

On co-polynomials on the real line

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Abstract

In this paper, we study new algebraic and analytic aspects of orthogonal polynomials on the real line when finite modifications of the recurrence coefficients, the so-called co-polynomials on the real line, are considered. We investigate the behavior of their zeros, mainly interlacing and monotonicity properties. Furthermore, using a transfer matrix approach we obtain new structural relations, combining theoretical and computational advantages. Finally, a connection with the theory of orthogonal polynomials on the unit circle is pointed out.

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Orthogonal polynomials on the real line, co-polynomials on the real line, zeros, transfer matrices

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1. Introduction

Let $d\mu$ be a non-trivial probability measure with an infinity support on some subset $A \subseteq \mathbb{R}$, such that

$$\int_A x^{2n} d\mu(x) < \infty, \quad n \geq 0.$$

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The application of Gram-Schmidt's orthogonalization procedure to $\{x^n\}_{n \geq 0}$ yields a unique sequence of monic polynomials $\{P_n\}_{n \geq 0}$,

$$P_n(x) = x^n + (\text{lower degree terms}),$$

and a sequence, $\{\gamma_n\}_{n \geq 0}$, of positive real numbers such that

$$\int_A P_n P_m d\mu = \gamma_n \delta_{n,m}, \quad m \geq 0, \quad (1.1)$$

where $\delta_{n,m}$ is the Kronecker delta. These polynomials are known in the literature as *orthogonal polynomials on the real line* (OPRL, in short), also known as *Chebyshev polynomials* before the book of Szegő [26] when the terminology was reserved for four special cases of trigonometric OPRL [26, Sec. 1.12].

It is very well known that the zeros of P_n , $\{x_{n,k}\}_{k=1}^n$, are real, simple and are located in the interior of the convex hull of the support A of the measure $d\mu$ and the zeros of P_n and P_{n+1} strictly interlace. The notation for zeros is

$$x_{n,n} < x_{n,n-1} < \cdots < x_{n,2} < x_{n,1}.$$

We suggest the reader to consult [2, 4, 13, 20, 21, 26], where a complete presentation of the classical theory of OPRL can be found.

Associated with any sequence of OPRL there exist sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ of positive real numbers and real numbers, respectively, such that

$$P_{n+1}(x) = (x - b_{n+1})P_n(x) - a_n P_{n-1}(x), \quad a_0 := 1, \quad n \geq 0, \quad (1.2)$$

with initial conditions $P_{-1} := 0$ and $P_0 := 1$. We set $P_n := 0$ for $n < 0$ and $a_n := b_n := 0$ for $n < 1$, then (1.2) holds for every $n \in \mathbb{Z}$. Set

$$\mathbf{P}_{n+1} := [P_{n+1}, P_n]^T, \quad \mathbf{A}_n := \begin{bmatrix} x - b_{n+1} & -a_n \\ 1 & 0 \end{bmatrix}.$$

Notice that from (1.2), we get

$$\mathbf{P}_{n+1} = \mathbf{A}_n \mathbf{P}_n, \quad \mathbf{P}_0 := [P_0, P_{-1}]^T,$$

as well as

$$\mathbf{P}_{n+1} = (\mathbf{A}_n \cdots \mathbf{A}_0) \mathbf{P}_0. \quad (1.3)$$

\mathbf{A}_n is said to be the transfer matrix. This representation will be the central object in Section 3. The converse of the previous result is the so-called Favard's theorem or Spectral Theorem in the OPRL theory. In other words, given a sequence of polynomials, $\{P_n\}_{n \geq 0}$, generated by (1.2) with recurrence coefficients $\{a_n\}_{n \geq 1}$ positive real and $\{b_n\}_{n \geq 1}$ real numbers, then there exists a nontrivial probability measure $d\mu$ supported on the real line so that the orthogonality conditions (1.1) hold. Moreover, if $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$ are bounded sequences,

then $d\mu$ is unique. From now on, we will assume that the recurrence coefficients always satisfy the hypothesis of Favard's theorem.

The theory of OPRL has attracted an increasing interest from the pioneer works of Legendre, Gauss, Jacobi, Chebyshev, Christoffel, Stieltjes and Markov, among others. The construction of new sequences of OPRL by modifying the original sequence is a powerful tool, with many applications to theoretical and applied problems, such as asymptotic analysis, zero behavior, integrable systems, birth-and-death process, quadrature, and quantum mechanics, among others. In particular, the study of the properties of new sequences of OPRL with respect to finite modifications (by changing or shifting) of the recursion coefficients is a classical topic. For example, associated polynomials appear in Stieltjes' works [24, 25] related to the convergence of certain continued fractions. Given the sequences $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, one defines for a fixed positive integer, k , the associated polynomials of order k , $\{P_n^{(k)}\}_{n \geq 0}$, by the recurrence relation

$$P_{n+1}^{(k)}(x) = (x - b_{n+k+1})P_n^{(k)}(x) - a_{n+k}P_{n-1}^{(k)}(x), \quad n \geq 0, \quad (1.4)$$

with initial conditions $P_{-1}^{(k)} := 0$ and $P_0^{(k)} := 1$. As previously, we set $P_n^{(k)} := 0$ for $n < 0$ and $a_n := b_n := 0$ for $n < 1$, then (1.4) holds for every $n \in \mathbb{Z}$. General results on such associated polynomials can be found in [3, 28].

On the other hand, OPRL associated with finite perturbations of recurrence coefficients, in what follows denoted as *co-polynomials on the real line* (COPRL, in short), are firstly considered by Allaway [1] and Chihara [5], who studied the case when only the first recursion coefficient b_1 is perturbed by adding a constant. This kind of perturbations is not artificial in any sense. We recall that the modification of a finite number of the recurrence coefficients corresponding to Chebyshev's polynomials of second kind leads to Bernstein-Szegő's polynomials [26]. Some results concerning finite perturbations of Chebyshev's polynomials can be found in [19]. The algebraic and analytic properties of general COPRL have been studied mainly by Marcellán, Dehesa and Ronveaux [15], Maroni [16], and Peherstorfer [18], see also [1, 8]. Some applications can be also found in [9, 10, 14, 23].

