# Quadratures and integral transforms arising from generating functions

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Abstract

By using the explicit form of the eigenvectors of the finite Jacobi matrix associated to a family of orthogonal polynomials and some asymptotic expressions, we obtain quadrature formulas for the integral transforms arising from linear generating functions of the classical orthogonal polynomials. As a bypass product, we obtain simple and accurate Riemann-Steklov quadrature formulas.

# 1 Introduction

This note has arisen as a sequel of a previous work [1], where quadratures for the integral transforms generated by bilinear functions of orthogonal polynomials have been found. In particular, the authors of [1] have given explicit quadratures for Fractional Fourier, Bessel, Appell, and Bateman-Pasternak transforms. The procedure to obtain a pair quadrature-transform uses the solutions of the eigenvalue problem of the Jacobi matrix, asymptotic formulas for orthogonal polynomials and their zeros, and a bilinear generating function for such polynomials. According to these findings, bilinear generating functions appear to be necessary to obtain specific forms of the integral transforms or, Poisson integrals as they are named elsewhere [2, 3], establishing a connection between the quadrature problem and the expansion of functions in terms of orthogonal systems. A fine example of the quadratures for integral transforms generated by this technique is given by the so-called XFT [4], a fast $\mathcal{O}(N \log N)$ discrete fractional Fourier transform which has found many applications [5]-[8]. Additionally, this quadrature method is not restricted to the classical orthogonal polynomials and also allows the use of other kind of bilinear generating functions, not necessarily of the Mehler's type. However, to find a suitable bilinear generating function may be a drawback of the above technique. This is because bilinear generating functions are generally more difficult to find than those linear generating functions. Therefore, the aim of this paper is to extend the scope of this method to find quadratures and integral transforms by including also linear generating functions, getting us closer to the problem of the expansion of functions in terms of orthogonal polynomials. In this way we find new quadrature formulas for the Gass transform, a Laplace transform, and for a Poisson integral. As a bypass product, we obtain simple and accurate Riemann-Steklov quadrature formulas for the classical polynomials (Hermite, Laguerre, and Jacobi). These extremely simple "general-purpose" formulas are Riemann sums for integrals on the intervals $(-\infty, \infty)$, $(0, \infty)$, and $(-1, 1)$ and therefore, they can also be used for linear or bilinear kernels, generalizing these quadrature problems.

# 2 Key points of the method

Since this paper is heavily based on [1], we summarize only some key results.
First, we will give a basic background about orthogonal polynomials an their role in Gaussian quadratures which will needed in the sequel.
Let $\mu$ be a nontrivial probability measure supported on an infinite subset $E$ of the real line. Let consider the canonical basis $\{x^n\}_{n\geq 0}$ in the linear space $\mathbb{P}$ of polynomials with real coefficients. By using the Gram-Schmidt orthogonalization process you get an orthonormal polynomial basis$\{p_n(x)\}_{n\geq 0}$ of $\mathbb{P}$. We will assume the leading coefficient of $p_n(x)$ is positive in order to have uniqueness of such a sequence of orthonormal polynomials. It is very well known (see [10], [16]) that these polynomials satisfy a three term recurrence

relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x), n \geq 0, \tag{1}$$

where $a_n > 0$ and assuming $p_{-1}(x) = 0$. Furthermore,

$$b_n = \int_E xp_n^2(x)d\mu, n \geq 0, a_n = \int_E xp_n(x)p_{n-1}(x)d\mu, n \geq 1. \tag{2}$$

Notice that the above recurrence relation reads as

$$x\mathfrak{P}_N(x) = J_N\mathfrak{P}_N(x) + a_N p_N(x)\mathbf{e}_N, n \geq 0, \tag{3}$$

where $\mathfrak{P}_N(x) = (p_0(x), p_1(x), \cdots, p_{N-1}(x))^t$ and $J_N$ is the Jacobi matrix of size $N \times N$

$$J_N = \begin{pmatrix} b_0 & a_1 & 0 & \cdots & 0 \\ a_1 & b_1 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & b_{N-1} \end{pmatrix}. \tag{4}$$

If $\alpha$ is a zero of $p_N(x)$, from (3) you get $\alpha\mathfrak{P}_N(\alpha) = J_N\mathfrak{P}_N(\alpha)$, i.e. $\alpha$ is an eigenvalue of the symmetric matrix $J_N$ and the corresponding unitary eigenvector is $\frac{1}{\sqrt{K_{N-1}(\alpha,\alpha)}}\mathfrak{P}_N(\alpha)$, where $K_{N-1}(\alpha, \alpha) = \sum_{k=0}^{N-1} |p_k(\alpha)|^2$. It is very well known (see [10], [11]) that the zeros of $p_N(x)$ are real, simple, and located in the interior of the convex hull of $E$. Let denote them by $x_{1,N} < x_{2,N} < \cdots < x_{N,N}$. On the other hand, let $u_k$ be the corresponding unitary eigenvector associated with $x_{k,N}$ defined as above. Thus, the matrix $U_N = (u_1|u_2|\cdots|u_N)$ is unitary.

