;n-1), \end{aligned} \tag{4.9}$$

$$\begin{aligned} C_2(x;n) &= \frac{-1}{(n-1+\alpha)(n-1)} D_1(x;n-1), \\ D_2(x;n) &= C_1(x;n-1) + \frac{(x+1-2n-\alpha)}{(n-1+\alpha)(n-1)} D_1(x;n-1). \end{aligned} \tag{4.10}$$

Proof. The proof of (4.7)-(4.8) is a straightforward consequence of (4.1)–(4.3) and the three term recurrence relation (2.1) for the monic Laguerre polynomials. □

The following lemma shows the converse relation of (4.1)–(4.7) for the polynomials $\widehat{L}^\alpha_n(x)$ and $\widehat{L}^\alpha_{n-1}(x)$

Lemma 4.3.

$$\widehat{L}_n^{\alpha}(x) = \frac{R_m(x)}{\Delta(x;n)}\left(B_2(x;n)\widehat{Q}_n(x) - B_1(x;n)\widehat{Q}_{n-1}(x)\right), \tag{4.11}$$

$$\widehat{L}_{n-1}^{\alpha}(x) = \frac{R_m(x)}{\Delta(x;n)}\left(-A_2(x;n)\widehat{Q}_n(x) + A_1(x;n)\widehat{Q}_{n-1}(x)\right). \tag{4.12}$$

where

$$\Delta(x;n) = A_1(x;n)B_2(x;n) - B_1(x;n)A_2(x;n), \quad \deg\Delta(x;n) = 2m.$$

Proof. Note that (4.1)–(4.7) is a system of two linear equations with two unknowns $\widehat{L}_n^{\alpha}(x)$ and $\widehat{L}_{n-1}^{\alpha}(x)$ and from the Cramer's rule the lemma follows. □

Lemma 4.4.

$$G(x;n)\widehat{Q}_n(x) + F(x;n)[\widehat{Q}_n]'(x) = H(x;n)\widehat{Q}_{n-1}(x), \tag{4.13}$$

$$J(x;n)\widehat{Q}_{n-1}(x) + F(x;n)[\widehat{Q}_{n-1}]'(x) = K(x;n)\widehat{Q}_n(x), \tag{4.14}$$

where

$$\begin{aligned}
F(x;n) &= x\Delta(x;n)R_m(x),\\
G(x;n) &= x\Delta(x;n)R_m'(x) + R_m(x)\left[D_1(x;n)A_2(x;n)\right.\\
&\quad \left.-C_1(x;n)B_2(x;n)\right],\\
H(x;n) &= R_m(x)[D_1(x;n)A_1(x;n) - C_1(x;n)B_1(x;n)],\\
J(x;n) &= x\Delta(x;n)R_m'(x) + R_m(x)\left[C_2(x;n)B_1(x;n)\right.\\
&\quad \left.-D_2(x;n)A_1(x;n)\right],\\
K(x;n) &= R_m(x)[C_2(x;n)B_2(x;n) - D_2(x;n)A_2(x;n)].
\end{aligned} \tag{4.15}$$

Proof. Replacing (4.11)–(4.12) in (4.2) and (4.8), (4.13) and (4.14) hold. □

From (4.13)

$$\widehat{Q}_{n-1}(x) = \frac{1}{H(x;n)}(G(x;n)\widehat{Q}_n(x) + F(x;n)[\widehat{Q}_n]'(x)),$$

and replacing this polynomial in (4.14), after some cumbersome computations, we obtain

Theorem 4.5 (The Holonomic equation)**.** *The n-th monic orthogonal polynomial with respect to the inner product (2.5), $\widehat{Q}_n(x) = \widehat{Q}_n^{(\alpha,\mathbf{c})}(x)$, is a polynomial solution of the second order linear differential equation with rational functions as coefficients*

$$[\widehat{Q}_n]''(x) + \mathcal{A}(x;n)[\widehat{Q}_n]'(x) + \mathcal{B}(x;n)\widehat{Q}_n(x) = 0, \tag{4.16}$$

where

$$\begin{aligned}
\mathcal{A}(x;n) &= -\frac{u_{2m}'(x;n)}{u_{2m}(x;n)} + 2\frac{R_m'(x)}{R_m(x)} + \frac{\alpha+1}{x} - 1\\
\mathcal{B}(x;n) &= \frac{H(x;n)G'(x;n) - G(x;n)H'(x;n)}{H(x;n)F(x;n)}\\
&\quad + \frac{J(x;n)G(x;n) - K(x;n)H(x;n)}{F^2(x;n)}
\end{aligned}$$

$$u_{2m}(x;n) = D_1(x;n)A_1(x;n) - C_1(x;n)B_1(x;n) \tag{4.17}$$

Note that $u_{2m}(x;n)$ is a polynomial of degree $2m$.