The goal of our research is to study new properties of the polynomials which satisfy a recurrence relation as (1.2) with new recurrence coefficients, perturbed in a (generalized) co-dilated and/or co-recursive way, $\{c_n\}_{n \geq 1}$ and $\{d_n\}_{n \geq 1}$, i.e.,

$$u_{n+1}(x) = (x - d_{n+1})u_n(x) - c_n u_{n-1}(x), \quad c_0 := 1, \quad n \geq 0,$$

with initial conditions $u_{-1} := 0$ and $u_0 := 1$. In other words, we consider arbitrary single modifications of the recurrence coefficients as follows:

$$c_n = \lambda_k^{\delta_{n,k}} a_n, \quad \lambda_k > 0, \quad (\text{co-dilated case}) \quad (1.5)$$

$$d_n = b_n + \tau_{k+1} \delta_{n,k+1}, \quad \tau_{k+1} \in \mathbb{R}. \quad (\text{co-recursive case}) \quad (1.6)$$

where k is a fixed non-negative integer number. Moreover, we will consider the finite composition of the above perturbations. In Section 2, we study some

new inequalities for the zeros of COPRL by following the approach presented in [6] for the study of the monotonicity of zeros of a class of para-orthogonal polynomials on the unit circle including the Askey hypergeometric polynomials ${}_2F_1(-n, a + bi; 2a; 1 - z)$, $a, b \in \mathbb{R}$. In Section 3, we obtain a new structural relation based on a transfer matrix approach proposed recently in [7] for similar perturbations in the theory of *orthogonal polynomials on the unit circle* (OPUC, in short). Finally, in Section 3 we point out the connection with the OPUC.

2. Zeros and inequalities

It is very well-known that the orthonormal version of (1.2), for recurrence coefficients depending on a parameter ϵ , can be written in an operator form by using a symmetric Jacobi matrix, $\mathbf{J}(\epsilon)$,

$$\mathbf{J}(\epsilon) = \begin{bmatrix} b_1 & d_1 & & & \\ d_1 & b_2 & d_2 & & \\ & d_2 & b_3 & d_3 & \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

where $d_n^2 = a_n$ (for simplicity, we omit here the dependence of ϵ). In a matrix form,

$$x\mathbf{p} = \mathbf{J}(\epsilon)\mathbf{p},$$

where $p_n = \gamma_n^{-1/2}P_n$ and $\mathbf{p} = [p_0, p_1, \dots]^T$. According to a version of Hellmann-Feynman's theorem [13, Sec. 7.3], if $\partial\mathbf{J}_n(\epsilon)/\partial\epsilon$ is strictly positive (resp. negative) definite, then the zeros of the corresponding OPRL are strictly increasing (resp. decreasing) functions of ϵ . But for some cases related with COPRL, we can obtain more information on the behavior of zeros following a different approach recently proposed in [6].

In [15], using the theory of difference equations, the authors deduced the explicit expression of the COPRL associated with the perturbation (1.5) and/or (1.6) in terms of the initial OPRL and their associated polynomials of order k .

Let us define

$$D(u_n, v_n) := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix}, \quad (2.7)$$

the Casorati determinant associated with two arbitrary sequences $\{u_n\}_{n \geq 1}$ and $\{v_n\}_{n \geq 1}$. From the theory of linear difference equations, we know that if the Casorati determinant is different from zero for every n , then these two sequences are said to be linearly independent [17]. Notice that $\{P_{n-k}^{(k)}\}_{n \geq 0}$, is a solution of the recurrence relation (1.2). It is easy to verify that

$$\begin{bmatrix} P_{n+1} & P_{n-k+1}^{(k)} \\ P_n & P_{n-k}^{(k)} \end{bmatrix} = \mathbf{A}_n \begin{bmatrix} P_n & P_{n-k}^{(k)} \\ P_{n-1} & P_{n-k-1}^{(k)} \end{bmatrix}.$$

Hence,

$$D(P_n, P_{n-k}^{(k)}) = a_n D(P_{n-1}, P_{n-k-1}^{(k)}).$$

Let X denote the set of zeros of P_{k-1} . From the above equalities, we get

$$D(P_n, P_{n-k}^{(k)}) = \left(\prod_{j=k}^n a_j \right) P_{k-1}, \quad (2.8)$$

which means that P_n and $P_{n-k}^{(k)}$ with $n > k$, are linearly independent in $\mathbb{C} \setminus X$. If we denote by $\{P_n(\cdot; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1})\}_{n \geq 0}$ the COPRL associated with the finite composition of perturbations (1.5) and (1.6) from order m to order k , $m \leq k$ then, after elementary calculations, for $m = k$ we have

THEOREM 2.1. *For $x \in \mathbb{C} \setminus X$ the following formulas hold:*

$$\begin{aligned} P_n(x; \lambda_k, \tau_{k+1}) &= P_n(x), & n \leq k, \\ P_n(x; \lambda_k, \tau_{k+1}) &= P_n(x) - Q_k(x) P_{n-k}^{(k)}(x), & n > k, \end{aligned}$$

where $Q_k(x) = \tau_{k+1} P_k(x) + a_k(\lambda_k - 1) P_{k-1}(x)$.

As a consequence of the last result, we get

COROLLARY 2.1. *$P_n(\cdot; \lambda_k, \tau_{k+1})$ and P_n share at most the zeros of Q_k and P_{k-1} .*

Proof. Suppose that $P_n(\cdot; \lambda_k, \tau_{k+1})$ and P_n have a common zero, α , different from the zeros of Q_k and P_{k-1} . Let Y denote the set of zeros of Q_k . Since $\alpha \in \mathbb{C} \setminus (X \cup Y)$, Theorem 2.1 implies $P_{n-k}^{(k)}(\alpha) = 0$, a contradiction. \square

From the interlacing property of two consecutive OPRL, we can easily deduce that Q_k , P_k , and P_{k-1} are coprime. But we can go a step further.