For every polynomial $p(x)$ of degree at most $2N-1$ the Gaussian quadrature rule means that $\int_E p(x)d\mu = \sum_{k=1}^{N} p(x_{k,N})\Lambda_{k,N}$, where the Cotes-Christoffel numbers are given by $\Lambda_{k,N} = \frac{1}{K_{N-1}(x_{k,N},x_{k,N})}, k = 1, \cdots, N$. Notice that according to the confluent Christoffel-Darboux formula (see [10], [12]) you get $K_{N-1}(x_{k,N}, x_{k,N}) = \frac{p_N'(x_{k,N})p_{N-1}(x_{k,N})}{a_N}$.

An interesting family of orthogonal polynomials which has received an increasing interest in the last years is related to measures whose Radom-Nikodyn derivative with respect to the Lebesgue measure supported on $E$, i.e. its absolutely continuous component $\omega$, satisfies a Pearson equation $(\Phi(x)\omega)' = \Psi(x)\omega$, where $\Phi(x)$ is a monic polynomial and $\Psi(x)$ is a polynomial of degree at least 1. Some boundary conditions are needed in order to have an integration by parts. In such a way, for every polynomial $p(x)$ $\int_E \Psi(x)p(x)\omega(x)dx + \int_E \Phi(x)p'(x)\omega(x)dx = 0$ holds.

Such families are said to be semiclassical (see [15]) and they satisfy a differential-difference equation (structure relation) (see [15], [13])

$$\Phi(x)p_n'(x) = A(x;n)p_n(x) + B(x;n)p_{n-1}(x), \tag{5}$$

where $A(x;n)$ and $B(x;n)$ are polynomials of constant degree (independent of $n$). The classical orthogonal polynomials (Hermite, Laguerre, Jacobi) are semiclassical with $\Phi(x) =$

1, $\Phi(x) = x$, $\Phi(x) = x^2 - 1$, respectively. As a straightforward consequence, for semiclassical polynomials the expression of the Cotes-Christoffel numbers is $\Lambda_{k,N} = \frac{a_N \Phi(x_{k,N})}{B(x_{k,N};N)[p_{N-1}(x_{k,N})]^2}$. In other words, you need to know the evaluation of the polynomials $p_{N-1}(x)$ and $B(x;N)$ at the zeros of $p_{N-1}(x)$. This idea will be very useful in the sequel.
Second, we will deal with linear generating functions of orthogonal polynomials $G(x,z) = \sum_{n=0}^{\infty} c_n(z) p_n(x)$. For some choices of the functions is possible to deduce a compact formula of this generating function. For instance, in the case of classical orthogonal polynomials as well as for other families of orthogonal hypergeometric polynomials you have very well known expressions of such a generating function (see, for instance, [14]). Our aim is to find quadrature formulas for some integral transforms associated with these generating functions. Our basic ingredients will be asymptotic estimates of zeros of orthogonal polynomials as well as asymptotic estimates of the Cotes- Christoffel numbers in such a way we get a Riemann sum of the integral transform. In the next sections we will illustrate our method for the classical orthogonal polynomials and some integral transforms associated with them. The comparison between the exact and the approximate expression based on quadrature formulas of such integral transformations is shown with some examples.

# 3 Hermite case

Consider the $n$th component of the $k$th orthonormal eigenvector $u_k = (u_{0,k}, u_{1,k}, \cdots u_{N-1,k})$, $k = 1, 2, \cdots, N$, of the Jacobi matrix $J_N$ for the Hermite case, which is given by

$$u_{n,k} = (-1)^{N+k} \left( \frac{2^{N-n-1}(N-1)!}{N\, n!} \right)^{1/2} \frac{H_n(x_{k,N})}{H_{N-1}(x_{k,N})}, \quad n = 0, \ldots, N-1, \tag{6}$$

where $x_{k,N}$ stands for the $k$th zero of $H_N(x)$. Here we use the normalization for Hermite polynomials given in [16]. Let $U_N$ be the matrix whose $k$th column is the vector $u_k$ given above. Denote by $d(z)$ the row vector $(d_0(z), d_1(z), \cdots, d_{N-1}(z))$, where $d_n(z)$, for each $n = 0, 1, \cdots, N-1$, is an auxiliary function to be determined later. Consider the row-matrix product $t(z) = d(z)U_N$, i.e., the row vector defined by

$$t(z) = (d_0(z), d_1(z), \cdots, d_{N-1}(z)) \begin{pmatrix} u_{0,1} & u_{0,2} & \cdots & u_{0,N} \\ u_{1,1} & u_{12} & \cdots & u_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ u_{N-1,1} & u_{N-1,2} & \cdots & u_{N-1,N} \end{pmatrix}. \tag{7}$$

