Relative asymptotics of matrix orthogonal polynomials for Uvarov perturbations. The degenerate case

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Abstract. Let α be a square matrix of measures, and $\{P_n(x;\alpha)\}_{n\geq 0}$ the associated sequence of orthonormal matrix polynomials satisfying the three-term recurrence relation $x\,P_n(x;\alpha) = A_{n+1}(\alpha)P_{n+1}(x;\alpha) + B_n(\alpha)P_n(x;\alpha)+A_n^*(\alpha)P_{n-1}(x;\alpha), \;\; n\geq 0$. Let $d\beta(u) \stackrel{\text{def}}{=} d\alpha(u)+M\delta(u-c)$, where M is a positive definite matrix, $\delta(u-c)$ is the Dirac measure supported at c that is located outside the support of $d\alpha$. We study the outer relative asymptotics of the sequence $\{P_n(x;\beta)\}_{n\geq 0}$ with respect to the sequence $\{P_n(x;\alpha)\}_{n\geq 0}$ under quite general assumption on the coefficients of the three term recurrence relation $\{A_n(\alpha)\}_{n\geq 0}$ and $\{B_n(\alpha)\}_{n\geq 0}$.

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1. Introduction

Let $\alpha(x) = (\alpha_{i,j}(x))_{i,j=1}^N$ be an $N\times N$ positive definite matrix of measures supported on an infinite subset Ω of the real line, i.e. for every Borel set $A\subset\Omega$ the numerical matrix $\alpha(A) = (\alpha_{i,j}(A))_{i,j=1}^N$ is positive semi-definite. Notice that the diagonal entries of α are positive measures and the non-diagonal entries are complex measures with $\alpha_{i,j} = \overline{\alpha_{j,i}}$.

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The support of a positive definite matrix of measures α is the support of the trace measure $\tau(\alpha) = \sum_{i=1}^{N} \alpha_{i,i}$.

A matrix polynomial of degree m is a mapping $P: \mathbb{C} \to \mathbb{C}^{(N,N)}$ such that

$$P(x) = \gamma_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$$

where $(b_k)_{k=0}^{m-1} \in \mathbb{C}^{(N,N)}$ and $\gamma_m \in \mathbb{C}^{(N,N)} \setminus \{0\}$.

Assuming that $\int_\Omega P(x)d\alpha(x)P(x)^*$ (resp. $\int_\Omega P(x)^* d\alpha(x)P(x)$) is nonsingular for every matrix polynomial P with nonsingular leading coefficient, we can define a sequence of matrix orthonormal polynomials with respect to an inner product in the linear space of matrix polynomials $\mathbb{C}^{(N,N)}[x]$ as follows

$$< P_n(\alpha), P_m(\alpha) >_L \stackrel{\text{def}}{=} \int_\Omega P_n(x;\alpha)d\alpha(x)P_m(x;\alpha)^* = \delta_{n,m}I_N, \qquad n,m = 0,1,2,\cdots, \quad (1.1)$$

(P_n is said to be a left orthonormal matrix polynomial of degree n) or,

$$< R_n(\alpha), R_m(\alpha) >_R \stackrel{\text{def}}{=} \int_\Omega R_n(x;\alpha)^* d\alpha(x)R_m(x;\alpha) = \delta_{n,m}I_N, \qquad n,m = 0,1,2,\cdots, \quad (1.2)$$

(R_n is said to be a right orthonormal matrix polynomial of degree n).

Orthogonal matrix polynomials have been studied since the second half of last century. M. G. Krein obtained some results on matrix moment problems from the point of view of operator theory [23]. During the 80's orthogonal matrix polynomials were connected to scattering theory by J. S. Geronimo [22]. An analog of Favard's theorem was established for the associated three-term recurrence relation with matrix coefficients by A. I. Aptekarev and E. M. Nikishin [2]. Later on, many results concerning the location of their zeros ([5, 17]), quadrature formulae ([6, 7, 18, 26]), asymptotic behavior ([5, 8, 10, 27, 29, 30]), spectral theory for linear difference ([1, 19, 20]) and differential ([9, 11, 12, 13, 14, 16]) operators with matrix polynomials as coefficients, and lowering and raising operators ([15]) have been obtained. A good and updated survey on this topic is [3].

Matrix orthogonal polynomials appear in a natural way when we deal with different kind of non-standard inner products. Indeed, in [21] discrete Sobolev-type orthogonal polynomials are interpreted as orthogonal matrix polynomials with respect to matrix of measures containing an absolutely continuous and a discrete part. In [4] matrix orthogonal polynomials can be associated to polynomials in two variables, orthogonal with respect to a measure supported in a domain of $\mathbf{R}^2$ when you consider the lexicographical order in the monomials and you apply the standard Gram-Schmidt method.

In [24] we gave an overview of recent results on analytic properties of matrix orthonormal polynomials. We focussed our attention in the distribution of zeros as well as in the asymptotic behavior of such polynomials under some restrictions about the measure of orthogonality.

On the other hand, in [28] a connection between the matrix coefficients of the recurrence relations satisfied by matrix orthogonal polynomials on the unit circle and on a finite interval of the real line has been established, and some outer relative asymptotics for matrix orthogonal polynomials was deduced.

In [30] we studied the outer relative asymptotics of two sequences of orthonormal matrix polynomials $\{P_n(x;\alpha)\}_{n\geq 0}$ and $\{P_n(x;\beta)\}_{n\geq 0}$ with respect to α and β assuming that the coefficients of the three term recurrence relation $\{A_n(\alpha)\}_{n\geq 0}$ and $\{B_n(\alpha)\}_{n\geq 0}$ given in (1.3) are convergent and $\lim_n A_n(\alpha)$ is non-singular).

In [27] we dealt with the outer relative asymptotics between two sequences of orthonormal matrix polynomials $\{P_n\}_{n\geq 0}$ (with unbounded three-term recurrence coefficients) and $\left\{\widetilde{P}_n\right\}_{n\geq 0}$, associated with a matrix of measures and its perturbation by the addition of Dirac matrix measure, respectively.

Let $\{P_n(x;\alpha)\}_{n\geq 0}$ be a sequence of orthonormal polynomials with respect to the matrix inner product (1.1). This sequence satisfies a three-term recurrence relation

$$\begin{aligned} x\,P_n(x;\alpha) = {} & A_{n+1}(\alpha)P_{n+1}(x;\alpha) \\ & + B_n(\alpha)P_n(x;\alpha) + A_n^*(\alpha)P_{n-1}(x;\alpha), \;\; n\geq 0, \end{aligned} \tag{1.3}$$

where

$$B_n = \langle xP_n, P_n\rangle_L = \langle P_n, xP_n\rangle_L = B_n^*, \quad n\geq 0,$$

and

$$A_n = \langle xP_{n-1}, P_n\rangle_L\,, \;\; n\geq 1.$$

Notice that if the leading coefficient γ_n of P_n is nonsingular, then $A_n = \gamma_{n-1}\gamma_n^{-1}$, i.e., A_n is a nonsingular matrix. On the other hand, if $\{U_n\}_{n\geq 0}$ is a sequence of unitary matrices then $\{U_nP_n\}_{n\geq 0}$ is another sequence of orthonormal matrix polynomials. In such a case the corresponding coefficients in the three-term recurrence relation are $\tilde{B}_n = U_nB_nU_n^*$ and $\tilde{A}_n = U_{n-1}A_nU_n^*$.
Conversely, given two sequences of matrices $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ of dimension $N\times N$, such that A_n are nonsingular matrices for every $n\geq 0$ and B_n are Hermitian matrices for every $n\geq 0$, then there exists a positive definite matrix of measures W such that the matrix polynomials defined by the recurrence relation

$$xY_n = A_{n+1}Y_{n+1} + B_nY_n + A_n^*Y_{n-1}, \;\; n\geq 0, \tag{1.4}$$

with the initial conditions $Y_{-1} = 0$ and $Y_0 = I_N$ constitute a sequence of matrix polynomials orthonormal with respect to the inner product (1.1) associated with W.