PROPOSITION 2.1. *Let assume $\lambda_k \neq 1$ and $\tau_{k+1} \neq 0$ and define $c := (\lambda_k - 1)/\tau_{k+1}$. Let $\{y_{k,j}(c)\}_{j=1}^k$ be the zeros of Q_k . The following statements hold:*

i) *If $c > 0$, then*

$$x_{k-1,j-1} < y_{k,j}(c) < x_{k,j}; \quad x_{k-1,0} := -\infty.$$

Moreover, $y_{k,j}(c)$ (for a fixed value of j) is a strictly increasing (resp. decreasing) function of λ_k (resp. τ_{k+1}).

ii) *If $c < 0$, then*

$$x_{k,j} < y_{k,j}(c) < x_{k-1,j}; \quad x_{k-1,k} := \infty.$$

Also, $y_{k,j}(c)$ (for a fixed value of j) is a strictly decreasing (resp. increasing) function of λ_k (resp. τ_{k+1}).

Furthermore,

$$\lim_{\lambda_k \rightarrow 1} y_{k,j}(c) = x_{k-1,j}, \quad \lim_{\tau_{k+1} \rightarrow \infty} y_{k,j}(c) = x_{k-1,j}.$$

Proof. The interlacing in the first part of the theorem follows in a straightforward way from [4, Ch. 1, Ex. 5.4]. Furthermore, in the same way, the monotonicity is a consequence of the interlacing property for the zeros of Q_k and P_{k-1} . Let

$$Q_k(x; \epsilon) := P_k(x) + (c + \epsilon)a_k P_{k-1}(x), \quad \epsilon > 0.$$

Hence,

$$Q_k(x; \epsilon) = Q_k(x) + \epsilon a_k P_{k-1}(x),$$

and the expected result on monotonicity follows as previously. The second part of the theorem is a direct consequence of Hurwitz's theorem [26, Thm. 1.91.3]. \square

We recall that the zeros of the polynomial Q_k lie in (a, b) , with the exception of the extreme zeros. The location of the extreme zeros with respect to the orthogonality interval A can be given by using [26, Thm. 3.3.4].

The next theorem has direct consequences in the interlacing and monotonicity of zeros of COPRL.

THEOREM 2.2. *Let $x_{n,j+1}$ and $x_{n,j}$ be two consecutive zeros of P_n , then the following holds. If there are no zeros of $Q_k P_k$ in $I_j := (x_{n,j+1}, x_{n,j})$ that are not zeros of $P_n(\cdot; \lambda_k, \tau_{k+1})$, then the interval I_j contains at most an odd number of zeros of $P_n(\cdot; \lambda_k, \tau_{k+1})$. Moreover, if there are zeros of $Q_k P_k$ in I_j that are not zeros of $P_n(\cdot; \lambda_k, \tau_{k+1})$, then the interval I_j contains at most an even number of zeros of $P_n(\cdot; \lambda_k, \tau_{k+1})$.*

Proof. With the notation of Proposition 2.1, we can assume $c > 0$ without loss of generality. In such a situation

$$\begin{aligned} D(P_n(x), P_n(x; \lambda_k, \tau_{k+1})) &= P_n(x)P_{n+1}(x; \lambda_k, \tau_{k+1}) - P_n(x; \lambda_k, \tau_{k+1})P_{n+1}(x), \\ &= a_n D(P_{n-1}(x), P_{n-1}(x; \lambda_k, \tau_{k+1})), \end{aligned}$$

yields

$$d(x) := D(P_n(x), P_n(x; \lambda_k, \tau_{k+1})) = - \left(\prod_{j=k-1}^n a_j \right) Q_k(x) P_k(x). \quad (2.9)$$

Obviously, $(-1)^j P_{n+1}(x_{n,j}) > 0$. Now, there are two cases depending on the sign of d to be considered. Denote by S_- the system of intervals indicated by thick solid lines in Figure 1. According to our assumptions and Proposition 2.1, we first consider the case for which $d(I_j) > 0$, i.e., $I_j \subset S_-$.

By (2.9), $-P(x_{n,j}; \lambda_k, \tau_{k+1})P_{n+1}(x_{n,j}) > 0$, which yields

$$(-1)^{j+1} P_n(x_{n,j}; \lambda_k, \tau_{k+1}) > 0.$$

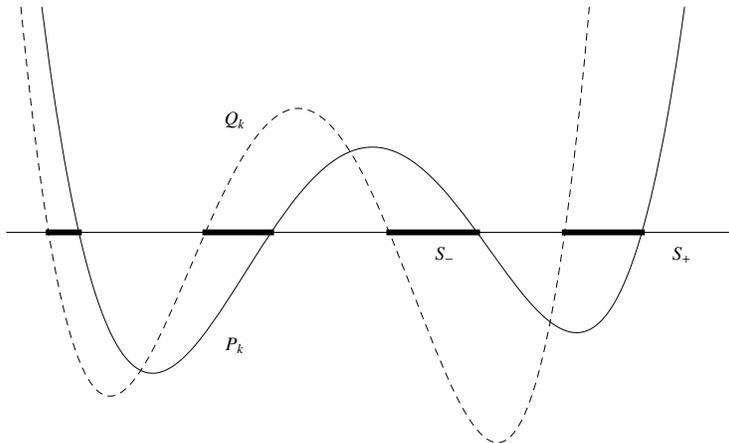


Figure 1: Graphs of Q_k and P_k for $c > 0$ and $k = 4$.

Therefore, in this situation the theorem holds.

On the other hand, a similar result can be obtained for $I_j \subset S_+$, where S_+ is the system of intervals not indicated by thick solid lines in Figure 1. The rest of the proof follows directly from the previous analysis. \square

Note that the previous result contains as a particular case the interlacing obtained in [5]. Let us consider the co-recursive case, that is, $\lambda_k := 1$. In this situation, the system of intervals satisfies $S_- = \{0\}$, or equivalently $S_+ = \{0\}$. Hence we have the following interlacing property.