5. Electrostatic Interpretation

In this section, we present an electrostatic interpretation of the distribution of the zeros of $\{\widehat{Q}_n(x)\}_{n\geq 0}$ as the logarithmic potential interaction of unit positive charges in the presence of an external field. We use the fact that this family of monic polynomials satisfies the second-order linear differential equation (4.16). Notice that the zeros of $\widehat{Q}_n(x)$ are real, simple and belong to the interior of the convex hull of $R_+ \cup \{c_1, c_2, \ldots, c_m\}$, because $d\nu$ is a positive Borel measure. Now we evaluate (4.16) at $x_{n,k}$, where $\{x_{n,k}\}_{k=1}^n$ are the zeros of $\widehat{Q}_n(x)$ arranged in an increasing order, yielding

$$\frac{[\widehat{Q}_n]''(x_{n,k})}{[\widehat{Q}_n]'(x_{n,k})} = -\mathcal{A}(x_{n,k}; n).$$

Using the explicit expressions of $\mathcal{A}(x_{n,k}; n)$ we get for $1 \leq k \leq n$,

$$\frac{[\widehat{Q}_n]''(x_{n,k})}{[\widehat{Q}_n]'(x_{n,k})} = \frac{u'_{2m}(x_{n,k}; n)}{u_{2m}(x_{n,k}; n)} - 2\frac{R'_m(x_{n,k})}{R_m(x_{n,k})} - \frac{\alpha+1}{x_{n,k}} + 1. \tag{5.1}$$

Taking into account the fact that the zeros of $\widehat{Q}_n(x)$ are simple, then

$$[\widehat{Q}_n]'(x) = \sum_{i=1}^n \prod_{\substack{j=1,\\ j\neq i}}^n (x - x_{n,j}), \qquad [\widehat{Q}_n]''(x) = \sum_{i=1}^n \sum_{\substack{j=1,\\ j\neq i}}^n \prod_{\substack{l=1,\\ i\neq l\neq j}}^n (x - x_{n,l}),$$

$$[\widehat{Q}_n]'(x_{n,k}) = \prod_{\substack{j=1,\\ j\neq k}}^n (x_{n,k} - x_{n,j}), \qquad [\widehat{Q}_n]''(x_{n,k}) = 2\sum_{\substack{i=1,\\ i\neq k}}^n \prod_{\substack{j=1,\\ i\neq j\neq k}}^n (x_{n,k} - x_{n,j}).$$

Consequently, (5.1) reads as an "electrostatic equilibrium condition" (see [7] and [9] for other examples).

Indeed, for $1 \leq k \leq n$,

$$\sum_{\substack{j=1\\ j\neq k}}^n \frac{1}{x_{n,j} - x_{n,k}} + \frac{1}{2}\frac{u'_{2m}(x_{n,k}; n)}{u_{2m}(x_{n,k}; n)} - \frac{R'_m(x_{n,k})}{R_m(x_{n,k})} - \frac{\alpha+1}{2x_{n,k}} + \frac{1}{2} = 0. \tag{5.2}$$

We should notice that according to Lemma 4.1 and the fact that the zeros of $\widehat{Q}_n(x)$ are simple, then $u_{2m}(x_{n,k}; n) \neq 0$.

The above equation means that, the zeros $\{x_{n,k}\}_{1\leq k\leq n}$ of the family $\{\widehat{Q}_n(x)\}_{n\geq 1}$ are the critical points of the gradient of the total energy.

