A matrix characterization for the D_ν-semiclassical and D_ν-coherent orthogonal polynomials

Lino G. Garza[a], Luis E. Garza[b], Francisco Marcellán[a,c], Natalia C. Pinzón-Cortés[d]

[a]*Departamento de Matemáticas, Universidad Carlos III de Madrid, Avda. Universidad 30, 28911 Leganés, Spain.*
[b]*Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, 28045 Colima, México.*
[c]*Instituto de Ciencias Matemáticas (ICMAT), C/Nicolás Cabrera, No. 13-15, Campus de Cantoblanco UAM, Spain.*
[d]*Departamento de Matemáticas, Facultad de Ciencias, Universidad Nacional de Colombia, Of. 404-310, Bogotá, Colombia.*

Abstract

We present a new structure relation for the sequence of orthogonal polynomials associated with a D_ν-semiclassical linear functional of class s, and then we use it to obtain a matrix characterization of the D_ν-semiclassical orthogonal polynomials in terms of the Jacobi matrix associated with the multiplication operator in the basis of orthonormal polynomials, and the nonsingular lower triangular matrix that represents the orthogonal polynomials with respect to some bases of polynomials. We also provide a matrix characterization of D_ν-coherent pairs of linear functionals.

Keywords: Semiclassical discrete orthogonal polynomials, matrix representation, structure relation, D_ν-coherent pairs.
2000 MSC: 42C05, 15A23.

---

1. Preliminaries

Every sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ with $\deg(P_n(x)) = n$ is a basis of $\mathbb{C}[x]$, the linear space of polynomials with complex coefficients. Then, there exists a unique sequence of linear functionals $\{\mathfrak{p}_n\}_{n\geq 0}$, called

Email addresses: lggarza@math.uc3m.es (Lino G. Garza), garzaleg@gmail.com (Luis E. Garza), pacomarc@ing.uc3m.es (Francisco Marcellán), ncpinzonco@unal.edu.co (Natalia C. Pinzón-Cortés)

the dual basis of $\{P_n(x)\}_{n\geq 0}$, such that $\langle \mathfrak{p}_n, P_m(x)\rangle = \delta_{n,m}$, $n, m \geq 0$, where $\delta_{n,m}$ denotes the Kronecker delta. As a consequence, a linear functional $\mathcal{U} : \mathbb{C}[x] \to \mathbb{C}$ can be expressed as $\mathcal{U} = \sum_{n\geq 0} \langle \mathcal{U}, P_n(x)\rangle\, \mathfrak{p}_n$.

Lemma 1.1. *Let* $\{P_n(x)\}_{n\geq 0}$ *and* $\{R_n(x)\}_{n\geq 0}$ *be sequences of monic polynomials, and let* $\{\mathfrak{p}_n\}_{n\geq 0}$ *and* $\{\mathfrak{r}_n\}_{n\geq 0}$ *be their respective dual sequences. If*

$$R_n(x) = P_n(x),\ n \leq i+k, \quad R_n(x) = \sum_{j=n-\iota}^{n} \gamma_{n,j} P_j(x),\ n \geq i+k+1,\ \gamma_{n,n} \neq 0,$$

then

$$\mathfrak{p}_m = \begin{cases} \mathfrak{r}_m & \text{if } 0 \leq m \leq k, \\ \mathfrak{r}_m + \sum_{n=i+k+1}^{m+i} \gamma_{n,m}\mathfrak{r}_n & \text{if } k+1 \leq m \leq i+k, \\ \sum_{n=m}^{m+i} \gamma_{n,m}\mathfrak{r}_n & \text{if } m \geq i+k+1. \end{cases}$$

Proof. For $m \geq 0$, we have

$$\mathfrak{p}_m = \sum_{n\geq 0} \langle \mathfrak{p}_m, R_n(x)\rangle\, \mathfrak{r}_n = \sum_{n=0}^{i+k} \delta_{m,n}\mathfrak{r}_n + \sum_{n\geq i+k+1} \sum_{j=n-i}^{n} \gamma_{n,j}\delta_{m,j}\mathfrak{r}_n,$$

which establishes the result. □

We can associate with a linear functional $\mathcal{U}$ a sequence of complex numbers $\{u_n\}_{n\geq 0}$, where $u_n = \langle \mathcal{U}, x^n\rangle$, $n \geq 0$, which is called the *sequence of moments* of $\mathcal{U}$. In this context, $\mathcal{U}$ is said to be *quasi-definite or regular* if $\det\left([u_{i+j}]_{i,j=0}^{n}\right) \neq 0$, for $n \geq 0$. This condition is equivalent to the existence of a sequence of monic polynomials $\{P_n(x)\}_{n\geq 0}$ such that

$$\deg(P_n(x)) = n,\ n \geq 0, \quad \text{and} \quad \langle \mathcal{U}, P_n(x)P_m(x)\rangle = k_n\, \delta_{n,m},\ k_n \neq 0,\ n, m \geq 0.$$

$\{P_n(x)\}_{n\geq 0}$ is said to be a *sequence of monic orthogonal polynomials (SMOP)* with respect to the linear functional $\mathcal{U}$. The linear functional $p(x)\mathcal{U}$, where $p(x)$ is a polynomial with complex coefficients, is defined as $\langle p(x)\mathcal{U}, q(x)\rangle = \langle \mathcal{U}, p(x)q(x)\rangle$ for all $q \in \mathbb{C}[x]$. In particular, $P_n(x)\mathcal{U} = \langle \mathcal{U}, P_n^2(x)\rangle\, \mathfrak{p}_n$, for $n \geq 0$.

For each $\ell \in \mathbb{C}$, let $\mathcal{B}_\ell$ be the following basis of $\mathbb{C}[x]$

$$\mathcal{B}_\ell = \left\{ u_{\ell,k}(x) \mid u_{\ell,0} = 1, u_{\ell,k}(x) = x(x-\ell)\cdots\big(x-(k-1)\ell\big), k = 1, 2, \ldots \right\}.$$

Notice that $\mathcal{B}_0$ is the canonical basis $\{u_{0,k}(x) = x^k\}_{k\geq 0}$. In this way, if $X_{\mathcal{B}}$ denotes the matrix representation of the multiplication by x on $\mathbb{C}[x]$ with respect to some basis $\mathcal{B} \subset \mathbb{C}[x]$, then it follows that

$$x \begin{bmatrix} u_{\ell,0}(x) \\ u_{\ell,1}(x) \\ \vdots \end{bmatrix} = X_\ell \begin{bmatrix} u_{\ell,0}(x) \\ u_{\ell,1}(x) \\ \vdots \end{bmatrix}, \quad \text{where} \quad X_\ell = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & \ell & 1 & \cdots \\ 0 & 0 & 2\ell & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$\ell \in \mathbb{C}$, since $xu_{\ell,k}(x) = k\ell u_{\ell,k}(x) + u_{\ell,k+1}(x)$. Besides, $x^k\left[u_{\ell,0}(x), \cdots\right]^T = X_\ell^k\left[u_{\ell,0}(x), \cdots\right]^T$, $k \geq 0$.

Given a monic polynomial sequence $\{P_n(x)\}_{n\geq 0}$ and a basis $\mathcal{B} \subset \mathbb{C}[x]$, we can associate with $\{P_n(x)\}_{n\geq 0}$ a semi-infinite nonsingular lower triangular matrix $A_{\mathcal{B}}$ whose n-th row contains the coefficients of the n-th degree polynomial $P_n(x)$ with respect to the basis $\mathcal{B}$. For example, if

$$P_n(x) = \sum_{j=0}^{n} a_{n,j} u_{\ell,j}, \quad n \geq 0, \quad \ell \in \mathbb{C} \text{ fixed},$$

then the entries of the matrix $A_{\mathcal{B}_\ell}$ are $a_{n,j}$, for $0 \leqslant j \leqslant n, n \geqslant 0$, and zero otherwise. Since P_n is monic, the diagonal entries are $a_{n,n} = 1$ and, as a consequence, $A_{\mathcal{B}_\ell}$ is nonsingular.