The $k$th element of this vector can be written as

$$t_{k,N}(z) = \mu_{k,N} \sum_{n=0}^{N-1} d_n(z) H_n(x_{k,N}), \tag{8}$$

where

$$\mu_{k,N} = (-1)^{N+k} \frac{\sqrt{2^{N-1}(N-1)!/N}}{H_{N-1}(x_{k,N})}, \quad k = 1, 2, \cdots, N, \tag{9}$$

and

$$d_n(z) = \sqrt{2^n n!}\; c_n(z).$$

Note that $\mu_{k,N}$ is positive and equal to the square root of a Cotes-Christoffel number of the Gauss-Hermite quadrature [16] up to a constant, i.e., $\mu_{k,N} = \sqrt{\Lambda_{k,N}}\,/\pi^{1/4}$.
By using some asymptotic formulas [1],

$$\mu_{k,N} \simeq \pi^{-1/4} e^{x_{k,N}^2/2} (\Delta x_{k,N})^{1/2}, \tag{10}$$

where $x_{k,N} \simeq (2k - N - 1)\pi/(2\sqrt{2N})$, $k = 1, 2, \cdots, N$, and

$$\Delta x_{k,N} = \pi/\sqrt{2N},$$

is the difference of two consecutive points of the partition $\{x_{1,N}, x_{2,N}, \cdots, x_{N,N}\}$.
Consider now a linear generating function $G(x, z)$ for the Hermite polynomials (see [14]),

$$G(x, z) = \sum_{n=0}^{\infty} c_n(z) H_n(x),$$

and let $f(x)$ be an integrable function in $(-\infty, \infty)$. Taking into account (10), the right-hand side of Eq. (8) reads as

$$\pi^{-1/4} e^{-x_{k,N}^2/2} G(x_{k,N}, z) \Delta^{1/2} x_{k,N},$$

for sufficiently large $N$. Multiplying both sides of Eq. (8) by $f(x_{k,N})$ and $\Delta^{1/2} x_{k,N}$, we get

$$e^{-x_{k,N}^2/2} G(x_{k,N}, z) f(x_{k,N}) \Delta x_{k,N} = \frac{\pi^{3/4}}{(2N)^{1/4}}\, t_{k,N}(z) f(x_{k,N}). \tag{11}$$

Note that

$$\lim_{N\to\infty} \sum_{k=1}^{N} e^{-x_{k,N}^2/2} G(x_{k,N}, z) f(x_{k,N}) \Delta x_{k,N}$$

is the Riemann sum of the integral transform

$$\int_{-\infty}^{\infty} K(x, z) f(x) dx, \tag{12}$$

where $K(x, z) = e^{-x^2/2} G(x, z)$. Therefore, the quadrature

$$\int_{-\infty}^{\infty} K(x, z) f(x) dx \simeq \frac{\pi^{3/4}}{(2N)^{1/4}} \sum_{k=1}^{N} t_{k,N}(z) f(x_{k,N}) \tag{13}$$

follows from (11).

## 3.1 An example

The well-known generating function for Hermite polynomials (see [14])

$$G(x,z) = e^{2xz - z^2} = \sum_{n=0}^{\infty} \frac{z^n}{n!} H_n(x)$$

yields a quadrature formula for the Gauss transform. With this selection for $G(x,z)$ and $f(x) = e^{-x^2/2} g(x)$, Eq. (13) becomes

$$\int_{-\infty}^{\infty} e^{-(x-z)^2} g(x) dx \simeq \frac{\pi^{3/4}}{(2N)^{1/4}} \sum_{k=1}^{N} t_{k,N}(z) e^{-x_{k,N}^2/2} g(x_{k,N}), \quad (14)$$

which gives a quadrature formula for the Gauss transform of the function $g(x)$. Fig. 1 illustrates the performance of (14).