A second polynomial sequence that is a solution of (1.4) appears for the initial conditions $Y_1 = A_1^{-1}$ and $Y_0 = 0$. If we denote it by $\{Q_n(x;\alpha)\}_{n\geq 1}$, then we get $deg\, Q_n = n-1$. Indeed,

$$Q_n(x;\alpha) = \int_\Omega \frac{P_n(x;\alpha) - P_n(s;\alpha)}{x-s} d\alpha(s). \tag{1.5}$$

Such a sequence of matrix polynomials is said to be the sequence of matrix orthonormal polynomials of the second kind with respect to the matrix of measures α, assuming the normalization $\int_\Omega d\alpha(s) = I_N$. These matrix polynomials satisfy the following Liouville-Ostrogradski formula

$$Q_n(x;\alpha)P_{n-1}^*(x;\alpha) - P_n(x;\alpha)Q_{n-1}^*(x;\alpha) = A_n^{-1}(\alpha). \tag{1.6}$$

Furthermore, for the n-th kernel polynomial associated with α

$$K_{n+1}(x,y;\alpha) \overset{\text{def}}{=} \sum_{j=0}^{n} P_j^*(y;\alpha)P_j(x;\alpha),$$

we get the Christoffel - Darboux formula

$$\begin{aligned}(x-y)K_{n+1}(x,y;\alpha) = P_n^*(y;\alpha)A_{n+1}(\alpha)P_{n+1}(x;\alpha)\\ -P_{n+1}^*(y;\alpha)A_{n+1}^*(\alpha)P_n(x;\alpha).\end{aligned} \tag{1.7}$$

By means of a straightforward computation we get

$$\begin{aligned}K_{n+1}(x,x;\alpha) = P_{n+1}^*(x;\alpha)'\, A_{n+1}^*(\alpha)P_n(x;\alpha)\\ -P_n^*(x;\alpha)'\, A_{n+1}(\alpha)P_{n+1}(x;\alpha).\end{aligned} \tag{1.8}$$

The matrix $K_n(x,y;\alpha)$ is called the $(n-1)$-th reproducing kernel since the following property holds. For every matrix polynomial $\Pi_m(x)$ of degree $m \leq n-1$, we have

$$\langle \Pi_m(x), K_n(x,y;\alpha)\rangle_\alpha = \int_\Omega \Pi_m(x)d\alpha(x)K_n^*(x,y;\alpha) = \Pi_m(y). \tag{1.9}$$

We denote by Δ_n the set of zeros of the matrix polynomial P_n, i.e.,

$$\Delta_n(\alpha) \overset{\text{def}}{=} \{x_{n,k};\ k = 1,\dots,nN,\ \det P_n(x_{n,k};\alpha) = 0\}$$

and setting

$$\Gamma = \bigcap_{N>0} M_N \qquad \text{where} \qquad M_N = \overline{\bigcup_{n\geq N} \Delta_n},$$

then we have $supp(\alpha) \subset \Gamma$ (α can be found as a weak limit of a sequence of discrete positive definite matrices of measures cf. [17]).

The aim of this contribution is to extend the results obtained in [27, 30] to the degenerate case. More precisely, we will deal with the case where there exist the limit A given in (2.1) and (4.3) and we assume no conditions about its nonsingular character.

The structure of this paper is the following. In Section 2, we assume that $\{A_n(\alpha)\}_{n\geq 0}$ and $\{B_n(\alpha)\}_{n\geq 0}$ converge and $\lim_n A_n(\alpha)$ is a singular matrix. Then we deduce the behavior of $\gamma_n(\beta)\gamma_n(\alpha)^{-1}$ as well as the outer relative asymptotics $P_n(x;\beta)P_n(x;\alpha)^{-1}$ when $d\beta(u) = d\alpha(u) + M\delta(u-c)$, M is positive definite matrix, and δ is the Dirac matrix measure supported at $c \in \mathbb{R}\setminus\widehat{\Gamma}$ ($\widehat{\Gamma}$ is the convex hull of the support of α). In Section 3, we deal with similar questions when the orthonormal matrix polynomials are associated with a varying matrix measure. In section 4, we consider $\{P_n\}_{n\geq 0}$ a sequence of orthonormal matrix polynomials associated with sequences of the coefficients of the three term recurrence relation $\{A_n\}_{n\geq 0}, \{B_n\}_{n\geq 0}$. We assume that the above matrix parameters diverge and (4.3) holds for A singular. Then we deduce the outer relative asymptotics of two sequences of orthonormal matrix polynomials $\{P_n(x)\}_{n\geq 0}$ and $\left\{\tilde{P}_n(x)\right\}_{n\geq 0}$, which are associated respectively, with a matrix of measures and its perturbation by the addition of a Dirac matrix measure.

2. Relative asymptotics for orthogonal matrix polynomials with convergent recurrence coefficients: Degenerate Case

Let $\{P_n(x;\alpha)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to a matrix of measures α satisfying (1.3). Assume that

$$\lim_{n\longrightarrow\infty} A_n(\alpha) = A \text{ and } \lim_{n\longrightarrow\infty} B_n(\alpha) = B. \tag{2.1}$$

Using [30, Lemma 2.1.], there exists a positive constant $a > 0$ such that $supp(\alpha) \subseteq \Gamma \subseteq [-a, a]$. We denote by $\widehat{\Gamma}$ the convex hull of the support of α. Let $W_{A,B}$ be the matrix weight corresponding to the Chebyshev matrix polynomials of the second kind (cf. [8, Section 3] for an explicit expression).

Let β be a matrix of measures, given by $d\beta(u) = d\beta_N(u) \stackrel{\text{def}}{=} d\alpha(u) + \sum_{k=1}^{N} M_k\delta(u-c_k)$ where M_k are positive definite matrices and $c_k \in \mathbb{R}\setminus\hat{\Gamma}$. If A is a non-singular matrix, we studied in [30] the limit of the leading coefficients $\gamma_n(\beta)\gamma_n(\alpha)^{-1}$ as well as the outer relative asymptotics $P_n(x;\beta)P_n^{-1}(x;\alpha)$ for n-th orthonormal matrix polynomials with respect to β and the n-th orthonormal matrix polynomials with respect to α, respectively. We proved in [30, Theorem 4.2] the following result.

Theorem 2.1. *Let $\{P_n(x,\alpha)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to the matrix of measures α. Assume that* (2.1) *holds with A a non-singular matrix. Then there exists a sequence of orthonormal matrix polynomials $\{P_n(x,\beta)\}_{n\geq 0}$ with respect to β such that for $x \in$*

$\mathbb{R} \setminus \left\{ \hat{\Gamma} \cup \{c_k; k = 1, \cdots, N\} \right\}$

$$\lim_{n \longrightarrow \infty} P_n(x,\beta) P_n^{-1}(x,\alpha) = \prod_{k=N}^{\overset{\curvearrowright}{1}} \left\{ \Lambda_k(c_k)^{-1} + \frac{1}{c_k - x} \left[\Lambda_k(c_k)^* - \Lambda_k(c_k)^{-1} \right] \left[\left(\int \frac{dW_{A,B}(t)}{c_k - t} \right)^{-*} - \left(\int \frac{dW_{A,B}(t)}{x - t} \right)^{-1} \right] \right\}$$

where $\prod_{k=r+s}^{\overset{\curvearrowright}{r}} T_k = T_{r+s} T_{r+s-1} \cdots T_r$, $\Lambda_k(c_k) = \lim_{n \to \infty} \left[\gamma_n(\beta_k) \gamma_n^{-1}(\beta_{k-1}) \right]^*$.