Following the notation used in [26], a matrix B is said to be a lower semi-matrix if there exists an integer number m such that $b_{i,j} = 0$ if $i - j < m$. The entry $b_{i,j}$ belongs to the $m-$th diagonal if $i - j = m$. If B is non zero, we say that B has index m if m is the minimum integer number such that B has at least one nonzero entry in the $m-$th diagonal. Furthermore, B is said to be $(n, m)-$banded if there exists a pair of integers numbers (n, m) with $n \leq m$ and all the nonzero entries of B lie between the diagonals of indices n and m. Finally, B is called monic if all the entries in its diagonal of index m are equal to 1. Notice that the set of banded matrices is closed under addition and multiplication, and that the inverse of a banded matrix might not be banded.

The following result characterizes the orthogonality of a sequence of monic polynomials with respect to some linear functional $\mathcal{U}$ in terms of its corresponding matrix $A_{\mathcal{B}}$.

Theorem 1.2 ([26, 27]). *Let $\{P_n(x)\}_{n\geq 0}$ be a monic polynomial sequence and let $A_{\mathcal{B}}$ be its associated matrix with respect to some basis $\mathcal{B} \subset \mathbb{C}[x]$ of monic polynomials. Then, $\{P_n(x)\}_{n\geq 0}$ is orthogonal with respect to some linear functional if and only if $A_{\mathcal{B}} X_{\mathcal{B}} A_{\mathcal{B}}^{-1}$ is a (monic) element of $\mathcal{T}$, the set of $(-1, 1)$-banded matrices whose entries in the diagonals of indices 1 and -1 are all nonzero.*

Let us consider the difference operator D_ω, and the q-derivative operator D_q, defined, respectively, by

$$(D_\omega p)(x) = \frac{p(x+\omega)-p(x)}{\omega}, \quad \text{for } \omega \in \mathbb{C}\setminus\{0\},$$
$$(D_q p)(x) = \frac{p(qx)-p(x)}{(q-1)x}, \text{ for } x \neq 0, \ (D_q p)(0) = p'(0),$$

for $q \in \mathbb{C}\setminus\{0\}$ and $q^n \neq 1, n \in \mathbb{Z}^+$, for $p \in \mathbb{C}[x]$. From now on, ν and ν^* will denote either ω and $-\omega$, or, q and q^{-1}, respectively. Other notation that we will also use is the following

$$\hbar_\nu = \begin{cases} 1, & \text{if } \nu=\omega, \\ \nu, & \text{if } \nu = q, \end{cases} \qquad x \star \nu = \begin{cases} x+\nu, & \text{if } \nu = \omega, \\ \nu x, & \text{if } \nu = q. \end{cases}$$

$$\jmath_\nu = \begin{cases} \nu, & \text{if } \nu=\omega, \\ 0, & \text{if } \nu = q, \end{cases} \qquad \eta_{k-1,\nu} = \begin{cases} k, & \text{if } \nu = \omega, \\ [k]_q = \frac{q^k-1}{q-1}, & \text{if } \nu = q. \end{cases}$$

Some properties of such operators are listed in the following lemma. They can be shown using easy computations.

Lemma 1.3. *For $p, r \in \mathbb{C}[x]$, we have*

- ***(i)*** $D_\omega\big[p(x+a)\big] = \big(D_\omega p\big)(x+a)$, $a \in \mathbb{C}$, $D_q\big[p(bx)\big] = b\big(D_q p\big)(bx)$, $b \in \mathbb{C}\setminus\{0\}$.
- ***(ii)*** $\big(D_\nu\,[p\,r]\big)(x) = r(x)\big(D_\nu p\big)(x) + p(x\star\nu)\big(D_\nu r\big)(x)$.
- ***(iii)*** $\big(D_{\nu^*} p\big)(x \star \nu) = \big(D_\nu p\big)(x)$.
- ***(iv)*** $D_\nu D_{\nu^*} = \hbar_{\nu^*} D_{\nu^*} D_\nu$.

For a linear functional $\mathcal{U}$, the (distributional) D_ν-derivative of $\mathcal{U}$, $D_\nu\mathcal{U}$, is given by

$$\langle D_\nu \mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, D_{\nu^*} p(x)\rangle, \quad p \in \mathbb{C}[x].$$

Notice that the derivative operator D_ν yields the usual derivative operator when $q \to 1$ and $\omega \to 0$. Indeed, when $\omega \to 0$ and $q \to 1$, $(D_\nu p)(x)$ converges to $\frac{d}{dx}p(x)$ in $\mathbb{C}[x]$, and $D_\nu\mathcal{U}$ converges to $D\mathcal{U}$ in $(\mathbb{C}[x])^*$, respectively, where $D\mathcal{U}$ is defined by $\langle D\mathcal{U}, p(x)\rangle = -\langle \mathcal{U}, p'(x)\rangle$.

The structure of the manuscript is as follows. In Section 2, we obtain a new structure relation for D_ν-semiclassical polynomials, and then we use it to characterize the D_ν-semiclassical character of a linear functional in terms of banded matrices. A similar characterization for D_ν-coherent pairs is presented in Section 3.

2. A matrix characterization for D_ν-semiclassical polynomials

A pair of non zero polynomials $\phi(x) = a_t x^t + \ldots + a_0$ and $\psi(x) = b_r x^r + \cdots + b_0$, such that $a_t b_r \neq 0$, $t \geq 0, r \geq 1$, is said to be an admissible pair if either $t-1 \neq r$ or $t-1 = r$ and $\eta_{n-1,\nu} a_{r+1} + b_r \neq 0, n \geq 0$. In this way, a quasi-definite linear functional $\mathcal{U}$ is called D_ν-semiclassical if there exists an admissible pair (ϕ, ψ) such that

$$D_\nu [\phi(x)\mathcal{U}] = \psi(x)\mathcal{U}, \quad \deg(\phi) \geq 0, \, \deg(\psi) \geq 1, \tag{2.1}$$

holds. The class of $\mathcal{U}$ is the nonnegative number $s = \min\big\{ \max\{\deg(\phi) - 2, \deg(\psi) - 1\} : (\phi, \psi) \text{ is an admissible pair satisfying (2.1)} \big\}$, and the corresponding SMOP is also called D_ν-semiclassical of class s.

The D_ν-semiclassical linear functionals were introduced by J. A. Sohat in [25]. In the last decades, they have been extensively studied by P. Maroni and his coworkers in [5, 13, 21, 23]. In [22], Maroni developed a complete study of these functionals, showing how they act on polynomials and giving some structure forms that will be vital in the present work. The D-semiclassical linear functionals of class one were classified by S. Belmehdi in [7] through a distributional study and by giving an integral representation for the canonical cases, except for the Bessel case. The study of the distributional equation (2.1) has revealed many families of D-semiclassical orthogonal polynomials of class greater than one, see for instance, [6, 10], where the authors study the case $s = 2$ for symmetric and positive linear functionals and [20] where F. Marcellán et al. obtain all the semiclassical linear functionals of class two and their integral representations. For a complete survey on this topic see [14].

The following result provides a criterion for determining the class of a D_ν-semiclassical linear functional.

Theorem 2.1 ([21, 24] when $\nu = \omega$, and [15] when $\nu = q$)**.** *Let $\mathcal{U}$ be a D_ν-semiclassical linear functional satisfying* (2.1)*. Then, the class of $\mathcal{U}$ is s if and only if*

$$\prod_{\substack{c\in\mathbb{C},\\ \phi(c)=0}} \left[\left| \hbar_{\nu^*}\psi(c\star\nu^*) - \big(D_{\nu^*}\phi\big)(c) \right| + \left| \big\langle \mathcal{U}, \hbar_{\nu^*}\big(\theta_{c\star\nu^*}\psi\big)(x) - \big(\theta_{c\star\nu^*}\circ\theta_c\phi\big)(x) \big\rangle \right| \right] > 0$$

holds, where $\theta_\alpha p(x) = \frac{p(x)-p(\alpha)}{x-\alpha}$, $\alpha \in \mathbb{C}, p \in \mathbb{P}$. If there exists $c \in \mathbb{C}$ such that $\phi(c) = 0$ and

$$\hbar_{\nu^*}\psi(c \star \nu^*) - \big(D_{\nu^*}\phi\big)(c) = \big\langle \mathcal{U}, \hbar_{\nu^*}\big(\theta_{c\star\nu^*}\psi\big)(x) - \big(\theta_{c\star\nu^*} \circ \theta_c\phi\big)(x) \big\rangle = 0,$$

then the D_ν-Pearson equation (2.1) *becomes*

$$D_\nu\big[\theta_c\phi(x)\mathcal{U}\big] = \Big[\hbar_{\nu^*}\big(\theta_{c\star\nu^*}\psi\big)(x) - \big(\theta_{c\star\nu^*}\circ\theta_c\phi\big)(x)\Big]\mathcal{U}.$$

A D_ν-semiclassical linear functional of class $s=0$ and its corresponding SMOP are called D_ν-classical. In this case, for polynomials satisfying (2.1) we get $\deg(\phi)\le 2$ and $\deg(\psi)=1$.