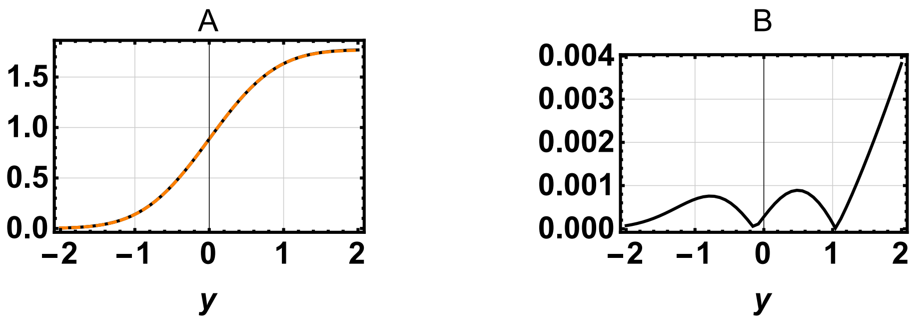


Figure 1: (A) Exact (solid line) and approximate (dashed line) Gauss transform of $g(x) = u(x)$, the unit step function. The quadrature was computed with $N = 200$ nodes. (B) Absolute error.

# 4 Laguerre case

The $n$th component of the $k$th orthonormal eigenvector $u_k$ of the Jacobi matrix $J_N$ for this case is given by

$$u_{n,k} = (-1)^{k+1} \left( \frac{n!\Gamma(N+\alpha)\, x_{k,N}}{N!(N+\alpha)\Gamma(n+\alpha+1)} \right)^{1/2} \frac{L_n^{(\alpha)}(x_{k,N})}{L_{N-1}^{(\alpha)}(x_{k,N})}, \quad n = 0, 1, \ldots, N-1, \quad (15)$$

where $x_{k,N}$ is the $k$th zero of $L_N^{(\alpha)}(x)$. Here we use the normalization for Laguerre polynomials given in [16]. The asymptotic form of $x_{k,N}$ is

$$x_{k,N} \simeq (k + \alpha/2 - 1/4)^2 \frac{\pi^2}{4N}, \quad k = 1, 2, \cdots, N,$$

for fixed $\alpha$. Proceeding as above, let $U_N$ be again the matrix whose $k$th column is the vector $u_k$, denote by $d(z)$ the row vector $(d_0(z), d_1(z), \cdots, d_{N-1}(z))$ and consider the row-matrix product $t(z) = d(z)U_N$, i.e., the row vector defined by (7). The $k$th element of this vector can be written now as

$$t_{k,N}(z) = \mu_{k,N} \sum_{n=0}^{N-1} c_n(z) L_n^{(\alpha)}(x_{k,N}), \tag{16}$$

where

$$\mu_{k,N} = (-1)^{k+1} \left( \frac{\Gamma(N+\alpha)\, x_{k,N}}{N!(N+\alpha)} \right)^{1/2} \frac{1}{L_{N-1}^{(\alpha)}(x_{k,N})}, \quad k = 1, 2, \cdots, N, \tag{17}$$

and $d_n(z)$ is given now by

$$d_n(z) = \left( \frac{\Gamma(\alpha+n+1)}{n!} \right)^{1/2} c_n(z).$$

In this case, $\mu_{k,N} = \sqrt{\Lambda_{k,N}}$, where $\Lambda_{k,N}$ is a Cotes-Christoffel number of the Gauss-Laguerre quadrature [16]. According to [1], the asymptotic formula[1]

$$\mu_{k,N} \simeq \sqrt{2}\, x_{k,N}^{\alpha/2+1/4} e^{-x_{k,N}/2} \Delta^{1/2}\sigma(x_{k,N}), \tag{18}$$

holds for sufficiently large $N$. Here, $\sigma(x) = \sqrt{x}$, and this means that

$$\Delta\sigma(x_{k,N}) = \sqrt{x_{k+1,N}} - \sqrt{x_{k,N}} = \frac{\pi}{2\sqrt{N}}.$$

Proceeding as above, consider now a linear generating function $G(x,z)$ for the Laguerre polynomials (see [14])

$$G(x,z) = \sum_{n=0}^{\infty} c_n(z) L_n^{(\alpha)}(x),$$

and let $f(x)$ be an integrable function in $(0,\infty)$. Taking into account (18), the right-hand side of Eq. (16) reads as

$$\sqrt{2}\, x_{k,N}^{\alpha/2+3/4} e^{-x_{k,N}/2} G(x_{k,N}, z) \Delta^{1/2}\sigma(x_{k,N})$$

for sufficiently large $N$. Multiplying both sides of Eq. (16) by $f(x_{k,N})$ and $(\Delta\sigma(x_{k,N})/2)^{1/2}$, we get

$$x_{k,N}^{\alpha/2+1/4} e^{-x_{k,N}/2} G(x_{k,N}, z) f(x_{k,N}) \Delta\sigma(x_{k,N}) = \frac{\pi^{1/2}}{2N^{1/4}} t_{k,N}(z) f(x_{k,N}). \tag{19}$$