In this section we study the degenerate case of the above problem when we assume A is a singular matrix.
To establish our main result, we need the following Lemma (cf. [8]) which deals with the outer ratio asymptotics for orthonormal matrix polynomials in a matrix Nevai class $M(A,B)$ (cf. [29, Def 2.1]) when A is a singular matrix.

Lemma 2.2. *Let* $\{P_n(x)\}_{n \geq 0}$ *be a sequence of orthonormal matrix polynomials satisfying the three-term recurrence relation* (1.3)*. Assume that* $\lim_n A_n = A$ *and* $\lim_n B_n = B$*, with* A *singular. Then there exists a positive definite matrix of measures* ν *such that*

$$\lim_{n \longrightarrow \infty} P_{n-1}(z) P_n^{-1}(z) A_n^{-1} = \int \frac{1}{z-t} \, d\nu(t) \stackrel{def}{=} F_{A,B}(z); \quad z \in \mathbb{C} \setminus \Gamma. \quad (2.2)$$

Moreover, the analytic function $F_{A,B}$ *satisfies*

$$A^* F_{A,B}(z) A F_{A,B}(z) + (B - zI_N) F_{A,B}(z) + I_N = 0, \qquad z \in \mathbb{C} \setminus \Gamma. \quad (2.3)$$

Now we are ready to establish a first theorem which deals with outer relative asymptotic results related to orthonormal matrix polynomials which are associated with a matrix of measures $d\beta(u) = d\alpha(u) + M\delta(u-c)$, where M is a positive definite matrix, and $\delta(u-c)$ is the Dirac matrix measure supported at $c \in \mathbb{R} \setminus \widehat{\Gamma}$.

Theorem 2.3. *Let* $\{P_n(x,\alpha)\}_{n \geq 0}$ *be a sequence of orthonormal matrix polynomials with respect to* α $(P_n(x;\alpha) = \gamma_n(\alpha) x^n + \cdots)$ *and which satisfies the three-term recurrence relation* (1.3)*. We assume that*

$$\lim_{n \longrightarrow \infty} A_n(\alpha) = A \ \textit{and} \ \lim_{n \longrightarrow \infty} B_n(\alpha) = B, \ \textit{with } A \textit{ singular.} \quad (2.4)$$

Then, there exists a positive definite matrix of measures ν *such that the following statements hold.*

(i) $\lim_{n \to \infty} \left[\gamma_n(\beta) \gamma_n(\alpha)^{-1} \right]^* \left[\gamma_n(\beta) \gamma_n(\alpha)^{-1} \right] =$

$$. \quad I_N + \left(\int \frac{d\nu(t)}{c-t} \right) \left(\frac{d}{dx} \left(\int \frac{d\nu(t)}{x-t} \right)(c) \right)^{-1} \left(\int \frac{d\nu(t)}{c-t} \right).$$

(ii) *Let* $\{P_n(x,\beta)\}_{n \geq 0}$ *be a sequence of orthonormal matrix polynomials with respect to* β*. Then*

(a) $\lim_{n \longrightarrow \infty} \left[\gamma_n(\beta) \gamma_n(\alpha)^{-1} \right]^* \times P_n(c;\beta) M P_n^*(c;\alpha) =$

$$. \quad - \left(\int \frac{d\nu(t)}{c-t} \right) \left(\frac{d}{dx} \left(\int \frac{d\nu(t)}{x-t} \right)(c) \right)^{-1} \left(\int \frac{d\nu(t)}{c-t} \right).$$

(b) For $x \in \mathbb{R} \setminus \left\{ \hat{\Gamma} \cup \{c\} \right\}$

$$\lim_{n \longrightarrow \infty} \left[\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right]^* \times P_n(x;\beta)P_n^{-1}(x;\alpha)$$
$$= I_N + \tfrac{1}{c-x}\left(\left(\int \tfrac{d\nu(t)}{c-t}\right)\left(\tfrac{d}{dx}\left(\int \tfrac{d\nu(t)}{x-t}\right)(c)\right)^{-1}\left(\int \tfrac{d\nu(t)}{c-t}\right)\right)$$
$$\times\left(\left(\int \tfrac{d\nu(t)}{c-t}\right)^{-*} - \left(\int \tfrac{d\nu(t)}{x-t}\right)^{-1}\right).$$

Remark 2.4. If $A_n(\alpha)$ and $B_n(\alpha)$ converge, respectively, to A and B with A a non-singular matrix, the limit given in the right hand side of statement i of the previous theorem is positive definite, and then there exists a sequence of orthonormal matrix polynomials $\{P_n(x,\beta)\}_{n\geq 0}$ for which the sequence $\left\{\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right\}_{n\geq 0}$ is convergent.

Remark 2.5. Let us assume that the sequences $A_n(\alpha)$ and $B_n(\alpha)$ converge, respectively, to A and B with A a non-singular matrix. Then from Theorem 2.3 we deduce [30, Theorem 3.1; Theorem 4.3; Theorem 4.1] with $\nu = W_{A,B}$ ($W_{A,B}$ is the matrix weight corresponding to the Chebyshev matrix polynomials of the second kind $U_n^{A,B}(t)$).

The basic tools to prove Theorem 2.3 are the three-term recurrence relation (1.3), the reproducing kernel property (1.9), the Liouville-Ostrogradski formula (1.6), and the outer ratio asymptotics (2.2). First, we state some results whose proofs can be deduced easily from [30, pages 7-9].

Lemma 2.6. *Let α and β be two matrix of measures, and let M be a positive definite matrix such that $d\beta(u) = d\alpha(u) + M\delta(u-c)$, where c is a real number. Then*

$$P_n(x;\beta) = \left[\gamma_n(\beta)\gamma(\alpha)^{-1}\right]^{-*} \times \left[P_n(x;\alpha) - P_n(c;\alpha)\left(I_N + MK_{n+1}(c,c;\alpha)\right)^{-1} MK_{n+1}^*(c,x;\alpha)\right]. \tag{2.5}$$

Before to start the proof of Theorem 2.3, we need the following Proposition which is a generalization of [30, Lemma 3.2.] (A is assumed to be a singular matrix).