On the other hand, for $k\ge 0$, $\big(D_\nu u_{\jmath_\nu,k}\big)(x) = \eta_{k-1,\nu}u_{\jmath_\nu,k-1}(x)$. Hence, if $D_{\nu,\jmath_\nu}$ is the matrix representation of D_ν with respect to $\mathcal{B}_{\jmath_\nu}$, then

$$\begin{array}{l} X_{\jmath_\nu}D_{\nu,\jmath_\nu} - D_{\nu,\jmath_\nu}\Big(X_{\jmath_\nu}\star\nu I\Big) = I, \\ \widehat{D}_{\nu,\jmath_\nu}D_{\nu,\jmath_\nu} = I, \end{array} \qquad D_{\nu,\jmath_\nu}\widehat{D}_{\nu,\jmath_\nu} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix},$$

$$\text{with}\quad D_{\nu,\jmath_\nu} = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \eta_{0,\nu} & 0 & 0 & \cdots \\ 0 & \eta_{1,\nu} & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \widehat{D}_{\nu,\jmath_\nu} = \begin{bmatrix} 0 & 1/\eta_{0,\nu} & 0 & \cdots \\ 0 & 0 & 1/\eta_{1,\nu} & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

If $A_{\jmath_\nu}$ is the matrix whose entries of its k-th row, $k\ge 0$, are the coefficients of a polynomial $p_k(x)$ of degree k with respect to the basis $\mathcal{B}_{\jmath_\nu}$, then the entries of the k-th row of $A_{\jmath_\nu}D_{\nu,\jmath_\nu}$ are the coefficients of $D_\nu p_k(x)$ which is a polynomial of degree $k-1$. Therefore, if $p_k(x)$ is monic, then the k-th row, $k\ge 0$, of $A_{\nu,\jmath_\nu} = \widehat{D}_{\nu,\jmath_\nu}A_{\jmath_\nu}D_{\nu,\jmath_\nu}$ contains the coefficients of the monic polynomial $p_k^{[1,\nu]}(x) = \frac{D_\nu p_{k+1}(x)}{\eta_{k,\nu}}$ with respect to the basis $\mathcal{B}_{\jmath_\nu}$. The following result characterizes D_ν-classical SMOPs in terms of their corresponding semi-infinite lower triangular matrices.

Theorem 2.2. *Let $A_{\jmath_\nu}$ be the matrix associated with the monic polynomials sequence $\{P_n(x)\}_{n\ge0}$ with respect to $\mathcal{B}_{\jmath_\nu}$. Then $\{P_n(x)\}_{n\ge0}$ is D_ν-classical if and only if $A_{\jmath_\nu}A^{-1}_{\nu,\jmath_\nu}$ is a $(0,2)$-banded monic matrix.*

Proof. In [27], the author proved that

$$A_{\jmath_\nu}A^{-1}_{\nu,\jmath_\nu} = L_{\jmath_\nu}D_{\nu,\jmath_\nu} + D_{\nu,\jmath_\nu}\big[(-M_{\nu,\jmath_\nu})\star\nu I\big],$$

where $L_{\jmath_\nu} = A_{\jmath_\nu}X_{\jmath_\nu}A^{-1}_{\jmath_\nu}\in\mathcal{T}$ and $M_{\nu,\jmath_\nu} = A_{\nu,\jmath_\nu}X_{\jmath_\nu}A^{-1}_{\nu,\jmath_\nu}$. Thus, $M_{\nu,\jmath_\nu}$ is a monic element of $\mathcal{T}$ if and only if $A_{\jmath_\nu}A^{-1}_{\nu,\jmath_\nu}$ is a $(0,2)$-banded monic matrix. Therefore, the result follows from Theorem 1.2. □

On the other hand, D_ν-semiclassical linear functionals can be characterized, in terms of structure relations, as follows.

Theorem 2.3. *Let $\mathcal{U}$ be a quasi-definite linear functional and let $\{P_n(x)\}_{n\geq 0}$ be its corresponding SMOP. Then, the following statements are equivalent*

- *There exist non zero polynomials ϕ, ψ, of degrees $t \geq 0, l \geq 1$, respectively, such that* (2.1) *holds.*
- *[22] (First structure relation) There exist a polynomial ϕ of degree t and sequences $\{a_{n,k}\}_{n\geq s}$ such that $\{P_n(x)\}_{n\geq 0}$ satisfies*

$$\phi(x)P_n^{[1,\nu^*]}(x) = \sum_{k=n-s}^{n+t} a_{n,k}P_k(x), \quad n \geq s, \quad a_{n,n-s} \neq 0, n \geq s+1, \tag{2.2}$$

  where s is a positive integer number with $t \leq s+2$.

- *[8] (Second structure relation) There exist non-negative integer numbers t, s, and sequences $\{\tilde{a}_{n,k}\}$, $\{\tilde{b}_{n,k}\}$, such that*

$$\sum_{k=n-s}^{n+s} \tilde{a}_{n,k}P_k(x) = \sum_{k=n-t}^{n+s} \tilde{b}_{n,k}P_k^{[1,\nu^*]}(x), \quad n \geq \max\{s, t+1\},$$

  holds, where $\tilde{a}_{n,n+s} = \tilde{b}_{n,n+s} = 1, n \geq \max\{s, t+1\}$.

In the next theorem, we provide another structure relation that characterizes D_ν-semiclassical linear functionals. It will be used later to express the D_ν-semiclassical character in terms of semi-infinite banded matrices.

Theorem 2.4. *For a nonzero monic polynomial $\phi(x)$ of degree t let $\mathcal{U}$ and $\{P_n(x)\}_{n\geq 0}$ be a linear functional and its corresponding SMOP respectively. The following statements are equivalent*

(i) *$\mathcal{U}$ is a D_ν-semiclassical linear functional of class s satisfying* (2.1), *i.e., there exists a polynomial $\psi(x)$ of degree $r \geq 1$ such that (ϕ, ψ) is an admissible pair and $D_\nu[\phi(x)\mathcal{U}] = \psi(x)\mathcal{U}$ holds.*

(ii) *There exist a non-negative integer s, an integer $r \geq 1$, and sequences $\{b_{n,j}\}_{n\geq s+1}$ and $\{c_{n,j}\}_{n\geq s+1}$ such that $s = \max\{t-2, r-1\}$ and $\{P_n(x)\}_{n\geq 0}$ satisfies the structure relation*

$$\sum_{j=0}^{s} b_{n,n-j}P_{n-j}(x) = \sum_{j=0}^{s+2} c_{n,n-j}P_{n-j}^{[1,\nu^*]}(x), \quad b_{n,n} = c_{n,n} = 1, \quad n \geq s+1, \tag{2.3}$$

where $\langle D_\nu [\phi(x)\mathcal{U}], P_n(x)\rangle = 0$, $r+1 \le n \le 2s+3$, $\langle D_\nu [\phi(x)\mathcal{U}], P_r(x)\rangle \neq 0$, *and* $\lim\limits_{\nu\to\delta_{\nu,q}} \frac{\langle D_\nu[\phi(x)\mathcal{U}],P_r(x)\rangle}{\langle \mathcal{U},P_r^2(x)\rangle} \neq -n$, *for* $n \ge 0$ *and* $r = t-1$.