Note that

$$\lim_{N\to\infty} \sum_{k=1}^{N} x_{k,N}^{\alpha/2+1/4} e^{-x_{k,N}/2} G(x_{k,N}, z) f(x_{k,N}) \Delta\sigma(x_{k,N})$$

[1] The value of the parameter $A$ of Ref. [1] for the Laguerre case should be 2, and not 1/2, as printed.

is the Riemann-Stieltjes sum of the integral transform

$$\int_0^\infty K(x,z)f(x)d\sigma(x), \tag{20}$$

where $K(x,z) = x^{\alpha/2+1/4}e^{-x/2}G(x,z)$. Therefore, the quadrature

$$\int_0^\infty K(x,z)f(x)d\sigma(x) \simeq \frac{\pi^{1/2}}{2N^{1/4}}\sum_{k=1}^{N} t_{k,N}(z)f(x_{k,N}) \tag{21}$$

follows from (19).

## 4.1 An example

The substitution of the well-known generating function for Laguerre polynomials (see [14])

$$G(x,z) = \frac{e^{xz/(z-1)}}{(1-z)^{\alpha+1}} = \sum_{n=0}^{\infty} z^n L_n^{(\alpha)}(x), \quad |z|<1,$$

in (21) and the use of $s = z/(1-z)$ yield

$$\int_0^\infty e^{-x(s+1/2)}x^{\alpha/2+1/4}f(x)d\sigma(x) \simeq \frac{\pi^{1/2}}{2N^{1/4}(1+s)^{\alpha+1}}\sum_{k=1}^{N} t_{k,N}\left(\frac{s}{1+s}\right)f(x_{k,N}), \quad s>0.$$

This expression can be written as a quadrature for the Laplace transform

$$\int_0^\infty e^{-(s+1/2)x}f(x)dx \simeq \frac{\pi^{1/2}}{N^{1/4}(1+s)^{\alpha+1}}\sum_{k=1}^{N} t_{k,N}\left(\frac{s}{1+s}\right)\frac{f(x_{k,N})}{x_{k,N}^{\alpha/2-1/4}}, \quad s>0, \tag{22}$$

where use of $d\sigma(x) = dx/2\sqrt{x}$ has been made. Fig. 2 illustrates the performance of (22) for $f(x) = x^2\sin x$ and $\alpha = 0$.

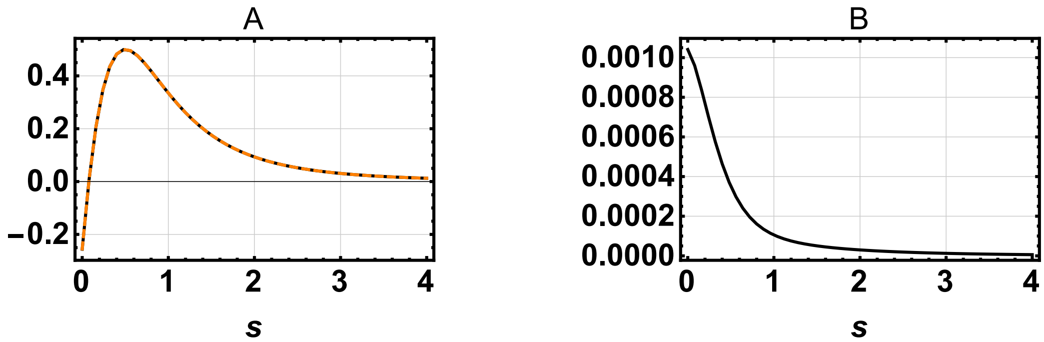


Figure 2: (A) Exact (solid line) and approximate (dashed line) Laplace transform given in (22) of $f(x) = x^2\sin x$. The quadrature was computed with $N = 200$ nodes and $\alpha = 0$. (B) Absolute error.

# 5 Jacobi case

In this case, the $n$th component of the $k$th eigenvector $u_k$ of the Jacobi matrix $J_N$ can be written as

$$u_{n,k} = \mu_{k,N} \left( \frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right)^{1/2} P_n^{(\alpha,\beta)}(x_{k,N}), \tag{23}$$

where $n = 0, 1, \ldots, N-1$, and

$$\mu_{k,N} = (-1)^{N+k} \left( \frac{(2N+\alpha+\beta)^2\Gamma(N+\alpha)\Gamma(N+\beta)}{4N!(N+\alpha)(N+\beta)\Gamma(N+\alpha+\beta+1)} \right)^{1/2} \frac{\sqrt{1-x_{k,N}^2}}{P_{N-1}^{(\alpha,\beta)}(x_{k,N})}. \tag{24}$$