COROLLARY 2.2. *Let $l < k$ be the number of no common zeros between $P_n(\cdot; 1, \tau_{k+1})$ and P_n . Denote by $\{y_{n,j}(1, \tau_{k+1})\}_{j=1}^l$ and $\{y_{n,j}\}_{j=1}^l$, these zeros. If $\tau_{k+1} < 0$, then*

$$y_{n,n}(1, \tau_{k+1}) < y_{n,l} < y_{n,l-1}(1, \tau_{k+1}) < y_{n,l-1} < \cdots < y_{n,1}(1, \tau_{k+1}) < y_{n,1}, \quad (2.10)$$

where the role of the zeros $\{y_{n,j}(1, \tau_{k+1})\}_{j=1}^l$ and $\{y_{n,j}\}_{j=1}^l$, is reversed when $\tau_{k+1} > 0$.

COROLLARY 2.3. *The zeros of the polynomial $P_n(\cdot; 1, \tau_{k+1}; 1, \tau_{k+2})$ (for a fixed value of k and $n > k$) are strictly increasing functions of τ_{k+1} and τ_{k+2} .*

The previous results for the co-recursive case reduce and give more information than Hellmann-Feynman's theorem. Notice that the existence of cases for which $\det(\partial \mathbf{J}_n(\epsilon)/\partial \epsilon) = 0$, mentioned in the beginning of the section, could imply strictly monotonicity of zeros. We recall that Corollary 2.3 was also perceived in [6] from the perturbation theory for symmetric matrices.

EXAMPLE 2.1. It is well known [4] that the monic Jacobi polynomials $\{P_n^{(\alpha,\beta)}\}_{n \geq 0}$ satisfy for any real value of α and β , the recurrence relation (1.2) where

$$a_n^{(\alpha,\beta)} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)(2n+\alpha+\beta)^2},$$

$$b_{n+1}^{(\alpha,\beta)} = \frac{\beta^2 - \alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}.$$

Furthermore, if $\alpha, \beta > -1$ the polynomials are orthogonal with respect to the weight $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1, 1]$. In order to illustrate Corollary 2.3, we consider a new sequence of Jacobi polynomials associated with two consecutive modification (1.6). Figure 2 is obtained by using Wolfram Mathematica[®] 9.0¹ with the aid of the function **JacobiP**[**n**, α , β , **x**] and the recurrence relation (1.2), and shows the polynomials $P_5^{(2,1)}$ (continuous line), $P_5^{(2,1)}(\cdot; 1, 0.2; 1, 0.25)$ (small-dashed line), and $P_5^{(2,1)}(\cdot; 1, 0.3; 1, 0.3)$ (large-dashed line). Observe that the zeros behave in accordance with our result. In other words, the monotonicity is 'strict' and it is not something that can be guaranteed by Hellmann-Feynman's theorem.

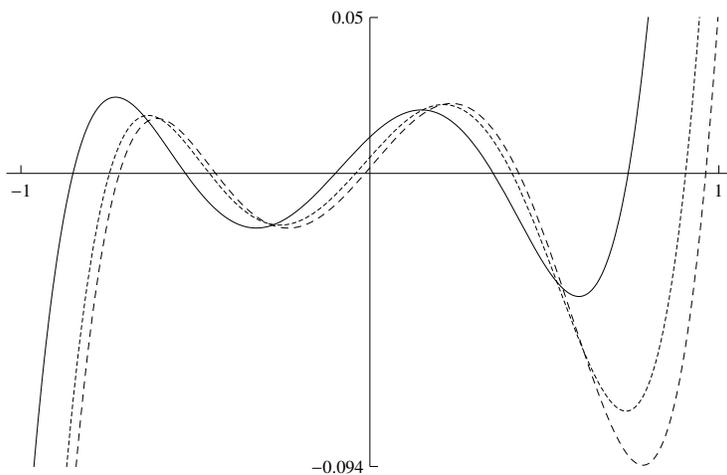


Figure 2: Graphs of $P_5^{(2,1)}$ (continuous line), $P_5^{(2,1)}(\cdot; 1, 0.2; 1, 0.25)$ (small-dashed line), and $P_5^{(2,1)}(\cdot; 1, 0.3; 1, 0.3)$ (large-dashed line).

According to Figure 1, a general result in the previous direction is more complicated because the zeros have different behavior depending on the intervals S_- and S_+ where they are located. In any case, for some sets of extreme zeros we can obtain more information.

¹Wolfram Mathematica is a registered trademark of Wolfram Research, Inc.

THEOREM 2.3. *With the notation of Proposition 2.1, let us define $y_1 := \max\{x_{k,1}, y_{k,1}(c)\}$. Let us denote by $\{x_{n,j}(\lambda_k, \tau_{k+1})\}_{j=1}^n$ the zeros of the polynomial $P_n(\cdot; \lambda_k, \tau_{k+1})$. If $c > 0$, then*

$$x_{n,l} < x_{n,l}(\lambda_k, \tau_{k+1}),$$

for all the zeros of $P_n(\cdot; \lambda_k, \tau_{k+1})$ and P_n in $\mathbb{R} \setminus [-\infty, y_1]$, where the role of the zeros $x_{n,l}$ and $x_{n,l}(\lambda_k, \tau_{k+1})$ is reversed when $c < 0$.

Proof. Without loss of generality we can assume, as in the proof of Theorem 2.2, $c > 0$. Hence, $y_1 = x_{k,1}$, see Figure 1. By (2.9), $P(x_{n,l}; \lambda_k, \tau_{k+1})P_{n+1}(x_{n,l}) > 0$, which yields

$$(-1)^j P_n(x_{n,l}; \lambda_k, \tau_{k+1}) > 0.$$

The result can be deduced as above. □

The usual tool dealing with the inequalities concerning the largest (or the smallest zero) of OPRL is the Perron-Frobenius Theorem [13, Thm. 7.4.1]. Notice that the previous result gives more information.