We now consider n unit positive charges located in the real line, with a logarithmic interaction under an external field $V(x)$. For $x \in \mathbb{R}\setminus\{c_j\}$ the total potential is

$$V(x) = \frac{1}{2}\ln u_{2m}(x; n) - \ln R_m(x) - \frac{\alpha+1}{2}\ln x + \frac{1}{2}x. \tag{5.3}$$

The term $-\ln R_m(x)$ is the potential field due to the mass points of our measure. Thus, we have in (5.3)

$$V(x) = \frac{1}{2}\ln u_{2m}(x; n) - \frac{1}{2}\ln\left(R_m^2(x) x^{\alpha+1} e^{-x}\right).$$

Following [10], the term

$$\upsilon_{long}(x) = \frac{-1}{2} \ln \left(R_m^2(x) x^{\alpha+1} e^{-x} \right),$$

is said to be a long range potential, which is associated with a polynomial perturbation of the Laguerre weight function. Similarly,

$$\upsilon_{short}(x) = \frac{1}{2} \ln u_{2m}(x;n)$$

represents a short range potential (or varying external potential) corresponding to $2m$ unit negative charges located at the zeros of $u_{2m}(x;n)$. Notice that these charges will be "floating" with n at each zero of the polynomial $u_{2m}(z)$. Their behavior for n large constitutes an open problem. Nevertheless, taking into account we are interested in such a behavior for any fixed n, we include some numerical examples in order to show the location of the zeros of the polynomial $u_{2m}(x;n)$.

6. Numerical experiment

Next we give some numerical experiments using Mathematica®, dealing with the least zeros of Laguerre-type polynomials. We are interested to show the location of their zeros outside the interval $[0, +\infty)$ and the position of the source-charges of the short range potential $\upsilon_{short}(x)$, which are the roots of the polynomial $u_4(x;n)$. In these experiments we consider in the inner product (1.5) two fixed mass points (that is, $m = 2$) at points $c_1 = -1$ and $c_1 = -2$. The parameter $\alpha = 0$ and the masses are always $a_1 = a_2 = 1$. Notice that in the examples shown, the zeros of the Laguerre-type polynomials never match the zeros of u_4 given in (4.17), i.e. the polynomial u_4 never vanishes at the zeros of any Laguerre-type polynomial. The negative zeros appear in bold.

Next, we show the position of the zeros of the Laguerre-type polynomial of degree $n = 4$ and the four real zeros of the polynomial $u_4(x;n)$. Notice that the polynomial $u_4(x;n)$ have four negative real roots, but there is only one zero of the Laguerre-type polynomial on $\mathbb{R}_-$.

zero	*1st*	*2nd*	*3rd*	*4th*
$\widehat{Q}_4(x)$	$\mathbf{-1.84565}$	0.0122706	2.65152	7.49184
$u_4(x;4)$	$\mathbf{-1.93302}$	$\mathbf{-1.48646}$	$\mathbf{-0.60338}$	$\mathbf{-0.000119291}$

As n increases, the situation changes as expected according to the Hurwitz's Theorem, and the mass points attract exactly one zero of the Laguerre-type polynomial in each gap between them.

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*
$\widehat{Q}_5(x)$	$\mathbf{-1.9219}$	$\mathbf{-0.439622}$	1.73422	5.20588	10.7544
$u_4(x;5)$	$\mathbf{-1.96394}$	$\mathbf{-1.56249}$	$\mathbf{-0.767607}$	$\mathbf{-0.11943}$	–

Next two tables show the behavior of the zeros of Laguerre-type polynomials and u_4 for degrees $n = 6$ and $n = 10$ respectively. Notice that the two negative zeros of $\widehat{Q}_n(x)$ and the four zeros of u_4 become more negative approaching to the position of the mass points.

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*
$\widehat{Q}_6(x)$	$\mathbf{-1.96485}$	$\mathbf{-0.711952}$	1.23489	3.98228	8.03313	14.1729
$u_4(x;6)$	$\mathbf{-1.9831}$	$\mathbf{-1.61526}$	$\mathbf{-0.871511}$	$\mathbf{-0.275212}$	–	–

zero	*1st*	*2nd*	*3rd*	*4th*	*5th*	*6th*
$\widehat{Q}_{10}(x)$	$\mathbf{-1.99898}$	$\mathbf{-0.979076}$	0.515223	2.00183	4.11731	6.87812
$u_4(x;10)$	$\mathbf{-1.99949}$	$\mathbf{-1.69674}$	$\mathbf{-0.989683}$	$\mathbf{-0.54116}$	–	–

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