Here we use the normalization for Jacobi polynomials given in [16]. In (23) and (24), $x_{k,N}$ is the $k$th zero of $P_N^{(\alpha,\beta)}(x)$, whose asymptotic form is

$$x_{k,N} \simeq \cos\left( \frac{N-k+\alpha/2+3/4}{N+(\alpha+\beta+1)/2}\,\pi \right), \quad k = 1, 2\cdots, N,$$

for fixed $\alpha$ and $\beta$.
Note that $\mu_{k,N}$ is the square root of the Jacobi Cotes-Christoffel number $\Lambda_{k,N}$, up to a constant independent of $k$ and $N$, i.e., $\mu_{k,N} = \sqrt{\Lambda_{k,N}/2^{\alpha+\beta+1}}$.
Proceeding again as above, the $k$th element of the row-matrix product $t(z) = d(z)U_N$, where $U_N$ is the matrix whose $k$th column is the vector $u_k$ and $d(z)$ is the row vector of components

$$d_n(z) = \left( \frac{n!(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)} \right)^{-1/2} c_n(z),$$

can be written as

$$t_{k,N}(z) = \mu_{k,N} \sum_{n=0}^{N-1} c_n(z) P_n^{(\alpha,\beta)}(x_{k,N}). \tag{25}$$

By using some asymptotic formulas [1], $\mu_{k,N}$ can be written, for sufficiently large $N$, as

$$\mu_{k,N} \simeq \frac{(1-x_{k,N})^{\alpha/4+1/8}(1+x_{k,N})^{\beta/4+1/8}}{2^{(\alpha+\beta+1)/2}} |\Delta\sigma(x_{k,N})|^{1/2}, \tag{26}$$

where $\sigma(x) = \arccos(x)$. For fixed $\alpha$ and $\beta$,

$$\Delta\sigma(x_{k,N}) = \arccos(x_{k+1,N}) - \arccos(x_{k,N}) \simeq -\frac{\pi}{N}.$$

Let $G(x,z)$ be a linear generating function for the Jacobi polynomials (see [14]),

$$G(x,z) = \sum_{n=0}^{\infty} c_n(z) P_n^{(\alpha,\beta)}(x),$$

and let $f(x)$ be an integrable function in $(-1,1)$. Eq. (26) yields for the right-hand side of (25) the asymptotic expression

$$\frac{(1-x_{k,N})^{\alpha/4+1/8}(1+x_{k,N})^{\beta/4+1/8}}{2^{(\alpha+\beta+1)/2}}G(x_{k,N},z)|\Delta\sigma(x_{k,N})|^{1/2}.$$

Multiplying both sides of (25) by $-|\Delta\sigma(x_{k,N})|^{1/2}$ and taking into account that

$$\lim_{N\to\infty}\sum_{k=1}^{N}(1-x_{k,N})^{\alpha/4+1/8}(1+x_{k,N})^{\beta/4+1/8}G(x_{k,N},z)f(x_{k,N})\Delta\sigma(x_{k,N})$$

is the Riemann-Stieltjes sum for the integral transform

$$\int_{\pi}^{0} K(x,z)f(x)d\sigma(x),$$

where $K(x,z)=(1-x)^{\alpha/4+1/8}(1+x)^{\beta/4+1/8}G(x,z)$, we obtain the quadrature formula

$$\int_{0}^{\pi} K(x,z)f(x)d\sigma(x) \simeq 2^{(\alpha+\beta+1)/2}\sqrt{\frac{\pi}{N}}\sum_{k=1}^{N}t_{k,N}(z)f(x_{k,N}) \qquad (27)$$

## 5.1 An example

The well-known generating function for jacobi polynomials (see [14])

$$G(x,z)=\frac{2^{\alpha+\beta}}{R(1-z+R)^{\alpha}(1+z+R)^{\beta}}=\sum_{n=0}^{\infty}z^n P_n^{(\alpha,\beta)}(x) \quad |r|<1,$$

where $R=\sqrt{1-2xz+z^2}$, yields the quadrature for the Poisson integral

$$\int_{-1}^{1}\frac{(1-x)^{\alpha/4+1/8}(1+x)^{\beta/4+1/8}}{R(1-z+R)^{\alpha}(1+z+R)^{\beta}}f(x)dx \simeq \sqrt{\frac{2\pi}{2^{\alpha+\beta}N}}\sum_{k=1}^{N}\sqrt{1-x_{k,N}^2}\,t_{k,N}(z)f(x_{k,N}), \quad (28)$$

where use of $d\sigma(x)=-dx/\sqrt{1-x^2}$ has been made. The performance of this formula is illustrated in Fig. 3 for $f(x)=(1-x^2)^{5/8}$ and $\alpha=1$ and $\beta=1$.