EXAMPLE 2.2. *The monic Laguerre polynomials $\{L_n^{(\alpha)}\}_{n \geq 0}$, satisfy, for any real value of α , the recurrence relation (1.2) where*

$$\begin{aligned} a_n^{(\alpha)} &= n(n + \alpha), \\ b_{n+1}^{(\alpha)} &= 2n + 1 + \alpha. \end{aligned}$$

*Furthermore, if $\alpha > -1$ the polynomials are orthogonal with respect to the weight $x^\alpha e^{-x}$ on the interval $[0, \infty)$. In order to illustrate Theorem 2.3, we consider a new sequence of Laguerre polynomials associated with the modifications (1.5) and (1.6). Figure 3 is obtained by using Wolfram Mathematica[®] 9.0 with the aid of the function **LaguerreL**[n, α, \mathbf{x}] and the recurrence relation (1.2), and shows the polynomials $L_4^{(4)}$ (continuous line) and $L_4^{(4)}(\cdot; 1.4, 4)$ (dashed line). Observe that for this case with $c = 0.1$, all the zeros greater than $y_1 = 2.7965$ behave in accordance with Theorem 2.3. Notice that, the Perron-Frobenius Theorem can guarantee this result only for the largest zero.*

3. A transfer matrix approach

Theorem 2.1 has been successfully used in the study of zeros of COPRL but presents two main constraints. First, the structural relation is not useful if we are interested in the finite composition of perturbations, mainly from a computational point of view. Second, the structural relation is not valid on the whole complex plane. The aim of this section is to use a transfer matrix approach to avoid these constraints.

Using the matrix notation (1.3), we have

$$\mathbf{P}_{n+1}(x; \lambda_k, \tau_{k+1}) = (\mathbf{A}_n \cdots \mathbf{A}_{k+1}) \mathbf{A}_k(\lambda_k, \tau_{k+1}) (\mathbf{A}_{k-1} \cdots \mathbf{A}_0) \mathbf{P}_0, \quad (3.11)$$

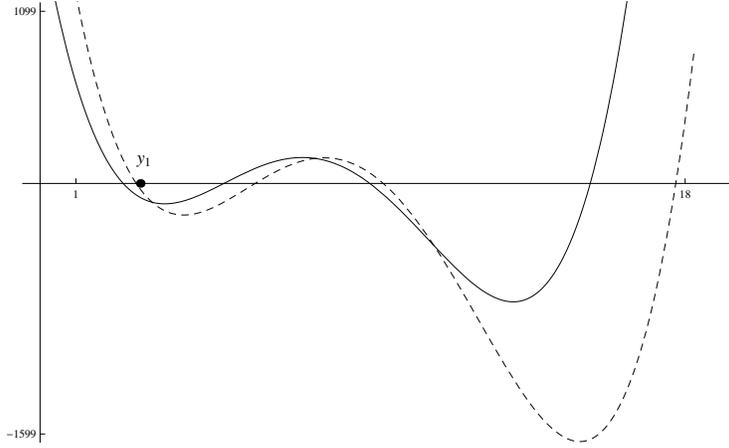


Figure 3: Graphs of $L_4^{(4)}$ (continuous line) and $L_4^{(4)}(\cdot; 1.4, 4)$ (dashed line).

where

$$\mathbf{A}_k(\lambda_k, \tau_{k+1}) = \begin{bmatrix} x - b_{k+1} - \tau_{k+1} & -\lambda_k a_k \\ 1 & 0 \end{bmatrix}.$$

Combining (1.3) and (3.11), we can deduce that the following formula holds on \mathbb{C}

$$\mathbf{P}_{n+1}(x; \lambda_k, \tau_k) = (\mathbf{A}_n \cdots \mathbf{A}_{k+1}) \mathbf{A}_k(\lambda_k, \tau_{k+1}) \mathbf{A}_k^{-1} (\mathbf{A}_n \cdots \mathbf{A}_{k+1})^{-1} \mathbf{P}_{n+1}(x).$$

The previous equation has some computational advantage as compared to Theorem 2.2 and it holds in \mathbb{C} . But we can improve this result by using an auxiliary sequence of polynomials.

Of course, the so-called first kind associated polynomials $\{r_n\}_{n \geq 0}$ are the unique solution of the recurrence relation (1.2) with initial conditions $r_{-1} := -1$ and $r_0 := 0$ or, equivalently, $r_0 := 0$ and $r_1 := 1/a_1$. Note that r_n is a polynomial of degree $n - 1$. We define $R_n := \gamma_n^{-1} r_n = P_{n-1}^{(1)}$ which is a monic polynomial.

THEOREM 3.1. *The following formulas hold in \mathbb{C} :*

$$\begin{pmatrix} \prod_{j=1}^k a_j \end{pmatrix} \begin{bmatrix} P_{n+1}(x; \lambda_k, \tau_{k+1}) \\ -R_{n+1}(x; \lambda_k, \tau_{k+1}) \end{bmatrix} = \mathbf{M}_k \begin{bmatrix} P_{n+1}(x) \\ -R_{n+1}(x) \end{bmatrix}, \quad n > k,$$

where \mathbf{M}_k is

$$\mathbf{M}_k = \begin{bmatrix} \left(\prod_{j=1}^k a_j \right) + Q_k R_k & Q_k P_k \\ \widehat{R}_k R_k & \left(\prod_{j=1}^k a_j \right) + \widehat{R}_k P_k \end{bmatrix},$$

with $\widehat{R}_k = -\tau_{k+1}R_k - (\lambda_k - 1)a_kR_{k-1}$.

Proof. Let us introduce the matrix \mathbf{B}_{n+1} , given by

$$\mathbf{B}_{n+1} = \begin{bmatrix} P_{n+1} & -R_{n+1} \\ P_n & -R_n \end{bmatrix}.$$

Since

$$D(P_{n+1}, -R_{n+1}) = a_n \det \mathbf{B}_n = \prod_{j=1}^n a_j$$

then \mathbf{B}_{n+1} is a nonsingular matrix.