Proposition 2.7. *Let $\{P_n(x,\alpha)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to the matrix of measures α. Let $A_n(\alpha)$ and $B_n(\alpha)$ be the matrix coefficients in the recurrence relation (1.3) such that*

$$\lim_{n\longrightarrow\infty} A_n(d\alpha) = A \;\; and \;\; \lim_{n\longrightarrow\infty} B_n(d\alpha) = B.$$

Then

$$\lim_{n\longrightarrow\infty} P_n^{-1}(x,d\alpha) = 0, \qquad for \quad x \in \mathbb{R}\setminus\hat{\Gamma}. \tag{2.6}$$

Proof. From (1.6)

$$P_n^{-1}(x;\alpha)A_n^{-1}(\alpha)P_{n-1}^{-*}(x;\alpha) = P_n^{-1}(x;\alpha)Q_n(x;\alpha) - (P_{n-1}^{-1}(x;\alpha)Q_{n-1}(x;\alpha))^*.$$

But from (1.5) we can prove that $(P_n^{-1}(x;\alpha)Q_n(x;\alpha))_n$ are Hermitian matrices. Since $\left\{P_n^{-1}(x;\alpha)Q_n(x;\alpha)\right\}_{n\geq 0}$ is a convergent sequence for $x\in\mathbb{C}\setminus\Gamma$ (see [7]), then we have

$$\lim_{n\longrightarrow\infty} P_n^{-1}(x;\alpha)A_n^{-1}(\alpha)P_{n-1}^{-*}(x;\alpha)=0. \tag{2.7}$$

Let $\{\varphi(n)\}_{n\geq 0}$ be an increasing sequence of positive integer numbers such that $L=\lim_{n\longrightarrow\infty} P_{\varphi(n)}^{-1}(x;\alpha)$ exists or is ∞, ($L(x;\alpha)=\infty$ means that at least one of its entries is ∞). Then

$$\begin{aligned} &P_{\varphi(n)+1}^{-1}(x;\alpha)A_{\varphi(n)+1}^{-1}(\alpha)P_{\varphi(n)}^{-*}(x;\alpha)= \\ &\qquad P_{\varphi(n)}^{-1}(x;\alpha)\left(P_{\varphi(n)}(x;\alpha)P_{\varphi(n)+1}^{-1}(x;\alpha)A_{\varphi(n)+1}^{-1}(\alpha)\right)P_{\varphi(n)}^{-*}(x;\alpha). \end{aligned} \tag{2.8}$$

Using (2.7), (2.8), and (2.2) we obtain

$$\begin{aligned} 0 &= \lim_{n\longrightarrow\infty} P_{\varphi(n)+1}^{-1}(x;\alpha)A_{\varphi(n)+1}^{-1}(\alpha)P_{\varphi(n)}^{-*}(x;\alpha) \\ &= L(x;\alpha)\left(\int \frac{1}{x-t}\,d\nu(t)\right)L(x;\alpha)^*. \end{aligned} \tag{2.9}$$

Since $x\in\mathbb{R}\setminus\hat{\Gamma}$ and $supp(\nu)$ is a compact subset of $\hat{\Gamma}$ we deduce as in [30, Prop.2.2] that the Markov matrix function $\int d\nu(t)/(x-t)$ is either positive or negative definite. Hence $L(x;\alpha)=0$. Thus $\left\{(P_n^{-1}(x;\alpha)\right\}_{n\geq 0}$ has no a subsequence that converges to a matrix (or ∞) other than the zero matrix 0. Hence (2.6) holds. ■

Proof. (of Theorem 2.3) If A is singular we use Lemma 2.6, and we have

$$\begin{aligned} &\int P_n(x;\beta)d\alpha(x)P_n^*(x;\alpha)=\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^{-*} \\ &\quad\times\Big[I_N-P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}M \\ &\qquad\qquad\times\int K_{n+1}^*(c,x;\alpha)d\alpha(x)P_n^*(x;\alpha)\Big]. \end{aligned}$$

This means that

$$\begin{aligned} \left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right] &= \left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^{-*} \\ &\quad\times\left[I_N-P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}MP_n^*(c;\alpha)\right], \end{aligned} \tag{2.10}$$

and, as a consequence,

$$\begin{aligned} &\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right] \\ &\quad= I_N-P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}MP_n^*(c;\alpha) \\ &\quad= I_N-\left[P_n^{-1}(c;\alpha)\right]^{-1}\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}\left[P_n^{-*}(c;\alpha)M^{-1}\right]^{-1} \\ &\quad= I_N-\left[P_n^{-*}(c;\alpha)M^{-1}P_n^{-1}(c;\alpha)+P_n^{-*}(c;\alpha)K_{n+1}(c,c;\alpha)P_n^{-1}(c;\alpha)\right]^{-1}. \end{aligned} \tag{2.11}$$

Notice that since M and $K_n(c,c,d\alpha)$ are positive definite, then

$$I_N-\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]$$

is also positive definite and, thus,

$$\begin{array}{rcl}\left\|\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right\|_B^2 & \overset{\text{def}}{=} & \sum_{i,j=1}^{N} | \left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]_{i,j} |^2 \\ & = & tr(\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^* \left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]) \leq N\end{array} \tag{2.12}$$

where $\|.\|_B$ is the Frobenius norm.
Taking into account [30], from (1.7) and (1.8) we get

$$\begin{aligned}&P_n^{-*}(c;\alpha)K_{n+1}(c,c;\alpha)P_n^{-1}(c;\alpha)\\ &\quad= -A_{n+1}(\alpha)\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)\right]^{-1}\\ &\qquad\times\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)\right]'\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)\right]^{-1}\\ &\quad= -\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)A_{n+1}(\alpha)^{-1}\right]^{-1}\\ &\qquad\times\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)A_{n+1}(\alpha)^{-1}\right]'\left[P_n(c;\alpha)P_{n+1}^{-1}(c;\alpha)A_{n+1}(\alpha)^{-1}\right]^{-1}.\end{aligned}$$

From (2.2) and [30, Corollary 2.1.]

$$\begin{aligned}&\lim_{n\longrightarrow\infty} P_n^{-*}(c;\alpha)K_{n+1}(c,c;\alpha)P_n^{-1}(c;\alpha)\\ &\qquad= -\left(\int \frac{d\nu(t)}{c-t}\right)^{-1}\left(\frac{d}{dx}\left(\int\frac{d\nu(t)}{x-t}\right)(c)\right)\left(\int\frac{d\nu(t)}{c-t}\right)^{-1}.\end{aligned} \tag{2.13}$$

Now using Lemma 2.7, we obtain

$$\lim_{n\longrightarrow\infty} P_n^{-*}(c;\alpha)M^{-1}P_n^{-1}(c;\alpha) = 0. \tag{2.14}$$

Finally, from (2.11), (2.13) and (2.14), the statement (i) is proved.

To prove (ii-a), we multiply $MP_n^*(c;\alpha)$ in the right side and $\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*$ in the left side of (2.5). Thus

$$\begin{aligned}&\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*\times P_n(c;\beta)MP_n^*(c;\alpha)\\ &\quad= [P_n(c;\alpha)MP_n^*(c;\alpha)\\ &\qquad -P_n(c;\alpha)(I_N+MK_{n+1}(c,c;\alpha))^{-1}MK_{n+1}(c,c;\alpha)MP_n^*(c;\alpha)]\\ &\quad= \left[P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}MP_n^*(c;\alpha)\right]\\ &\quad= \left[P_n^{-*}(c;\alpha)M^{-1}P_n^{-1}(c;\alpha)+P_n^{-*}(c;\alpha)K_{n+1}(c,c;\alpha)P_n^{-1}(c;\alpha)\right]^{-1}.\end{aligned}$$

Taking into account (2.13) and (2.14) we deduce (ii-a). To prove (ii-b), we multiply in (2.5) by $P_n^{-1}(x;\alpha)$ and we substitute $MK_{n+1}^*(c,x;\alpha)P_n^{-1}(x;\alpha)$ by (1.7). Thus

$$\begin{aligned}&\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*\times P_n(x;\beta)P_n^{-1}(x;\alpha)\\ &\quad= \left[I_N - P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}MK_{n+1}^*(c,x;\alpha)P_n(x;\alpha)^{-1}\right]\\ &\quad= \Big[I_N - P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}\tfrac{1}{c-x}M\left[P_{n+1}^*(c;\alpha)A_{n+1}^*\right.\\ &\qquad\qquad\left.-P_n^*(c;\alpha)A_{n+1}P_{n+1}(x;\alpha)P_n^{-1}(x;\alpha)\right]\Big]\\ &\quad= \Big[I_N - \tfrac{1}{c-x}P_n(c;\alpha)\left(I_N+MK_{n+1}(c,c;\alpha)\right)^{-1}MP_n^*(c;\alpha)\\ &\qquad\qquad\times\left[P_n^{-*}(c;\alpha)P_{n+1}^*(c;\alpha)A_{n+1}^* - A_{n+1}P_{n+1}(x;\alpha)P_n^{-1}(x;\alpha)\right]\Big].\end{aligned}$$