Proof. **(i)** $\Rightarrow$ **(ii)**: Let consider the Fourier expansion

$$P_{n+j}(x) = \sum_{k=0}^{n+j} a_{n+j,k} P_k^{[1,\nu^*]}(x), \quad a_{n+j,n+j} = 1,\ j = 0,\dots,s,\ n \ge 1.$$

Multiplying the above equation by $b_{n+s,n+j}$, with $b_{n+s,n+s} = 1$, and adding for $j = 0,\dots,s$, we obtain

$$\sum_{j=0}^{s} b_{n+s,n+j} P_{n+j}(x) = \sum_{k=0}^{n+s} \beta_{k,n+s} P_k^{[1,\nu^*]}(x), \quad n \ge 1, \tag{2.4}$$

where

$$\beta_{k,n+s} = a_{n+s,k} + \sum_{j=\max\{0,k-n\}}^{s-1} b_{n+s,n+j} a_{n+j,k}, \quad k = 0,\dots,n+s,\ n \ge 1.$$

If we apply $\langle P_m(x \star \nu^*)\phi(x)\mathcal{U},\,\cdot\rangle$ to the above equation, for $m+(s+2) \le n-1$, then we get

$$\begin{aligned}
&\sum_{k=0}^{n+s} \beta_{k,n+s} \left\langle \phi(x)\mathcal{U}, \frac{1}{\eta_{k,\nu^*}}\Big(D_{\nu^*}[P_{k+1}(x)P_m(x)] - P_{k+1}(x) D_{\nu^*} P_m(x)\Big)\right\rangle \\
&= -\sum_{k=0}^{n+s} \frac{\beta_{k,n+s}}{\eta_{k,\nu^*}} \Big\langle \mathcal{U}, \Big[\psi(x)P_m(x) + \phi(x) D_{\nu^*} P_m(x)\Big] P_{k+1}(x)\Big\rangle \\
&= -\sum_{k=0}^{m+s} \alpha_{m,k}\beta_{k,n+s}, \qquad m = 0,1,\dots,n-s-3,\ n \ge s+3,
\end{aligned} \tag{2.5}$$

where, for $k = 0,\dots,m+s$, $m = 0,1,\dots,n-s-3$, and $n \ge s+3$,

$$\alpha_{m,k} = \frac{\big\langle \mathcal{U}, \big[\psi(x)P_m(x) + \phi(x)D_{\nu^*}P_m(x)\big] P_{k+1}(x)\big\rangle}{\eta_{k,\nu^*}}.$$

In matrix form, (2.5) reads

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_{0,0} & \cdots & \alpha_{0,s} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{n-s-3,0} & \cdots & \alpha_{n-s-3,s} & \alpha_{n-s-3,s+1} & \cdots & \alpha_{n-s-3,n-3} \end{bmatrix} \begin{bmatrix} \beta_{0,n+s} \\ \beta_{1,n+s} \\ \vdots \\ \beta_{n-3,n+s} \end{bmatrix},$$

or, equivalently,

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_{0,0} & \cdots & \alpha_{0,s-1} \\ \vdots & \vdots & \vdots \\ \alpha_{n-s-3,0} & \cdots & \alpha_{n-s-3,s-1} \end{bmatrix}_{(n-s-2)\times s} \begin{bmatrix} \beta_{0,n+s} \\ \vdots \\ \beta_{s-1,n+s} \end{bmatrix}$$

$$+ \begin{bmatrix} \alpha_{0,s} & 0 & \cdots & 0 \\ \alpha_{1,s} & \alpha_{1,s+1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n-s-3,s} & \cdots & \cdots & \alpha_{n-s-3,n-3} \end{bmatrix}_{(n-s-2)\times(n-s-2)} \begin{bmatrix} \beta_{s,n+s} \\ \vdots \\ \beta_{n-3,n+s} \end{bmatrix}.$$

In this way, since $\alpha_{m,m+s} \neq 0$, for $m = 0, \ldots, n-s-3$ (by the admissibility condition of a SMOP), then $\beta_{k,n+s} = 0$, $k = 0, \ldots, s-1$, implies that $\beta_{k,n+s} = 0$, for $k = s, \ldots, n-3$. Thus (2.4) becomes

$$\sum_{j=0}^{s} b_{n+s,n+j} P_{n+j}(x) = \sum_{k=n-2}^{n+s} \beta_{k,n+s} P_k^{[1,\nu^*]}(x), \quad n \geq 1,$$

which is equivalent to (2.3), completing the proof. Now, let us prove that $\beta_{k,n+s} = a_{n+s,k} + \sum_{j=0}^{s-1} b_{n+s,n+j} a_{n+j,k} = 0$, for $k = 0, \ldots, s-1$, i.e., let us show that the system

$$\Gamma_{n,s} := -\begin{bmatrix} a_{n+s,0} \\ a_{n+s,1} \\ \vdots \\ a_{n+s,s-1} \end{bmatrix}$$

$$= \begin{bmatrix} a_{n,0} & a_{n+1,0} & \cdots & a_{n+s-1,0} \\ a_{n,1} & a_{n+1,1} & \cdots & a_{n+s-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,s-1} & a_{n+1,s-1} & \cdots & a_{n+s-1,s-1} \end{bmatrix}_{s\times s} \begin{bmatrix} b_{n+s,n} \\ b_{n+s,n+1} \\ \vdots \\ b_{n+s,n+s-1} \end{bmatrix} =: \Lambda_{n,s} \Upsilon_{n,s}$$

has a solution. If $\Lambda_{n,s}$ is a nonsingular matrix, the desired conclusion follows. On the other hand, if $\det(\Lambda_{n,s}) = 0$, the system $\Lambda_{n,s}\Upsilon_{n,s} = \Gamma_{n,s}$ has a solution if and only if the matrices $\Lambda_{n,s}$ and $[\Lambda_{n,s} \mid \Gamma_{n,s}]_{s\times(s+1)}$ have the same number of linearly independent rows.

Let us assume that the jth and the kth rows of $\Lambda_{n,s}$ are linearly dependent, then there exists $\lambda_n \in \mathbb{C}$ such that

$$\left(a_{n,j}, \quad a_{n+1,j}, \quad \cdots, \quad a_{n+s-1,j}\right) = \lambda_n \left(a_{n,k}, \quad a_{n+1,k}, \quad \cdots, \quad a_{n+s-1,k}\right).$$

Since n is arbitrary, then the algorithm described above holds for $n+1$, and thus we get

$$\left(a_{n+1,j}, \quad a_{n+2,j}, \quad \cdots, \quad a_{n+s,j}\right) = \lambda_{n+1}\left(a_{n+1,k}, \quad a_{n+2,k}, \quad \cdots, \quad a_{n+s,k}\right).$$

As a consequence, the jth and the kth rows of $[\Lambda_{n,s} \mid \Gamma_{n,s}]$ are also linearly dependent. Therefore, $\Lambda_{n,s}$ has the same number of independent rows as $[\Lambda_{n,s} \mid \Gamma_{n,s}]$, and then, $\Lambda_{n,s}\Upsilon_{n,s} = \Gamma_{n,s}$ has a solution.

(ii) $\Rightarrow$ **(i)**: Let us consider the sequence of monic polynomials $\{R_n(x)\}_{n\geq 0}$ defined by

$$R_n(x) = P_n(x),\ n \leq 2s{+}3, \quad R_{n+1}(x) = \sum_{j=0}^{s+2} \frac{\eta_{n,\nu^*} c_{n,n-j}}{\eta_{n-j,\nu^*}} P_{n-j+1}(x),\ n \geq 2s{+}3.$$

Then, from the definition of $R_{n+1}(x)$, $n \geq 2s+3$, and from (2.3) it follows that

$$\langle D_\nu\left[\phi(x)\mathcal{U}\right], R_{n+1}(x)\rangle = -\eta_{n,\nu^*} \sum_{j=0}^{s} b_{n,n-j} \langle \phi(x)\mathcal{U}, P_{n-j}(x)\rangle = 0,$$

for $n \geq \max\{s+t+1, 2s+3\} = 2s+3$. As a consequence, taking into account the assumption, we get

$$D_\nu\left[\phi(x)\mathcal{U}\right] = \sum_{n\geq 0} \langle D_\nu\left[\phi(x)\mathcal{U}\right], R_n(x)\rangle\, \mathfrak{r}_n = \sum_{n=0}^{r} \langle D_\nu\left[\phi(x)\mathcal{U}\right], P_n(x)\rangle\, \mathfrak{r}_n,$$

where $\{\mathfrak{r}_n\}_{n\geq 0}$ is the dual sequence of $\{R_n(x)\}_{n\geq 0}$.