We now apply the previous argument again, in order to obtain

$$\mathbf{B}_{n+1} = \mathbf{A}_n \mathbf{B}_n = \mathbf{A}_n \cdots \mathbf{A}_0, \quad (3.12)$$

the product of the transfer matrices for the associated polynomials of order k . Let us denote by $\mathbf{B}_{n+1}(\cdot; \lambda_k, \tau_{k+1})$ the polynomial matrices corresponding to the COPRL associated with the perturbations (1.5) and (1.6), and $\mathbf{B}_{n-k}^{(k)}$ the product of transfer matrix for associated polynomials of order k . Since

$$\mathbf{B}_{n-k}^{(k)} = \mathbf{A}_n \cdots \mathbf{A}_k,$$

from (3.12), we get

$$\mathbf{B}_{n-(k+1)}^{(k+1)} = \mathbf{B}_{n+1} \mathbf{B}_k^{-1} \mathbf{A}_k^{-1}, \quad (3.13)$$

$$\mathbf{B}_{n+1}(\lambda_k, \tau_{k+1}) = \mathbf{B}_{n-(k+1)}^{(k+1)} \mathbf{A}_k(\lambda_k, \tau_{k+1}) \mathbf{B}_k. \quad (3.14)$$

From (3.13) and (3.14), we get

$$\mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}) = (\mathbf{A}_k(\lambda_k, \tau_{k+1}) \mathbf{B}_k)^T (\mathbf{A}_k \mathbf{B}_k)^{-T} \mathbf{B}_{n+1}^T, \quad (3.15)$$

where

$$\begin{aligned} (\mathbf{A}_k(\lambda_k, \tau_{k+1}) \mathbf{B}_k)^T &= \begin{bmatrix} P_{k+1}(x; \lambda_k, \tau_{k+1}) & P_k(x) \\ -R_{k+1}(x; \lambda_k, \tau_{k+1}) & -R_k(x) \end{bmatrix}, \\ \left(\prod_{j=1}^k a_j \right) (\mathbf{A}_k \mathbf{B}_k)^{-T} &= \begin{bmatrix} -R_k & -P_k \\ R_{k+1} & P_{k+1} \end{bmatrix}. \end{aligned}$$

Then it is easy to check that

$$(\mathbf{A}_k(\lambda_k, \tau_{k+1}) \mathbf{B}_k)^T (\mathbf{A}_k \mathbf{B}_k)^{-T} = \mathbf{M}_k, \quad (3.16)$$

which, after some elementary calculations, proves the theorem. \square

EXAMPLE 3.1. *In this case [5], we have COPRL associated with the modifications (1.6) for $k := 0$. By (2.8), Theorem 2.1 and Theorem 3.1 are equivalent. It is easy to check that*

$$P_{n+1}(x; 1, \tau_1) = P_{n+1}(x) - \tau_1 R_{n+1}(x).$$

Next we give a relation between the COPRL associated with two modifications of different levels.

COROLLARY 3.1. *Let k, m be two fixed non-negative integer numbers with $m < k$. Then, the following relation holds*

$$\left(\prod_{j=m+1}^k a_j \right) \begin{bmatrix} P_{n+1}(x; \lambda_k, \tau_{k+1}) \\ -R_{n+1}(x; \lambda_k, \tau_{k+1}) \end{bmatrix} = \mathbf{M}_k \mathbf{M}_m^{-1} \begin{bmatrix} P_{n+1}(x; \lambda_m, \tau_{m+1}) \\ -R_{n+1}(x; \lambda_m, \tau_{m+1}) \end{bmatrix}, \quad n > k.$$

Proof. The proof is a straightforward consequence of (3.15) and (3.16). \square

For a finite composition of perturbations we have the following result.

THEOREM 3.2. *For $0 < m \leq k < \infty$ and for $n > m$ the following relation holds:*

$$\left(\prod_{j=m}^k \prod_{l=0}^j a_l \right) \begin{bmatrix} P_{n+1}(x; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1}) \\ -R_{n+1}(x; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1}) \end{bmatrix} = \left(\prod_{j=m}^k \mathbf{M}_j \right) \begin{bmatrix} P_{n+1}(x) \\ -R_{n+1}(x) \end{bmatrix}.$$

Proof. Since \mathbf{M}_k depends only on the first $k+1$ original recurrence coefficients and the perturbed a_k and b_{k+1} , we have

$$\begin{aligned} \left(\prod_{j=0}^k a_j \right) \mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}) &= \mathbf{M}_k \mathbf{B}_{n+1}^T, \quad n > k, \\ \left(\prod_{j=0}^{k-1} a_j \right) \mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}, \lambda_{k-1}, \tau_k) &= \mathbf{M}_{k-1} \mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}), \quad n > k-1, \\ &\vdots \\ \left(\prod_{j=0}^m a_j \right) \mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}, \dots, \lambda_m, \tau_{m+1}) &= \mathbf{M}_m \mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}, \dots, \lambda_{m-1}, \tau_m), \quad n > m. \end{aligned}$$

Clearly,

$$\mathbf{B}_{n+1}^T(\lambda_k, \tau_{k+1}, \dots, \lambda_m, \tau_{m+1}) = \mathbf{B}_{n+1}^T(\lambda_m, \tau_{m+1}, \dots, \lambda_k, \tau_{k+1}).$$

Thus,

$$\left(\prod_{j=m}^k \prod_{l=0}^j a_l \right) \mathbf{B}_{n+1}^T(\lambda_m, \tau_{m+1}, \dots, \lambda_k, \tau_{k+1}) = \left(\prod_{i=m}^k \mathbf{M}_i \right) \mathbf{B}_{n+1}^T, \quad n > m,$$

and the result follows. \square

4. Connection with the unit circle case

In [27], see also [26, Sec. 11.5], Szegő pointed out the relation between OPRL on $[-1, 1]$ and some sequences of OPUC, by using the Joukowski transformation mapping the exterior of the unit circle onto the exterior of the interval $[-1, 1]$ by the modification of the corresponding measure of orthogonality. For more details, see [26].