From (2.10)

$$\begin{aligned}&\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^* \times P_n(x;\beta)P_n^{-1}(x;\alpha)\\ &\qquad = I_N + \tfrac{1}{c-x}\left[\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right]^*\left[\gamma_n(\beta)\gamma_n^{-1}(\alpha)\right] - I_N\right]\\ &\qquad\qquad \times\left[P_n^{-*}(c;\alpha)P_{n+1}^*(c;\alpha)A_{n+1}^* - A_{n+1}P_{n+1}(x;\alpha)P_n^{-1}(x;\alpha)\right]\end{aligned}$$

and using (2.2) we deduce the statement (ii-b). ■

Example 2.8. Let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

From(2.3) we have

$$\int \frac{d\nu(t)}{z-t} = \begin{pmatrix} \frac{z-2}{\left(z+\frac{1}{2}(-3-\sqrt{5})\right)\left(z+\frac{1}{2}(-3+\sqrt{5})\right)} & 0 \\ 0 & \frac{1}{z-2} \end{pmatrix}. \tag{2.15}$$

Thus

$$\nu = \begin{pmatrix} \frac{5+\sqrt{5}}{10}\,\delta_{\frac{3-\sqrt{5}}{2}} + \frac{5-\sqrt{5}}{10}\,\delta_{\frac{3+\sqrt{5}}{2}} & 0 \\ 0 & \delta_2 \end{pmatrix}$$

with $supp(\nu) = \left\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 2\right\}$ and $\hat{\Gamma} = \left[\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right]$.
A straightforward computation yields

$$\lim_{n\to\infty}\left[\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right]^*\left[\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right] = \begin{pmatrix} \frac{1}{c^2-4c+5} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.16}$$

Notice that the matrix given in (2.16) is positive semi-definite when $c \in \mathbb{R}\setminus\widehat{\Gamma}$ (indeed it is a singular matrix). Then we deduce that if we replace the hypothesis in remark 2.4 by considering A a singular matrix, then we cannot guarantee the convergence of $\left\{\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right\}_{n\geq 0}$.

Now, taking into account (2.15) we compute the terms in the outer relative aymptotics in the right hand side of (ii-b) (Theorem 2.3). Then

$$\lim_{n\longrightarrow\infty}\left[\gamma_n(\beta)\gamma_n(\alpha)^{-1}\right]^* \times P_n(x;\beta)P_n^{-1}(x;\alpha) = \begin{pmatrix} \frac{x-c}{(c^2-4c+5)(x-2)} & 0 \\ 0 & 0 \end{pmatrix}.$$

3. Relative asymptotics for orthogonal matrix polynomials with varying recurrence coefficients: Degenerate Case

Let $\{P_n(x;\alpha_k)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to a varying matrix of measures α_k, $k = 1, 2, \cdots$, satisfying the following recurrence relation

$$\begin{aligned}xP_n(x;\alpha_k) &= A_{n+1}(\alpha_k)P_{n+1}(x;\alpha_k)\\ &\qquad + B_n(\alpha_k)P_n(x;\alpha_k) + A_n^*(\alpha_k)P_{n-1}(x;\alpha_k)\end{aligned} \tag{3.1}$$

where $P_{-1}(x;\alpha_k)=0,\;\; P_0(x;\alpha_k)=\int d\alpha_k = I_N$.
Δ_m will denote the set of zeros of $P_{n_m}(\,.\,;\alpha_{k_m})$ and $\Gamma \stackrel{\text{def}}{=} \bigcap_{N>0}\overline{\bigcup_{m\geq N}\Delta_m}$. We assume Γ is bounded and let $\hat{\Gamma}\supseteq\Gamma$ be the convex hull of Γ.

We consider a sequence of orthonormal matrix polynomials $\{P_n(x;\beta_k)\}_{n\geq 0}$ with respect to a varying matrix of measures defined by

$$d\beta_k(u) = d\alpha_k(u) + M\delta(u-c), \qquad \text{for } c\in\mathbb{R}\setminus\hat{\Gamma}, \tag{3.2}$$

where M is a positive definite matrix and δ is the Dirac matrix measure.

Now, we give the main result in this section which deals with the relative asymptotics for orthonormal matrix polynomials with respect to a varying matrix of measures.

Theorem 3.1. *Let $\{P_n(x;\alpha_k)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to a varying matrix of measures α_k satisfying the three term recurrence relation (3.1). Assume Γ to be bounded. Let $\{n_m\}_{m\geq 0}$, $\{k_m\}_{m\geq 0}$ be two increasing sequences of positive integers and assume there exist a singular matrix A and an Hermitian matrix B such that for all $l\geq 0$*

$$\lim_{m\to\infty} A_{n_m-l}(\alpha_{k_m}) = A, \qquad \lim_{m\to\infty} B_{n_m-l}(\alpha_{k_m}) = B. \tag{3.3}$$

Then there exists a positive definite matrix of measures ν, which is degenerate, such that the following statements hold:

(i) $$\lim_{m\to\infty}\left[\gamma_{n_m}(\beta_{k_m})\gamma_{n_m}(\alpha_{k_m})^{-1}\right]^*\left[\gamma_{n_m}(\beta_{k_m})\gamma_{n_m}(\alpha_{k_m})^{-1}\right] = I_N - \left(\int\frac{d\nu(t)}{c-t}\right)\left(\int\frac{d\nu(t)}{(c-t)^2}\right)^{-1}\left(\int\frac{d\nu(t)}{c-t}\right).$$

(ii) Let $\{P_n(x;\beta_k)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials with respect to $\beta_k(u)$ in (3.2) with $c\in\mathbb{R}\setminus\hat{\Gamma}$. Then, for $x\in\mathbb{R}\setminus\left\{\hat{\Gamma}\cup\{c\}\right\}$,

$$\lim_{m\to\infty}\left[\left[\gamma_{n_m}(d\beta_{k_m})\gamma_{n_m}(d\alpha_{k_m})^{-1}\right]^*\times P_{n_m-1}(x;\beta_{k_m})P_{n_m}^{-1}(x;\alpha_{k_m})A_{n_m}^{-1}(\alpha_{k_m})\right]$$
$$=\left(\int\frac{d\nu(t)}{x-t}\right)-\left(\int\frac{d\nu(t)}{c-t}\right)\left(\int\frac{d\nu(t)}{(c-t)^2}\right)^{-1}\left(\int\frac{d\nu(t)}{c-t}\right)\left(\int\frac{d\nu(t)}{c-t}\right)^{-*}\left(\int\frac{d\nu(t)}{(x-t)(c-t)}\right).$$

Remark 3.2. Under the hypothesis of Theorem 3.1, if A is a non-singular matrix, then the statements (i) and (ii) hold with $\nu = W_{A,B}$. $W_{A,B}$ is the matrix weight for the Chebyshev matrix polynomials of the second kind.

Proof. (of Remark 3.2) Use [27, Theorem 2.3.; Theorem. 2.5.]. ∎

To prove Theorem 3.1 we need the following preliminary results.

Proposition 3.3. *Under the hypothesis of Theorem 3.1, the following statements hold.*

(i) $\lim_{m\to\infty} P_{n_m}^{-1}(x;\alpha_{k_m}) = 0,\quad$ *for* $\quad x\in\mathbb{R}\setminus\hat{\Gamma}$.