In this way, using Lemma 1.1 with $\iota = s+2$, $\kappa = s+1$, and $\gamma_{n,j} = \dfrac{\eta_{n-1,\nu^*} c_{n-1,j-1}}{\eta_{j-1,\nu^*}}$, we obtain that $\mathfrak{r}_n = \mathfrak{p}_n$, for $n \leq s+1$. Hence, since $r \leq s+1$ and $P_n(x)\mathcal{U} = \langle \mathcal{U}, P_n^2(x)\rangle\, \mathfrak{p}_n$, it follows that

$$D_\nu\left[\phi(x)\mathcal{U}\right] = \psi(x)\mathcal{U}, \quad \text{where} \quad \psi(x) = \sum_{n=0}^{r} \frac{\langle D_\nu\left[\phi(x)\mathcal{U}\right], P_n(x)\rangle}{\langle \mathcal{U}, P_n^2(x)\rangle} P_n(x).$$

□

Notice that (2.3) can be expressed in matrix form as

$$BA_{\jmath_{\nu^*}} = CA_{\nu^*,\jmath_{\nu^*}}, \tag{2.6}$$

where B and C are monic $(0,s)-$banded and $(0,s+2)-$banded matrices, respectively, whose entries are the coefficients $b_{n,n-j}$ and $c_{n,n-j}$. As a consequence, the previous result means that $\{P_n(x)\}_{n\geqslant 0}$ is a D_ν-semiclassical SMOP of class s if and only if there exist matrices B and C such that (2.6) holds. Furthermore, if $\{P_n(x)\}_{n\geqslant 0}$ is D_ν-semiclassical of class s, it follows from (2.6) that $BA_{\jmath_{\nu^*}}A^{-1}_{\nu^*,\jmath_{\nu^*}}$ is a $(0,s+2)-$banded monic matrix. Conversely, if there exists a $(0,s)-$banded monic matrix B such that $BA_{\jmath_{\nu^*}}A^{-1}_{\nu^*,\jmath_{\nu^*}}$ is a $(0,s+2)-$banded monic matrix, then (2.6) holds and, therefore, $\{P_n(x)\}_{n\geqslant 0}$ is D_ν-semiclassical of class s. As a consequence, we have the following straightforward generalization of Proposition 2.4 for D_ν-semiclassical polynomials.

Theorem 2.5. *Let $\{P_n(x)\}_{n\geqslant 0}$ be a MOPS with respect to a linear functional $\mathcal{U}$. Then, $\mathcal{U}$ is D_ν-semiclassical of class s if and only if there exists a semi-infinite $(0,s)-$banded monic matrix B such that $BA_{\jmath_{\nu^*}}A^{-1}_{\nu^*,\jmath_{\nu^*}}$ is a $(0,s+2)-$banded monic matrix.*

Remark 2.6. When $s=0$, $\{P_n(x)\}_{n\geqslant 0}$ is D_ν-classical if and only if

$$BA_{\jmath_{\nu^*}}A^{-1}_{\nu^*,\jmath_{\nu^*}} = A_{\jmath_{\nu^*}}A^{-1}_{\nu^*,\jmath_{\nu^*}}$$

is a $(0,2)-$banded monic matrix, (which is the result stated in Theorem 2.2, since a linear functional is D_ν-semiclassical of class s if and only if it is D_{ν^*}-semiclassical of class s). In other words, $\{P_n(x)\}_{n\geqslant 0}$ satisfies (see [2])

$$P_n(x) = P_n^{[1,\nu^*]}(x) + c_{n,n-1}P_{n-1}^{[1,\nu^*]}(x) + c_{n,n-2}P_{n-2}^{[1,\nu^*]}(x), \quad n \geq 1.$$

Now, let us consider a positive definite D_ν-semiclassical linear functional $\mathcal{U}$ that satisfies the D_ν-Pearson equation (2.1), and let $\{p_n(x)\}_{n\geqslant 0}$ be its corresponding sequence of orthonormal polynomials. From the three-term recurrence relation, we get

$$x\mathbf{p}(x) = \tilde{J}\mathbf{p}(x), \tag{2.7}$$

where

$$\tilde{J} = \begin{pmatrix} b_0 & a_1 & 0 & \dots \\ a_1 & b_1 & a_2 & \ddots \\ 0 & a_2 & b_2 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \quad \mathbf{p}(x) = \begin{pmatrix} p_0(x), & p_1(x), & p_2(x), & \cdots \end{pmatrix}^T.$$

On the other hand, the matrix expression for the (normalized) first structure relation for D_ν-semiclassical polynomials given in (2.2) reads

$$\phi(x)D_{\nu^*}\mathbf{p}(x) = \widehat{H}\mathbf{p}(x), \quad \text{with} \quad \widehat{H} = X_0^T\tilde{H}, \tag{2.8}$$

where $\tilde{H}$ is a $(-t, s)-$banded matrix whose entries are the coefficients appearing in the right side of (2.2), and $D_\nu \mathbf{p}(x) = [D_\nu p_0(x), D_\nu p_1(x), \ldots]^T$. In this way, the relation between $\tilde{H}$ and $\tilde{J}$ is described in the following result.

Theorem 2.7. *Let $\{p_n(x)\}_{n\geqslant 0}$ be a D_ν-semiclassical sequence of orthonormal polynomials and let $\widehat{H}$ be the $(-t+1, s+1)-$banded matrix associated with the first structure relation* (2.8)*. Then*

(i) $[\tilde{J}, \widehat{H}]_{\nu^*} = \phi(\tilde{J})$,

(ii) $\widehat{H} + \widehat{H}^T = -\psi(\tilde{J})$,

where $[\tilde{J}, \widehat{H}]_{\nu^*} = \tilde{J}\widehat{H} - \widehat{H}(\tilde{J} \star \nu^* I)$, and ϕ, ψ are the polynomials appearing in the D_ν-Pearson equation.

Proof. (i): Taking the D_{ν^*}-derivative in (2.7), then multiplying by $\phi(x)$ and using (2.8) we get

$$(x \star \nu^*)\widehat{H}\mathbf{p}(x) + \phi(x)\mathbf{p}(x) = \tilde{J}\widehat{H}\mathbf{p}(x).$$

Thus, since $(x \star \nu^*)\mathbf{p}(x) = (\tilde{J} \star \nu^* I)\mathbf{p}(x)$ and $x^k\mathbf{p}(x) = \tilde{J}^k\mathbf{p}(x)$, $k \geq 0$, above equation becomes

$$\widehat{H}(\tilde{J} \star \nu^* I)\mathbf{p}(x) + \phi(\tilde{J})\mathbf{p}(x) = \tilde{J}\widehat{H}\mathbf{p}(x),$$

and therefore, (i) holds. (ii): From (2.7), we have

$$\begin{aligned}\Big\langle D_\nu\left[\phi(x)\mathcal{U}\right], \mathbf{p}(x)p_m(x)\Big\rangle &= \Big[\big\langle D_\nu\left[\phi(x)\mathcal{U}\right], p_0(x)p_m(x)\big\rangle, \big\langle D_\nu\left[\phi(x)\mathcal{U}\right], p_1(x)p_m(x)\big\rangle, \ldots\Big]^T \\ &= \Big\langle \psi(x)\mathcal{U}, \mathbf{p}(x)p_m(x)\Big\rangle = \Big\langle \mathcal{U}, \psi(\tilde{J})\mathbf{p}(x)p_m(x)\Big\rangle, \quad m \geq 0.\end{aligned}$$