The monic OPUC, $\{\Phi_n\}_{n \geq 0}$, is generated by the forward recurrence relation

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z),$$

with initial condition $\Phi_0 := 1$. Here, $\Phi_n^*(z) = z^n \bar{\Phi}_n(z^{-1})$ is the reversed polynomial and the complex numbers $\{\alpha_n\}_{n \geq 0}$, $\alpha_n = -\bar{\Phi}_{n+1}(0)$, are known as Schur or Verblunsky coefficients. The best general references on OPUC are the monographs of Geronimus [11, 12] and Simon [21]. Let us denote by $\partial\mathbb{D}$ the boundary of the open unit disk $\mathbb{D} := \{z \in \mathbb{C}; |z| < 1\}$. The Verblunsky theorem or Spectral Theorem in the OPUC theory states that when $|\alpha_n| < 1$, $n \geq 0$, then $\{\Phi_n\}_{n \geq 0}$ is a sequence of monic orthogonal polynomials with respect to a unique nontrivial probability measure supported on $\partial\mathbb{D}$.

In [6], the author studied the effect on OPRL when the Verblunsky coefficients $\{\alpha_n\}_{n \geq 0}$ are perturbed in the following way:

$$\beta_n := \begin{cases} \beta_k \in \mathbb{D}, & n = k, \\ \alpha_n, & \text{otherwise.} \end{cases} \quad (4.17)$$

Here, k is a fixed non-negative integer number. The polynomials associated with the perturbed Verblunsky coefficients (4.17) are known as *co-polynomials on the unit circle* (COPUC, in short). In the next result we consider the inverse situation.

THEOREM 4.1. *Let $\{\hat{\alpha}_n\}_{n \geq 0}$ be the Verblunsky coefficients for the corresponding COPUC, $\{\Phi_n(\cdot; \lambda_k, \tau_{k+1})\}_{n \geq 0}$, associated with (1.5) and (1.6) through the Szegő transformation. Let define $S_n := P_{n+1}/P_n$ and $S_n(x; \lambda_k, \tau_{k+1}) := P_{n+1}(x; \lambda_k, \tau_{k+1})/P_n(x; \lambda_k, \tau_{k+1})$. Then,*

$$\hat{\alpha}_{2n-1} = \alpha_{2n-1} + c_n, \quad \hat{\alpha}_{2n} = \alpha_{2n} + d_n,$$

where

$$c_n = S_n(1) - S_n(-1) + S_n(-1; \lambda_k, \tau_{k+1}) - S_n(1; \lambda_k, \tau_{k+1}),$$

$$d_n = 2 \left(\frac{S_n(-1)}{\alpha_{2n-1} + 1} - \frac{S_n(-1; \lambda_k, \tau_{k+1})}{\hat{\alpha}_{2n-1} + 1} \right).$$

Proof. It is very well-known [26, Sec. 11.5] that

$$\Phi_{2n}(0) = S_n(1) - S_n(-1) - 1, \quad \Phi_{2n+1}(0) = \frac{S_n(1) + S_n(-1)}{S_n(1) - S_n(-1)}.$$

Since,

$$S_n(1) + S_n(-1) = \Phi_{2n}(0) + 2S_n(-1) + 1,$$

we have

$$\begin{aligned} \Phi_{2n+1}(0) - \Phi_{2n+1}(0; \lambda_k, \tau_{k+1}) &= \frac{\Phi_{2n}(0) + 2S_n(-1) + 1}{\Phi_{2n}(0) + 1} \\ &\quad - \frac{\Phi_{2n}(0; \lambda_k, \tau_{k+1}) + 2S_n(-1; \lambda_k, \tau_{k+1}) + 1}{\Phi_{2n}(0; \lambda_k, \tau_{k+1}) + 1}, \end{aligned}$$

and thus the theorem is proved. \square

Note that the modifications (1.5) and (1.6) imply through the Szegő transformation the modification of all the Verblunsky coefficients greater than k . By the properties of zeros of OPRL, in order to obtain the value of the polynomials $S_n(\cdot; \lambda_k, \tau_{k+1})$ at -1 and 1 , we can use Theorem 2.1.

REMARKS 4.1. *From now on, we adopt the notation \doteq used in [6], i.e., for the homography mapping*

$$y = \frac{ax + b}{cx + d}, \quad ad - bc \neq 0,$$

we will write

$$y \doteq \mathbf{A}x, \quad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

The Stieltjes or Cauchy transformation of the orthogonality measure $d\mu$,

$$m_\mu(x) = \int_A \frac{d\mu(y)}{y - x}, \quad x \in \mathbb{C} \setminus A$$

has a particular interest in the theory of OPRL.

By a spectral transformation of the m -function m_μ , we mean a new m -function associated with a measure $d\sigma$, a modification of the original measure $d\mu$. We refer to pure rational spectral transformation as a transformation of m_μ given by

$$m_\sigma \doteq \mathbf{A}m_\mu, \tag{4.18}$$

where a , b , c , and d are non-zero polynomials that provide a 'true' asymptotic behavior to (4.18), see [29].

Let us denote by $m_\mu(\cdot; \lambda_k, \tau_{k+1})$ the m -function associated with the perturbations (1.5) and (1.6).

THEOREM 4.2. $m_\mu(\cdot; \lambda_k, \tau_{k+1})$ is a pure rational spectral transformation of m_μ , given by

$$m_\mu(x; \lambda_k, \tau_{k+1}) \doteq (\mathbf{J}_2 \mathbf{M}_k \mathbf{J}_2) m_\mu(x),$$

where

$$\mathbf{J}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Proof. From Theorem 3.1, we get

$$-\frac{R_{n+1}(x; \lambda_k, \tau_{k+1})}{P_{n+1}(x; \lambda_k, \tau_{k+1})} = \frac{(\mathbf{M}_k)_{2,1} - (\mathbf{M}_k)_{2,2} \frac{R_{n+1}(x)}{P_{n+1}(x)}}{(\mathbf{M}_k)_{1,1} - (\mathbf{M}_k)_{1,2} \frac{R_{n+1}(x)}{P_{n+1}(x)}},$$

and the theorem follows from Stieltjes' Theorem [26, 21]. \square

Note that the previous result was also obtained in [15].