(ii) There exists a positive definite matrix of measures ν, which is degenerate, (more precisely $\int(tI-B)d\nu(tI-B)^*$ is singular), such that

$$\lim_{m\to\infty} P_{n_m-1}(x;\alpha_{k_m})P_{n_m}^{-1}(x;\alpha_{k_m})A_{n_m}^{-1}(\alpha_{k_m}) = \int\frac{d\nu(t)}{x-t} \qquad \textit{for } x\in\mathbb{R}\setminus\hat{\Gamma}.$$

Proof. (of Proposition 3.3) To prove (ii), see [10, Sect. 4]. Now, to prove (i) we proceed as in [27, Lemma 2.4.] with some modifications.
From equation (1.6) we get

$$P_{n_m}^{-1}(x;\alpha_{k_m})A_{n_m}^{-1}(\alpha_{k_m})P_{n_m-1}^{-*}(x;\alpha_{k_m}) \\ = P_{n_m}^{-1}(x;\alpha_{k_m})Q_{n_m}(x;\alpha_{k_m}) - (P_{n_m-1}^{-1}(x;\alpha_{k_m})Q_{n_m-1}(x;\alpha_{k_m}))^*.$$

From [27, Theorem 2.1.] $(P_{n_m}^{-1}(x,\alpha_{k_m})Q_{n_m}(x,\alpha_{k_m}))_m$ is a convergent sequence and, as in [7] we can deduce that $P_n^{-1}(t)Q_n(t)$ is Hermitian. Then

$$\lim_{m\to\infty} P_{n_m}^{-1}(x;\alpha_{k_m})A_{n_m}^{-1}(\alpha_{k_m})P_{n_m-1}^{-*}(x;\alpha_{k_m}) = 0. \tag{3.4}$$

Let $\varphi(s)$ be an increasing sequence of nonnegative integer numbers such that

$$L(x) = \lim_{s\to\infty} P_{n_{\varphi(s)}-1}^{-1}(x;\alpha_{k_{\varphi(s)}})$$

either exists or is ∞. Here, $n_{\varphi(s)}$ is a subsequence of n_m and $k_{\varphi(s)}$ is a subsequence of k_m. Then

$$P_{n_{\varphi(s)}}^{-1}(x;d\alpha_{k_{\varphi(s)}})A_{n_{\varphi(s)}}^{-1}(d\alpha_{k_{\varphi(s)}})P_{n_{\varphi(s)}-1}^{-*}(x;\alpha_{k_{\varphi(s)}}) = P_{n_{\varphi(s)}-1}^{-1}(x;\alpha_{k_{\varphi(s)}}) \\ \times \left(P_{n_{\varphi(s)}-1}(x;\alpha_{k_{\varphi(s)}})P_{n_{\varphi(s)}}^{-1}(x;\alpha_{k_{\varphi(s)}})A_{n_{\varphi(s)}}^{-1}(\alpha_{k_{\varphi(s)}})\right) \\ \times\ P_{n_{\varphi(s)}-1}^{-*}(x;\alpha_{k_{\varphi(s)}}). \tag{3.5}$$

Using (3.4), (3.5), and statement (ii) we get

$$0 = L(x)\left(\int \frac{1}{x-t}\,d\nu(t)\right)L(x)^*. \tag{3.6}$$

Since $x \in \mathbb{R}\setminus\hat{\Gamma}$ and $supp(\nu) \subset \hat{\Gamma}$ we deduce as in [30, Prop.2.2] that the Markov matrix function $\int d\nu(t)/(x-t)$ is positive or negative definite for $x \in \mathbb{R}\setminus\hat{\Gamma}$. Hence $L(x) = 0$. Thus $\left\{P_{n_m}^{-1}(x;\alpha_{k_m})\right\}_{n\geq 0}$ has no a subsequence that converges to a matrix (or ∞) other than the zero matrix 0. Hence (i) follows. ■

Proof. (of Theorem 3.1) To prove (i) we proceed as in the proof of [27, Theorem 2.3] using Proposition 3.3 instead of [27, Lemma 2.4] and [10, Theorem 2.1].

To prove (ii) we proceed as in the proof of [27, Theorem 2.5]. Then we can obtain

$$\left[\gamma_{n-1}(\beta_k)\gamma_{n-1}^{-1}(\alpha_k)\right]^* \times P_{n-1}(x;\beta_k)P_n^{-1}(x;\alpha_k)A_n^{-1}\alpha_k) \\ = \Bigg[\sum_{j=1}^{p} \tfrac{1}{x-x_{n,k,j}}\, P_{n-1}(x_{n,k,j};\alpha_k)\Gamma_{n,k,j}P_{n-1}^*(x_{n,k,j};\alpha_k) \\ - \left(P_{n-1}(c;\alpha_k)\left(I_N + MK_n(c,c;\alpha_k)\right)^{-1} MP_n^*(c;\alpha_k)A_n^*(\alpha_k)\right) \\ \times \sum_{j=1}^{p} \tfrac{1}{(x-x_{n,k,j})(c-x_{n,k,j})} P_{n-1}(x_{n,k,j};\alpha_k)\Gamma_{n,k,j}P_{n-1}^*(x_{n,k,j};\alpha_k)\Bigg], \tag{3.7}$$

where $\Gamma_{n,k,j}$ are the matrix weights in the quadrature formula for the sequence of polynomials $\{P_n(x;\alpha_k)\}_{n\geq 0}$ (see [7, Thrm 3.1])

$$\Gamma_{n,k,j} \stackrel{\text{def}}{=} \frac{l_j}{(\det(P_n(x;\alpha_k)))^{(l_j)}(x_{n,k,j})} \left(Adj(P_n(x;\alpha_k))\right)^{(l_j-1)}(x_{n,k,j})Q_n(x_{n,k,j};\alpha_k). \tag{3.8}$$

Here, $x_{n,k,j}$, $j = 1, \cdots, p$, are the zeros of the polynomial $P_n(x;\alpha_k)$ and l_j will denote their multiplicity. Furthermore,

$$\begin{aligned} P_{n-1}(c;\alpha_k)\left(I_N + MK_n(c,c;\alpha_k)\right)^{-1} MP_n^*(c;\alpha_k)A_n^*(\alpha_k) \\ = P_{n-1}(c;\alpha_k)\left(I_N + MK_n(c,c;\alpha_k)\right)^{-1} MP_{n-1}^*(c;\alpha_k) \\ \times \left(P_{n-1}(c;\alpha_k)P_n^{-1}(c;\alpha_k)A_n^{-1}(\alpha_k)\right)^{-*} \end{aligned} \tag{3.9}$$

and from (2.11)

$$\begin{aligned} \left[\gamma_{n-1}(\beta_k)\gamma_{n-1}^{-1}(\alpha_k)\right]^* \left[\gamma_{n-1}(\beta_k)\gamma_{n-1}^{-1}(\alpha_k)\right] \\ = I_N - P_{n-1}(c;\alpha_k)\left(I_N + MK_n(c,c;\alpha_k)\right)^{-1} MP_{n-1}^*(c;\alpha_k). \end{aligned} \tag{3.10}$$

Using Theorem 3.1 (i) and Proposition 3.3, there exists a positive definite matrix of measures ν, which is degenerate, such that

$$\begin{aligned} \lim_{m\to\infty} P_{n_m-1}(c;\alpha_{k_m})\left(I_N + MK_{n_m}(c,c;\alpha_{k_m})\right)^{-1} MP_{n_m-1}^*(c;\alpha_{k_m}) \\ = -\left(\int \frac{d\nu(t)}{c-t}\right)\left\{\frac{d}{dx}\left(\int \frac{d\nu(t)}{x-t}\right)(c)\right\}^{-1}\left(\int \frac{d\nu(t)}{c-t}\right). \end{aligned}$$