Notice that we get the m-th column of $\psi(\tilde{J})$. On the other hand, using (2.8), we obtain

$$\begin{aligned}\langle D_\nu\left[\phi(x)\mathcal{U}\right], \mathbf{p}(x)p_m(x)\rangle &= -\big\langle \phi(x)\mathcal{U}, D_{\nu^*}\mathbf{p}(x)p_m(x \star \nu^*) + \mathbf{p}(x)D_{\nu^*}p_m(x)\big\rangle \\ &= -\Big\langle \mathcal{U}, \widehat{H}\mathbf{p}(x)p_m(x \star \nu^*)\Big\rangle - \Big\langle \mathcal{U}, \mathbf{p}(x)\phi(x)D_{\nu^*}p_m(x)\Big\rangle \\ &= -\Big[m\text{-th column of } \widehat{H}\Big] - \Big[m\text{-th row of } \widehat{H}\Big]^T,\end{aligned}$$

which is the m-th column of $-[\widehat{H} + \widehat{H}^T]$. Therefore, (ii) follows. □

Remark 2.8. Notice that

- From (i) we get

$$
\begin{aligned}
0 = \phi(\tilde{J}) - \phi(\tilde{J})^T &= \tilde{J}\widehat{H} + (\tilde{J} \star \nu^* I)\widehat{H}^T - \widehat{H}(\tilde{J} \star \nu^* I) - \widehat{H}^T \tilde{J} \\
&= \tilde{J}\left(\widehat{H} + \hbar_{\nu^*}\widehat{H}^T\right) - \left(\widehat{H} + \hbar_{\nu^*}\widehat{H}^T\right)^T \tilde{J} - \jmath_{\nu^*}\left(\widehat{H} - \widehat{H}^T\right),
\end{aligned}
$$

  or equivalently,

$$
\tilde{J}\left(\widehat{H} + \hbar_{\nu^*}\widehat{H}^T\right) - \left(\widehat{H} + \hbar_{\nu^*}\widehat{H}^T\right)^T \tilde{J} = \jmath_{\nu^*}\left(\widehat{H} - \widehat{H}^T\right).
$$

  Therefore, when $\nu = \omega$, $\tilde{J}(\widehat{H} + \widehat{H}^T)/2$ is a symmetric matrix, where $(\widehat{H} + \widehat{H}^T)/2$ is the symmetric component of $\widehat{H}$. On the other hand,

$$
\begin{aligned}
2\phi(\tilde{J}) = \phi(\tilde{J}) + \phi(\tilde{J})^T &= \tilde{J}\widehat{H} - (\tilde{J} \star \nu^* I)\widehat{H}^T - \widehat{H}(\tilde{J} \star \nu^* I) + \widehat{H}^T \tilde{J} \\
&= \tilde{J}\left(\widehat{H} - \hbar_{\nu^*}\widehat{H}^T\right) - \left(\hbar_{\nu^*}\widehat{H} - \widehat{H}^T\right)\tilde{J} - \jmath_{\nu^*}\left(\widehat{H} + \widehat{H}^T\right),
\end{aligned}
$$

  i.e.,

$$
\tilde{J}\,\frac{\widehat{H} - \hbar_{\nu^*}\widehat{H}^T}{2} - \frac{\hbar_{\nu^*}\widehat{H} - \widehat{H}^T}{2}\,\tilde{J} = \phi(\tilde{J}) + \jmath_{\nu^*}\frac{\widehat{H} + \widehat{H}^T}{2}.
$$

  Hence, if $\nu = \omega$, the skew-symmetric component of $\widehat{H}$ satisfies $\tilde{J}\frac{\widehat{H}-\widehat{H}^T}{2} - \frac{\widehat{H}-\widehat{H}^T}{2}\tilde{J} = \phi(\tilde{J})$.

- From (ii), the symmetric component of $\widehat{H}$ satisfies

$$
\frac{\widehat{H} + \widehat{H}^T}{2} = -\frac{1}{2}\psi(\tilde{J}).
$$

Finally, we state the relation between the matrices $A_{\jmath_{\nu^*}}$ and H.

Proposition 2.9. *Let $A_{\jmath_{\nu^*}}$ be the lower triangular matrix associated with the D_ν-semiclassical MOPS $\{P_n(x)\}_{n\geqslant 0}$, with respect to the basis $\mathcal{B}_{\jmath_{\nu^*}}$. If H is the $(-t, s)$-banded matrix associated with the first structure relation* (2.2), *i.e., $\phi(x)D_{\nu^*}\mathbf{p}(x) = X_0^T H\mathbf{p}(x)$, we have*

$$
H = X_0 A_{\jmath_{\nu^*}} D_{\nu^*,\jmath_{\nu^*}} \phi(X_{\jmath_{\nu^*}}) A_{\jmath_{\nu^*}}^{-1}.
$$

Proof. If $\mathbf{Y}_{\jmath_{\nu^*}}(x) = \left[1, x, x(x - \jmath_{\nu^*}), x(x - \jmath_{\nu^*})(x - 2\jmath_{\nu^*}), \ldots\right]^T$, then $\mathbf{p}(x) = A_{\jmath_{\nu^*}}\mathbf{Y}_{\jmath_{\nu^*}}(x)$, and therefore, $D_{\nu^*}\mathbf{p}(x) = A_{\jmath_{\nu^*}} D_{\nu^*}\mathbf{Y}_{\jmath_{\nu^*}}(x) = A_{\jmath_{\nu^*}} D_{\nu^*,\jmath_{\nu^*}}\mathbf{Y}_{\jmath_{\nu^*}}(x)$. Multiplying by $\phi(x)$ and comparing with (2.2), we obtain

$$
X_0^T H A_{\jmath_{\nu^*}}\mathbf{Y}_{\jmath_{\nu^*}}(x) = A_{\jmath_{\nu^*}} D_{\nu^*,\jmath_{\nu^*}}\phi(x)\mathbf{Y}_{\jmath_{\nu^*}}(x).
$$

Moreover, if $\phi(x) = \sum_{k=0}^{\deg(\phi)} \gamma_k x^k$, then

$$\phi(x)\mathbf{Y}_{\jmath_{\nu^*}}(x) = \left(\sum_{k=0}^{\deg(\phi)} \gamma_k x^k\right) \begin{bmatrix} u_{\jmath_{\nu^*},0} \\ u_{\jmath_{\nu^*},1} \\ \vdots \end{bmatrix} = \sum_{k=0}^{\deg(\phi)} \gamma_k \left[x^k \mathbf{Y}_{\jmath_{\nu^*}}(x)\right]$$

$$= \sum_{k=0}^{\deg(\phi)} \gamma_k X_{\jmath_{\nu^*}}^k \mathbf{Y}_{\jmath_{\nu^*}}(x) = \phi(X_{\jmath_{\nu^*}})\mathbf{Y}_{\jmath_{\nu^*}}(x),$$

and, taking into account $X_0 X_0^T = I$, the result follows. □

3. A matrix characterization of D_ν-Coherent pairs

A pair of linear functionals $(\mathcal{U}, \mathcal{V})$ is said to be a (M, N)-D_ν-coherent pair of order (m, k) if their corresponding SMOP $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ satisfy

$$\sum_{i=0}^{M} a_{i,n} D_\nu^m P_{n+m-i}(x) = \sum_{i=0}^{N} b_{i,n} D_\nu^k Q_{n+k-i}(x), \quad n \geq 0,$$

where M, N, m, $k \in \mathbb{N} \cup \{0\}$, $a_{i,n}, b_{i,n} \in \mathbb{C}$, for $n \geq 0$, $a_{M,n} \neq 0$ for $n \geq M$, $b_{N,n} \neq 0$ for $n \geq N$ and $a_{i,n} = b_{i,n} = 0$ for $i > n$. When $k = 0$, $(\mathcal{U}, \mathcal{V})$ is called a (M, N)-D_ν-coherent pair of order m, and if $(m, k) = (1, 0)$, we say it (M, N)-D_ν-coherent pair.

I. Area, E. Godoy and F. Marcellán in [3] and [4] extended the concept of coherence from the theory of orthogonal polynomials in one continuous variable to orthogonal polynomials in one discrete variable. They proved that if $(\mathcal{U}, \mathcal{V})$ constitutes a $(1, 0)$-D_ν-coherent pair, then either $\mathcal{U}$ or $\mathcal{V}$ is D_ν-classical and the other one is a rational modification of the first.