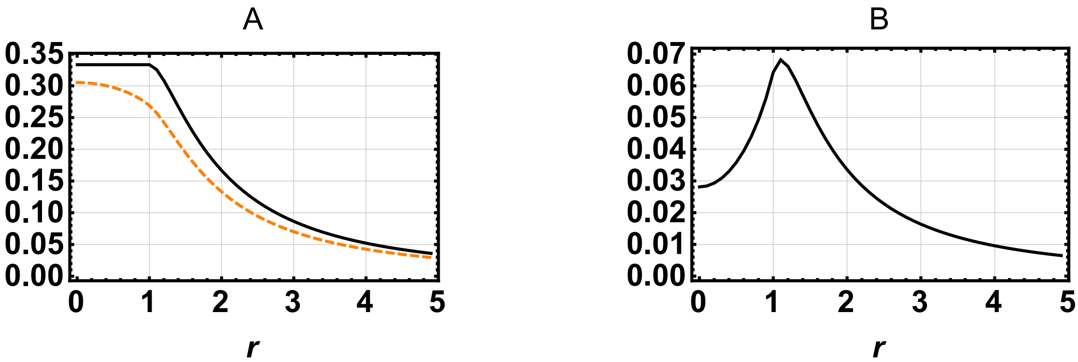


Figure 3: (A) Exact (solid line) and approximate (dashed line) of $f(x) = (1-x^2)^{5/8}$. The quadrature was computed with $N = 100$ nodes and $\alpha = 1$ and $\beta = 1$. (B) Absolute error.

# 6 Further quadrature formulas

Let us consider the quadratures for the Hermite, Laguerre, and Jacobi cases in more detail. These formulas are given by Eqs. (13), (21), and (27), respectively. In each one, the discrete kernel, $t_{k,N}$ equals the product of $\mu_{k,N}$ times a finite sum which approaches the generating function $G(x,z)$ for sufficiently large $N$. This function appears also in the corresponding integral transform, and therefore, the three quadrature formulas can be simplified by cancelling this function from both sides of the equations. In addition, $\mu_{k,N}$ contributes to the continuous kernel $K(x,z)$ with certain function appearing in the asymptotic form of $\mu_{k,N}$, and therefore, contained also in $\mu_{k,N}$. Thus, the quadrature formulas can be further simplified to the following Riemann-Steklov sums.

1. Quadrature for the interval $(-\infty,\infty)$.

$$\int_{-\infty}^{\infty} f(x)dx \simeq \sum_{k=1}^{N} f(x_{kN})\Delta_N, \tag{29}$$

where

$$\Delta_N = \frac{\pi}{\sqrt{2N}}, \quad \text{and} \quad x_{k,N} = \left(k - \frac{N+1}{2}\right)\frac{\pi}{\sqrt{2N}}.$$

2. Quadrature for the interval $(0,\infty)$.

$$\int_{0}^{\infty} f(x)dx \simeq \sum_{k=1}^{N} 2\sqrt{x_{k,N}}f(x_{k,N})\Delta_N, \tag{30}$$

where

$$\Delta_N = \frac{\pi}{2\sqrt{N}}, \quad \text{and} \quad x_{k,N} = (k+\alpha/2-1/4)^2\frac{\pi^2}{4N}.$$

3. Quadrature for the interval $(-1, 1)$.

$$\int_{-1}^{1} f(x)dx \simeq \sum_{k=1}^{N} \sqrt{1 - x_{k,N}^2} f(x_{k,N})\Delta_N, \tag{31}$$

where

$$\Delta_N = \frac{\pi}{N}, \quad \text{and} \quad x_{k,N} = \cos\left(\frac{N - k + \alpha/2 + 3/4}{N + (\alpha + \beta + 1)/2}\,\pi\right).$$

In order to compare the performance of these formulas, let us consider the integral transforms of the previous examples. Figs. 4-6 show that the simple quadrature formulas given in (29)-(31) are reliable and more accurate than those given in the previous section.