Thus

$$\begin{aligned} \lim_{m\to\infty} P_{n_m-1}(c;\alpha_{k_m})\left(I_N + MK_{n_m}(c,c;\alpha_{k_m})\right)^{-1} MP_{n_m}^*(c;\alpha_{k_m})A_{n_m}^*(\alpha_{k_m}) \\ = -\left(\int \frac{d\nu(t)}{c-t}\right)\left\{\frac{d}{dx}\left(\int \frac{d\nu(t)}{x-t}\right)(c)\right\}^{-1}\left(\int \frac{d\nu(t)}{c-t}\right)\left(\int \frac{d\nu(t)}{c-t}\right)^{-*}. \end{aligned} \tag{3.11}$$

Now, we proceed as in the proof of [27, Step 1 and Step 2 (pp. 48-49)] by replacing the condition A is non-singular by A is singular and Γ is bounded. We use the so-called method of moments (see [27, page 49] or [10, page 13] considering the matrix polynomial $t^n I_N$ instead of $U_n^{A,B}(t)$). Then there exists a positive definite matrix of measures ν such that

$$\begin{aligned} \lim_{m\to\infty} \sum_{j=1}^{p} \frac{1}{(x - x_{n_m,k_m,j})(c - x_{n_m,k_m,j})} \\ \times\ P_{n_m-1}(x_{n_m,k_m,j};\alpha_{k_m})\Gamma_{n_m,k_m,j}P_{n_m-1}^*(x_{n_m,k_m,j};\alpha_{k_m}) = \int \frac{d\nu(t)}{(x-t)(c-t)} \end{aligned} \tag{3.12}$$

for $x \in \mathbb{C} \setminus \hat{\Gamma}$ and

$$\begin{aligned} \lim_{m\to\infty} \sum_{j=1}^{p} \frac{1}{x - x_{n_m,k_m,j}} P_{n_m-1}(x_{n_m,k_m,j};\alpha_{k_m})\Gamma_{n_m,k_m,j}P_{n_m-1}^*(x_{n_m,k_m,j};\alpha_{k_m}) \\ = \int \frac{1}{x-t}\, d\nu(t), \end{aligned} \tag{3.13}$$

for $x \in \mathbb{C} \setminus \hat{\Gamma}$. Taking into account (3.7), (3.11), (3.12) and (3.13) we get (ii).
■

4. Relative asymptotics for orthogonal matrix polynomials with unbounded recurrence coefficients: Degenerate Case

When unbounded recurrence coefficients are considered in the scalar case (assuming a similar hypothesis given by (4.3)), the asymptotic behaviour is studied for the scaled polynomials $p_n(c_n x)$. In the matrix case, we define the scaled matrix polynomial $P(Cx)$ as in [10]. First we consider a sequence of matrix polynomials of one matrix variable $(\mathsf{P}_n(T))_n$ defined from the matrix three-term recurrence relation

$$T\mathsf{P}_n(T) = A_{n+1}\mathsf{P}_{n+1}(T) + B_n\mathsf{P}_n(T) + A_n^*\mathsf{P}_{n-1}(T), \qquad (4.1)$$

with initial conditions $\mathsf{P}_{-1}(T) = 0$, $\mathsf{P}_0(T) = P_0$. It is clear that $\mathsf{P}_n(xI_N) = P_n(x)$ ($\{P_n(x)\}_{n\geq 0}$ is the polynomial sequence which satisfies (1.4)). Then we define the matrix polynomials $P_n(C;x)$ of one real variable by $P_n(C;x) \stackrel{\text{def}}{=} \mathsf{P}_n(C\,x)$. They satisfy

$$C\,t\,P_n(C;t) = A_{n+1}P_{n+1}(C;t) + B_nP_n(C;t) + A_n^*P_{n-1}(C;t) \quad n \geq 0. \quad (4.2)$$

Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials satisfying (1.4) such that its matrix recurrence coefficients diverge in a particular way: we will assume that there exists a sequence $\{C_n\}_{n\geq 0}$ of positive definite matrices such that

$$\lim_{n\to\infty} C_n^{-1/2}A_nC_n^{-1/2} = A, \quad \lim_{n\to\infty} C_n^{-1/2}B_nC_n^{-1/2} = B, \quad \lim_{n\to\infty} C_n^{-1/2}C_{n-1}^{1/2} = I_N. \qquad (4.3)$$

In order to apply Theorem 3.1 let us consider the scaled sequence of matrix polynomials $\{P_n(C_k;t)\}_{n\geq 0}$. From (4.2), we have

$$\begin{aligned} t\,C_k^{1/2}P_n(C_k;t) = {} & C_k^{-1/2}A_{n+1}C_k^{-1/2}C_k^{1/2}P_{n+1}(C_k;t) \\ & + C_k^{-1/2}B_nC_k^{-1/2}C_k^{1/2}P_n(C_k;t) \\ & + C_k^{-1/2}A_n^*C_k^{-1/2}C_k^{1/2}P_{n-1}(C_k;t), \quad n \geq 0. \end{aligned} \qquad (4.4)$$

The sequence of matrix polynomials $\{P_n(C_k;x)\}_{n\geq 0}$ is orthonormal with respect to some varying matrix of measures which we denote by W_k. Δ_n will denote the set of zeros of $P_n(C_n;x)$ and $\Gamma = \bigcap_{N>0} M_N$, where $M_N = \overline{\bigcup_{n\geq N}\Delta_n}$.

Now we will analyze the degenerate case of the relative asymptotics for matrix orthonormal polynomials with unbounded matrix recurrence coefficients. That is, a sequence of matrix polynomials satisfying (3.1) under the condition (4.3) assuming A is singular and the sequence $\{C_n\}_{n\geq 0}$ is increasing (which, according to [10, Lemma 3.1], yields Γ is bounded). We denote by $\hat{\Gamma}$ the convex hull of Γ.

Theorem 4.1 (Outer relative asymptotics). *Let $\{P_n(x)\}_{n\geq 0}$ be a sequence of orthonormal matrix polynomials such that (1.4) holds. Consider $P_n(C_n;x) = \gamma(W_n)x^n + \cdots$, the n-th orthogonal polynomial satisfying (4.2) associated with W_n and $P_{n-1}(x;\beta_n)$, the $(n-1)$-th orthonormal polynomial associated with $d\beta_n(u) \stackrel{def}{=} dW_n(u) + M\delta(u-c)$, $c \in \mathbb{R}\setminus\hat{\Gamma}$, and M is a positive definite* *matrix. Assume that* (4.3) *holds for* A *singular. Then there exists a positive definite matrix of measures* ν *such that*

$$\lim_{n\to\infty}\left[\left[\gamma_n(\beta_n)\gamma_n(W_n)^{-1}\right]^* P_{n-1}(x;\beta_n)P_n^{-1}(C_n;x)A_n^{-1}C_n^{1/2}\right] = \left(\int \frac{d\nu(t)}{x-t}\right) - \left(\int \frac{d\nu(t)}{c-t}\right)\left(\int \frac{d\nu(t)}{(c-t)^2}\right)^{-1}\left(\int \frac{d\nu(t)}{c-t}\right)\left(\int \frac{d\nu(t)}{c-t}\right)^{-*}\left(\int \frac{d\nu(t)}{(x-t)(c-t)}\right) \tag{4.5}$$

for $x \in \mathbb{R}\setminus\left\{\hat{\Gamma}\cup\{c\}\right\}$.