COROLLARY 4.1. $m_\mu(\cdot; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1})$ is a pure rational spectral transformation of m_μ given by

$$m_\mu(x; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1}) \doteq \left(\mathbf{J}_2 \prod_{j=m}^k \mathbf{M}_j \mathbf{J}_2 \right) m_\mu(x).$$

On the other hand, the Riesz–Herglotz transform of a nontrivial probability measure supported on $\partial\mathbb{D}$,

$$F(z) = \int_{\partial\mathbb{D}} \frac{y+z}{y-z} d\sigma(y),$$

is the so-called \mathcal{C} -function in the OPUC theory. This function plays an analogous role to the m -function in the OPRL theory. We recall that there is also a relation between the corresponding m -function for OPRL in $[-1, 1]$ and \mathcal{C} -function, as follows

$$F(z) = \frac{1-z^2}{2z} m_\mu(x),$$

or, equivalently,

$$m_\mu(x) = \frac{F(z)}{\sqrt{x^2-1}},$$

with $2x = z + z^{-1}$ and $z = x - \sqrt{x^2-1}$.

THEOREM 4.3. Let $F(\cdot; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1})$ be the \mathcal{C} -function associated with the finite composition of perturbations (1.5) and (1.6) through the Szegő transformation. Then,

$$F(z; \lambda_m, \tau_{m+1}; \dots; \lambda_k, \tau_{k+1}) \doteq \left(\mathbf{J}_2 \prod_{j=m}^k \mathbf{M}_j \mathbf{J}_2 \right) F(z),$$

with $2x = z + z^{-1}$.

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References

- [1] W. R. Allaway, The identification of a class of orthogonal polynomial sets, Ph.D. Thesis, Univ. Alberta, Edmonton, 1972.
- [2] G.E. Andrews, R. Askey, R. Roy, Special Functions, Vol. 71, Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2000.
- [3] S. Belmehdi, On the associated orthogonal polynomials, J. Comput. Appl. Math. 32 (1990) 311-319.
- [4] T. S. Chihara, An Introduction to Orthogonal Polynomials, Gordon and Breach, New York - London - Paris, 1978.
- [5] T. S. Chihara, On co-recursive orthogonal polynomials, Proc. Amer. Math. Soc. 8 (1957) 899-905.
- [6] K. Castillo, Monotonicity of zeros for a class of polynomials including hypergeometric polynomials. Submitted.
- [7] K. Castillo, On perturbed Szegő recurrences, J. Math. Anal. Appl. 411 (2014) 742-752.
- [8] J. Dini, P. Maroni and A. Ronveaux, Sur une perturbation de la récurrence vérifiée par une suite de polynômes orthogonaux, Portugal. Math. 46 (1989) 269-282.
- [9] W. Erb, Optimally space localized polynomials with applications in signal processing, J. Fourier Anal. Appl. 18 (2012) 45-66.
- [10] W. Erb, Accelerated Landweber methods based on co-dilated orthogonal polynomials, Numer. Algorithms 68 (2015) 229-260.

- [11] Ya. L. Geronimus, *Orthogonal polynomials: Estimates, asymptotic formulas, and series of polynomials orthogonal on the unit circle and on an interval*, Consultants Bureau, New York, 1961.
- [12] Ya. L. Geronimus, *Polynomials orthogonal on a circle and their applications*, Amer. Math. Soc. Translation, Series 1, Vol. 3 (1962) 1-78. Translation of the Russian original 1948.
- [13] M. E. H. Ismail, *Classical and quantum orthogonal polynomials in one variable*, Vol. 98, *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, 2005.
- [14] J. Letessier, *Some results on co-recursive associated Laguerre and Jacobi polynomials*, SIAM J. Math. Anal. 25 (1994) 528-548.
- [15] F. Marcellán, J.S. Dehesa, A. Ronveaux, *On orthogonal polynomials with perturbed recurrence relations*, J. Comput. Appl. Math. 30 (1990) 203-212.
- [16] Maroni, P, *Une Théorie Algébrique des Polynômes Orthogonaux: Applications aux Polynômes Orthogonaux Semi-classiques*, in: C. Brezinski et al. (eds.), *Orthogonal Polynomials and Their Applications*, IMACS Annals on Comput. and Appl. Math. 9, J.C. Baltzer AG, Basel (1991) 98–130.
- [17] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Amer. Math. Soc. Chelsea Publishing, Providence, R. I., 2000.
- [18] F. Peherstorfer, *Finite perturbations of orthogonal polynomials*, J. Comput. Appl. Math. 44 (1992) 275-302.
- [19] G. Sansigre and G. Valent, *A large family of semi-classical polynomials: The perturbed Chebyshev*, J. Comput. Appl. Math. 57 (1995) 271-281.
- [20] J. Shohat, *Théorie Générale des Polynômes Orthogonaux de Tchebichef*, Vol. 61, *Mémorial des Sciences Mathématiques*, Paris, 1934.
- [21] B. Simon, *Orthogonal polynomials on the unit circle*, 2 vols., Amer. Math. Soc. Coll. Publ. Series, Vol. 54, Amer. Math. Soc., Providence, R. I., 2005.
- [22] B. Simon, *Szegő's theorem and its descendants: Spectral theory for L^2 perturbations of orthogonal polynomials*, Princeton University Press, Princeton and Oxford, 2011.
- [23] H. A. Slim, *On co-recursive orthogonal polynomials and their application to potential scattering*, J. Math. Anal. Appl. 136 (1988) 1-19.
- [24] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse 8 (1894) J1-122.
- [25] T. J. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Toulouse 9 (1895) A1-47.

- [26] G. Szegő, Orthogonal polynomials, Amer. Math. Soc. Coll. Publ., Vol. 23, Amer. Math. Soc., Providence, R. I., Fourth Edition 1975.
- [27] G. Szegő, Über die Entwicklung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems, Math. Ann. 82 (1921) 188-212.
- [28] W. Van Assche, Orthogonal polynomials, associated polynomials and functions of the second kind, J. Comput. Appl. Math. 37 (1991) 237-249.
- [29] A. Zhedanov, Rational spectral transformations and orthogonal polynomials, J. Comput. Appl. Math. 85 (1997) 67-86.