Later on, in [16] (2004), K. H. Kwon, J. H. Lee and F. Marcellán studied ($M + 1$-term) generalized D_ω-coherent pairs, (for us $(M, 0)$-D_ω-coherent pairs). In particular, they analyzed $(2, 0)$-Δ-coherent pairs, i.e., two SMOP $\{P_n(x)\}_{n\geqslant 0}$ and $\{R_n(x)\}_{n\geqslant 0}$ satisfying a relation

$$R_n(x) = \frac{1}{n+1}\Delta P_{n+1}(x) - \frac{\sigma_n}{n}\Delta P_n(x) - \frac{\tau_{n-1}}{n-1}\Delta P_{n-1}(x), \quad n \geq 2,$$

where σ_n and τ_n are arbitrary constants and Δ is the D_ω operator with $\omega = 1$ (forward difference operator). In this way, they showed that the linear functionals must be Δ-semiclassical and they are related by an expression of rational type. They also studied the case when one of the linear functionals is Δ-classical.

More recently, in [18] and [19], F. Marcellán and N. C. Pinzón Cortés studied the (1,1)-D_ν-coherent pairs proving that (1,1)-D_ν-coherence implies that the linear functionals are D_ν-semiclassical, one of class at most 1 and the other of class at most 5. Moreover, the functionals are related by a rational function. In [17] they also give a matrix interpretation in terms of the Jacobi matrices associated with the corresponding sequences of orthogonal polynomials. Finally, in [1], R. Álvarez-Nodarse, J. Petronilho, N. C. Pinzón-Cortés, and R. Sevinik-Adıgüzel analyzed the more general case, (M,N)-D_ν-coherent pairs of order (m,k), concluding that the linear functionals are related by a rational factor and, when $m \neq k$, then both $\mathcal{U}$ and $\mathcal{V}$ are D_ν-semiclassical functionals.

All the above constitute an extension of the results obtained for the continuous case given in [9] and in its introduction.

3.1. $(M,0)$-D_ν-Coherent pairs

Let us consider the simplest case of coherence, when $N = k = 0$ and $M = m = 1$, i.e. $\mathcal{U}$ and $\mathcal{V}$ constitute a $(1,0)$-D_ν-coherent pair. In such a case, the corresponding SMOP satisfy the structure relation

$$P_n^{[1,\nu]}(x) + c_{1,n} P_{n-1}^{[1,\nu]}(x) = Q_n(x), \quad n \geq 0. \tag{3.1}$$

From [12], we can characterize the (1,0)-D_ν-coherence by using banded matrices as follows.

Theorem 3.1. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be SMOPs with associated lower triangular matrices $A_{\jmath_\nu}$ and $B_{\jmath_\nu}$, respectively, with respect to the basis $\mathcal{B}_{\jmath_\nu}$. Then $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(1,0)$-D_ν-coherent pair if and only if $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}$ is a lower bidiagonal matrix with ones in the main diagonal and nonzero entries in the subdiagonal.*

Proof. Let assume $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$ is a $(1,0)$-D_ν-coherent pair. Since $A_{\nu,\jmath_\nu}$ is the lower triangular matrix associated with $\{P_n^{[1,\nu]}(x)\}_{n\geq 0}$, then (3.1) can be written in matrix form as

$$A_{\nu,\jmath_\nu} + C_1 X_0^T A_{\nu,\jmath_\nu} = B_{\jmath_\nu},$$

where $C_1 = \mathrm{diag}(c_{1,0}, c_{1,1}, \dots)$. Since $A_{\nu,\jmath_\nu}$ is nonsingular, we have

$$I + C_1 X_0^T = B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}.$$

As a consequence, $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}$ is lower bidiagonal with ones in the main diagonal and non zero entries in the subdiagonal, since $c_{1,n} \neq 0$, $n \geq 1$. Conversely, if $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1} = T$ is bidiagonal with ones in the main diagonal and non zero entries in the subdiagonal, then

$$T A_{\nu,\jmath_\nu} = B_{\jmath_\nu},$$

so $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(1,0)$-D_ν-coherent pair of SMOPs. □

The case of $(1,0)$-D_ν-coherence can be generalized to $(M,0)$-D_ν-coherence when we consider a finite number M of terms in the left-hand side of the structure relation (3.1), i.e.

$$P_n^{[1,\nu]}(x) + \sum_{k=1}^{M} c_{k,n} P_{n-k}^{[1,\nu]}(x) = Q_n(x), \quad n \geq 0. \tag{3.2}$$

In such a case, we have the following result.

Theorem 3.2. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be SMOPs with associated lower triangular matrices $A_{\jmath_\nu}$ and $B_{\jmath_\nu}$, respectively, with respect to the basis $\mathcal{B}_{\jmath_\nu}$. Then $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(M,0)$-D_ν-coherent pair if and only if $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}$ is a $(0,M)$-banded matrix with ones on the main diagonal.*

Proof. Assume $(\{P_n(x)\}_{n\geq 0}, \{Q_n(x)\}_{n\geq 0})$ is a $(M,0)$-D_ν-coherent pair given by (3.2). If for $1 \leq k \leq M$, C_k is a diagonal matrix with entries $c_{k,n}$, $n \geq 0$, then (3.2) can be written in matrix form as

$$\left[I + \sum_{k=1}^{M} C_k (X_0^T)^k \right] A_{\nu,\jmath_\nu} = B_{\jmath_\nu},$$

and, since $A_{\nu,\jmath_\nu}$ is nonsingular,

$$I + \sum_{k=1}^{M} C_k (X_0^T)^k = B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}.$$

As a consequence, $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1}$ is a $(0,M)$-banded matrix. Conversely, if $B_{\jmath_\nu} A_{\nu,\jmath_\nu}^{-1} = T$ is a $(0,M)-$banded matrix with ones in the main diagonal, then we have

$$T A_{\nu,\jmath_\nu} = B_{\jmath_\nu},$$

so that $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ constitute a $(M,0)$-D_ν-coherent pair. □

3.2. $(M,0)$-D_ν-Coherent pairs of order m

For $m \geq 1$, let us define the monic polynomial of degree n, $P_n^{[m,\nu]}(x)$, associated with the m-th D_ν-derivative of the monic polynomial $P_{n+m}(x)$ as

$$P_n^{[m,\nu]}(x) = \frac{D_\nu^m P_{n+m}(x)}{\eta_{n,m,\nu}}, \quad \text{where} \quad \eta_{n,m,\nu} = \begin{cases} (n+1)_m, & \text{if } \nu = \omega, \\ \frac{(q^{n+1};q)_m}{(1-q)^m}, & \text{if } \nu = q, \end{cases} \quad n \geq 0.$$

Here $(a)_n$ and $(a;q)_n$ denote the Pochhammer symbol and the q-Pochhammer symbol, respectively, given by

$$(a)_0 = 1, \quad (a)_n = a(a+1)\cdots(a+n-1), \quad n \geq 1,$$
$$(a;q)_0 = 1, \quad (a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}), \ n \geq 1.$$

Notice that if $A_{\jmath_\nu}$ is the nonsingular lower triangular matrix associated with the SMOP $\{P_n(x)\}_{n\geqslant 0}$ with respect to the basis $\mathcal{B}_{\jmath_\nu}$, then the nonsingular lower triangular matrix $A_{\nu,\jmath_\nu}^{[m]}$ associated with $\{P_n^{[m,\nu]}(x)\}_{n\geqslant 0}$ is $A_{\nu,\jmath_\nu}^{[m]} = \widehat{D}_{\nu,\jmath_\nu}^m A_{\jmath_\nu} D_{\nu,\jmath_\nu}^m$, where

$$D_{\nu,\jmath_\nu}^m = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & \cdots \\ \eta_{n,m,\nu} & \vdots & 0 & \cdots \\ 0 & \eta_{n+1,m,\nu} & \vdots & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{bmatrix} \leftarrow m\text{th row},$$

$$\widehat{D}_{\nu,\jmath_\nu}^m = \begin{bmatrix} 0 & \cdots & \frac{1}{\eta_{n,m,\nu}} & 0 & \cdots \\ 0 & 0 & \cdots & \frac{1}{\eta_{n+1,m,\nu}} & \cdots \\ 0 & 0 & 0 & \cdots & \ddots \\ 0 & 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

(Notice that $\eta_{n,1,\nu} = \eta_{n,\nu}$ given in previous sections). Then, considering the structure relation of $(1,0)$-D_ν-coherence

$$P_n^{[m,\nu]}(x) + c_{1,n} P_{n-1}^{[m,\nu]}(x) = Q_n(x), \quad n \geq 0,$$

and arguing as above, we have the following matrix characterization.