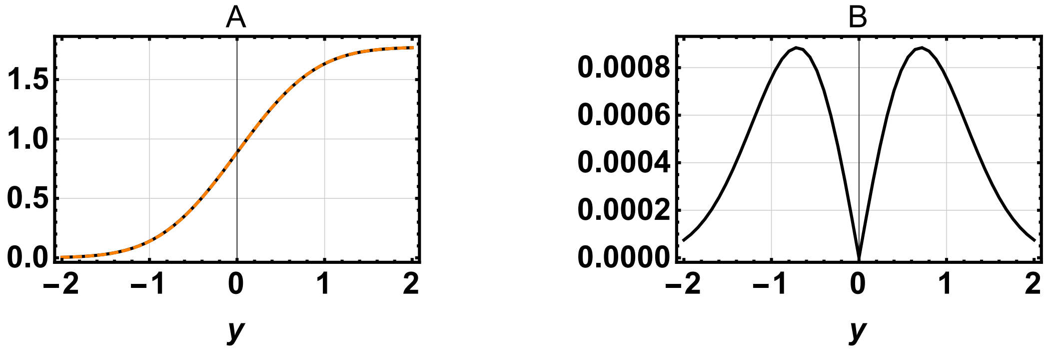


Figure 4: (A) Exact (solid line) and approximate (dashed line) ouput of (29) with $f(x) = e^{-(x-y)^2}(1-x^2)^{5/8}u(x)$. The quadrature was computed with $N = 200$ nodes. (B) Absolute error.

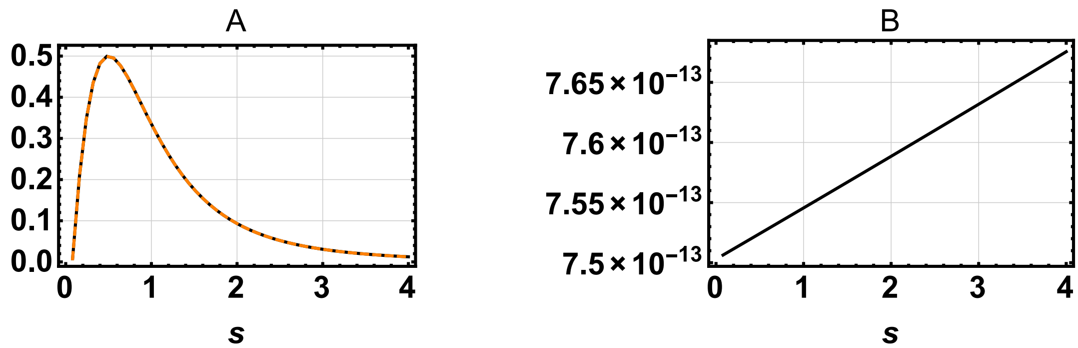


Figure 5: (A) Exact (solid line) and approximate (dashed line) output of (30) with $f(x) = e^{-(s+1/2)x}x^2 \sin x$. The quadrature was computed with $N = 200$ nodes and $\alpha = 0$. (B) Absolute error.

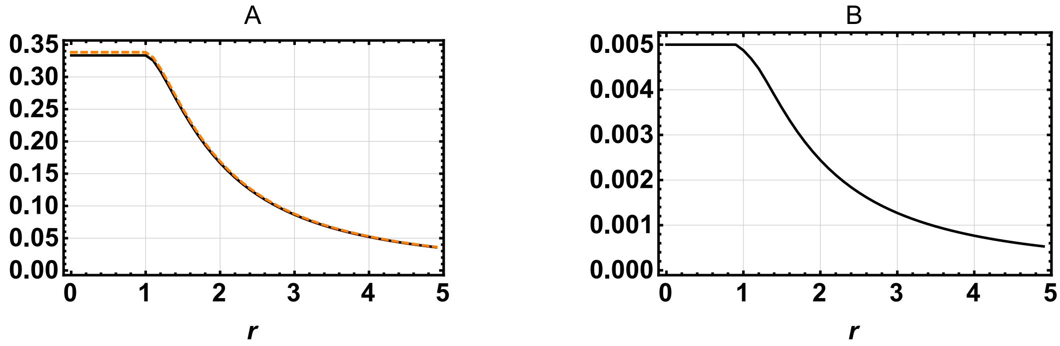


Figure 6: (A) Exact (solid line) and approximate (dashed line) output of (30) with $f(x) = (1-x^2)/(R(1-r+R)(1+r+R))$, where $R = \sqrt{1-2xr+r^2}$. The quadrature was computed with $N = 100$ nodes and $\alpha = 1$ and $\beta = 1$. (B) Absolute error.

# 7 Final Remarks

The quadrature formulas given in Section 6 are extremely simple to compute, but the cases of the positive real semi-axis and the finite interval depend on the parameters of the associated polynomials. The sets of nodes $x_{k,N}$ constructed with different values of these parameters will give different results for the integrals. However, the method to find quadrature rules associated to linear generating functions of the classical orthogonal polynomials presented in Sections 3-5 of this paper furnishes the way to find the optimum values of the parameters involved in the simple Riemann-Steklov quadratures of Sect. 6, in the cases of the positive real semi-axis and the finite interval, i.e., $\alpha$ for the Laguerre case and $\alpha$ and $\beta$ for the Jacobi case. This is a problem equivalent to the expansion problem of a function in terms of orthogonal polynomials.
It is important to remark that (30) is in agreement with some previous results [9].

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