Theorem 3.3. *Let $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ be SMOPs with associated matrices $A_{\jmath_\nu}$ and $B_{\jmath_\nu}$, respectively, with respect to $\mathcal{B}_{\jmath_\nu}$. Then $\{P_n(x)\}_{n\geq 0}$ and*

$\{Q_n(x)\}_{n\geq 0}$ *constitute a* $(1,0)$*-coherent pair if and only if* $B_{\jmath_\nu}\left(A^{[m]}_{\nu,\jmath_\nu}\right)^{-1}$ *is a lower bidiagonal matrix with ones on the main diagonal and nonzero entries in the subdiagonal.*

Finally, the following matrix characterization for the structure relation of $(1,0)$-D_ν-coherence of order m

$$P_n^{[m,\nu]}(x)+\sum_{k=1}^{M}c_{k,n}P_{n-k}^{[m,\nu]}(x)=Q_n(x),\quad n\geq 0,$$

can be easily obtained proceeding as in the proof of Theorem 3.2.

Theorem 3.4. *Let* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *be SMOPs with associated matrices* $A_{\jmath_\nu}$ *and* $B_{\jmath_\nu}$*, respectively, with respect to the basis* $\mathcal{B}_{\jmath_\nu}$*. Then* $\{P_n(x)\}_{n\geq 0}$ *and* $\{Q_n(x)\}_{n\geq 0}$ *constitute a* $(M,0)$*-*D_ν*-coherent pairs of order* m *if and only if* $B_{\jmath_\nu}\left(A^{[m]}_{\nu,\jmath_\nu}\right)^{-1}$ *is a* $(0,M)-$*banded matrix with ones on the main diagonal.*

Acknowledgements

The work of the first author was supported by a grant of the Secretaría de Educación Pública of México and the Mexican Government. The work of the second author was supported by Consejo Nacional de Ciencia y Tecnología of México, grant 156668. The work of the third author was supported by Dirección General de Investigación Científica y Técnica, Ministerio de Economía y Competitividad of Spain, grant MTM2012-36732-C03-01. The authors thank the anonymous referee for her/his valuable comments and suggestions. They contributed to improve the presentation of the manuscript.

References

References

[1] R. Álvarez-Nodarse, J. Petronilho, N. C. Pinzón-Cortés, R. Sevinik-Adıgüzel. *On linearly related sequences of difference derivatives of discrete orthogonal polynomials.* J. Comput. Appl. Math. **284** (2015), 26-37.

[2] I. Area. *Polinomios ortogonales de variable discreta: Pares coherentes. Problemas de conexión.* Doctoral Dissertation, Universidad de Vigo, 1999. In Spanish.

[3] I. Area, E. Godoy, F. Marcellán. *q-Coherent Pairs and q-Orthogonal Polynomials.* Appl. Math. Comput. **128** (2002), 191-216.

[4] I. Area, E. Godoy, F. Marcellán. Δ-*Coherent Pairs and Orthogonal Polynomials of a Discrete Variable*. Integral Transforms Spec. Funct. **14** (2003), 31-57.

[5] J. Alaya, P. Maroni. *Symmetric Laguerre-Hahn forms of class* $s = 1$. Integral Transforms Spec. Funct. **4** (1996), 301-320.

[6] J. Alaya, M. Sghaier. *Semiclassical Forms of Class* $s = 2$*: The Symmetric Case, when* $\Phi(0) = 0$, Methods Appl. Anal. **13** (2006), 387-410.

[7] S. Belmehdi. *On semi-classical linear functionals of class s = 1. Classification and integral representations.* Indag. Math. N. S. **3** (1992), 253-275.

[8] R. Costas-Santos, F. Marcellán. *Second structure relation for q-semiclassical polynomials of the Hahn Tableau.* J. Math. Anal. Appl. **329** (2007), 206-228.

[9] M. N. de Jesús, F. Marcellán, J. Petronilho, N. C. Pinzón-Cortés. *(M,N)-coherent pairs of order (m,k) and Sobolev orthogonal polynomials.* J. Comput. Appl. Math. **256** (2014), 16-35.

[10] A. M. Delgado. *Ortogonalidad No Estándar: Problemas Directos e Inversos.* Doctoral Dissertation, Universidad Carlos III de Madrid, 2006. In Spanish.

[11] A. M. Delgado, F. Marcellán. *On an Extension of Symmetric Coherent Pairs of Orthogonal Polynomials.* J. Comput. Appl. Math. **178** (2005), 155-168.

[12] L. G. Garza, L. E. Garza, F. Marcellán, N. C. Pinzón-Cortés. *A matrix approach for the semiclassical and coherent orthogonal polynomials.* Appl. Math. Comput. **256** (2015), 459-471.

[13] A. Ghressi, L. Khériji. *Orthogonal q-polynomials related to perturbed form.* Appl. Math. E-Notes **7** (2007), 111-120.

[14] A. Ghressi, L. Khériji. *A Survey on D-Semiclassical Orthogonal Polynomials.* Appl. Math. E-Notes **10** (2010), 210-234.

[15] L. Khériji. *An Introduction to the* H_q*-Semiclassical Orthogonal Polynomials.* Methods Appl. Anal. **10** (2003), 387-411.

[16] K. H. Kwon, J. H. Lee, F. Marcellán. *Generalized* Δ*-Coherent Pairs.* J. Korean Math. Soc. **41** (2003), 977-994.

[17] F. Marcellán, N. C. Pinzón-Cortés. *(M,N)-Coherent Pairs of Linear Functionals and Jacobi Matrices*. Appl. Math. Comput. **232** (2014), 76-83.

[18] F. Marcellán, N. C. Pinzón-Cortés. *$(1,1)$-D_ω-Coherent Pairs*. J. Difference Equ. Appl. **19** (2013), 1828-1848.

[19] F. Marcellán, N. C. Pinzón-Cortés. *$(1,1)$-q-Coherent Pairs*. Numer. Algorithms **60** (2012), 223-239.

[20] F. Marcellán, M. Sghaier, M. Zaatra. *On semiclassical linear functionals of class $s=2$. Classification and integral representations*. J. Difference Equ. Appl. **18** (2012), 973-1000.

[21] P. Maroni. *Une Théorie Algébrique des Polynômes Orthogonaux. Application aux Polynômes Orthogonaux Semi-Classiques*. In Orthogonal Polynomials and their Applications, C. Brezinski, L. Gori, and A. Ronveaux (Eds.). IMACS Annals Comput. Appl. Math. **9** (1991), 95-130.

[22] P. Maroni. *Semi-Classical Character and Finite-Type Relations Between Polynomial Sequences*. Appl. Numer. Math. **31** (1999), 295-330.

[23] P. Maroni. *Variations around classical orthogonal polynomials. Connected problems*. J. Comput. Appl. Math. **48** (1993), 133-155.

[24] P. Maroni, M. Mejri. *The symmetric D_ω-Semiclassical Orthogonal Polynomials of Class One*. Numer. Algorithms **49** (2008), 251-282.

[25] J. A. Shohat. *A differential equation for orthogonal polynomials*, Duke Math. J. **5** (1939), 401-417.

[26] L. Verde-Star. *Characterization and construction of classical orthogonal polynomials using a matrix approach.* Linear Algebra Appl. **438** (2013), 3635-3648.

[27] L. Verde-Star. *Recurrence coefficients and difference equations of classical discrete orthogonal and q-orthogonal polynomial sequences.* Linear Algebra Appl. **440** (2014), 293-306.