Proof. For each k, let us write

$$P_n(t;W_k) = C_k^{1/2}P_n(C_k;t), \quad A_n(W_k) = C_k^{-1/2}A_nC_k^{-1/2}, \quad B_n(W_k) = C_k^{-1/2}B_nC_k^{-1/2}. \tag{4.6}$$

Hence, from (4.4) and for each k, the sequence of matrix polynomials $\{P_n(t;dW_k)\}_{n\geq 0}$ satisfies (3.1). Under the assumption (4.3) it is easy to see that the limit conditions (3.3) hold for $n_m = k_m = m$. Indeed, for $l \geq 0$

$$\begin{aligned}\lim_{n\to\infty} A_{n-l}(W_n) &= \lim_{n\to\infty} C_n^{-1/2}A_{n-l}C_n^{-1/2}\\ &= \lim_{n\to\infty} C_n^{-1/2}C_{n-1}^{1/2}C_{n-1}^{-1/2}\cdots C_{n-l}^{1/2}C_{n-l}^{-1/2}A_{n-l}C_{n-l}^{-1/2}\\ &\quad C_{n-l}^{1/2}\cdots C_{n-1}^{-1/2}C_{n-1}^{1/2}C_n^{-1/2} = A.\end{aligned}$$

Using the same argument we prove that $\lim_{n\to\infty} A_{n-l}(W_n) = B$ for $l \geq 0$. Now we use Theorem 3.1. Then there exists a positive definite matrix of measures ν such that

$$\lim_{n\to\infty}\left[\left[\gamma_n(\beta_n)\gamma_n(W_n)^{-1}\right]^* P_{n-1}(x;\beta_n)P_n^{-1}(C_n;x)C_n^{-1/2}A_n^{-1}(W_n)\right]$$
$$= \left(\int \frac{d\nu(t)}{x-t}\right) - \left(\int \frac{d\nu(t)}{c-t}\right)\left(\int \frac{d\nu(t)}{(c-t)^2}\right)^{-1}\left(\int \frac{d\nu(t)}{c-t}\right)\left(\int \frac{d\nu(t)}{c-t}\right)^{-*}\left(\int \frac{d\nu(t)}{(x-t)(c-t)}\right).$$

Taking into account (4.6) we deduce (4.5). ■

Example 4.2. (Non degenerate case). We consider the following matrix recurrence coefficients

$$B_0 = \begin{pmatrix} b_{-1} & a_0 \\ a_0 & b_0 \end{pmatrix}, \qquad B_n = \begin{pmatrix} b_{-n-1} & 0 \\ 0 & b_n \end{pmatrix}, \qquad n = 1, \ldots.$$

and

$$A_n = \begin{pmatrix} -a_{-n} & 0 \\ 0 & a_n \end{pmatrix}, \qquad n = 0, \ldots.$$

They are closely related to the doubly infinite difference equation (see [25]),

$$b_{n+1}Y_{n+1}(z) - (z-a_n)Y_n(z) + b_nY_{n-1}(z) = 0, \quad n \in \mathbb{Z}.$$

The cases $a_n = dn$, $b_n^2 = an^2+bn+c$, $n \in \mathbb{Z}$, with a,b,c,d real numbers, ($a, c \neq 0$ and $b_n^2 > 0$), were studied in [25] and are related to associated Meixner ($d^2 > 4a > 0$), Meixner-Pollaczek ($d^2 < 4a$), and Laguerre ($d^2 = 4a \neq 0$) polynomials. For this case ($a, d > 0$) we take

$$C_n = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}.$$

The sequence $\{C_n\}_{n\geq 0}$ satisfies $\lim_n C_n^{-1/2} C_{n-1}^{1/2} = I_2$ and

$$\lim_n C_n^{-1/2} A_n C_n^{-1/2} = A = \lim_{n\to\infty} \begin{pmatrix} \frac{a_{-n}}{-n} & 0 \\ 0 & \frac{a_n}{n} \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix},$$
$$\lim_n C_n^{-1/2} B_n C_n^{-1/2} = B = \lim_{n\to\infty} \begin{pmatrix} \frac{b_{-n-1}}{n} & 0 \\ 0 & \frac{b_n}{n} \end{pmatrix} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{a} \end{pmatrix}.$$

From Remark 3.2, $\nu = W_{A,B}$ and from [8, Corollary 2.3] we get

$$\int \frac{d\nu(t)}{x-t} = \frac{z - \sqrt{a} - \sqrt{(\sqrt{a}-x)^2 - 4d^2}}{2d^2} I_2.$$

Then from (4.5) we get

$$\begin{aligned} &\lim_{n\to\infty} \tfrac{1}{d\sqrt{n}} \left[\left[\gamma_n(d\beta_n)\gamma_n(dW_n)^{-1}\right]^* \times P_{n-1}(x; d\beta_n) P_n^{-1}(C_n; x) \right] \\ &= \tfrac{1}{2d^2(c-x)} \Bigg[-c^2 + \left(x + \sqrt{(\sqrt{a}-c)^2 - 4d^2} - \sqrt{(\sqrt{a}-x)^2 - 4d^2} \right) c \\ &\qquad +4d^2 - a + \sqrt{a}(c+x) + \\ &\quad + \left(\sqrt{(\sqrt{a}-x)^2 - 4d^2} - x \right) \left(x + \sqrt{(\sqrt{a}-c)^2 - 4d^2} \right) \Bigg] I_2. \end{aligned}$$

Example 4.3. (Degenerate case). We consider the following matrix recurrence coefficients

$$A_n = \begin{pmatrix} \frac{1}{n} & 0 \\ n & \frac{1}{n^2} \end{pmatrix}, \qquad B_n = \begin{pmatrix} n + \frac{1}{n} & 0 \\ 0 & 2n + \frac{1}{n^2} \end{pmatrix}, \quad n > 1.$$

We take

$$C_n^{1/2} = \begin{pmatrix} \sqrt{n} & 0 \\ 0 & \sqrt{n} \end{pmatrix}.$$

The sequence $\{C_n\}_{n\geq 0}$ satisfies $\lim_n C_n^{-1/2} C_{n-1}^{1/2} = I_2$ and

$$\lim_n C_n^{-1/2} A_n C_n^{-1/2} = \lim_{n\to\infty} \begin{pmatrix} \frac{1}{n^2} & 0 \\ 1 & \frac{1}{n^3} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = A,$$

$$\lim_n C_n^{-1/2} B_n C_n^{-1/2} = \lim_{n\to\infty} \begin{pmatrix} 1 + \frac{1}{n^2} & 0 \\ 0 & 2 + \frac{1}{n^3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = B.$$

As in [8, page 343] we obtain

$$\int \frac{d\nu}{z-t} = \begin{pmatrix} \frac{z-2}{z^2-3z+1} & 0 \\ 0 & \frac{1}{z-2} \end{pmatrix},$$

from which we can find

$$\nu = \begin{pmatrix} \frac{5+\sqrt{5}}{10} \delta_{(3-\sqrt{5})/2} + \frac{5-\sqrt{5}}{10} \delta_{(3+\sqrt{5})/2} & 0 \\ 0 & \delta_2 \end{pmatrix}.$$

Then from (4.5) we get

$$\lim_{n\to\infty}\left[\left[\gamma_n(\beta_n)\gamma_n(W_n)^{-1}\right]^* P_{n-1}(x;\beta_n)P_n^{-1}(C_n;x)A_n^{-1}C_n^{1/2}\right]$$

$$= \begin{pmatrix} \frac{x-c}{(c^2-4c+5)(x^2-3x+1)} & 0 \\ 0 & 0 \end{pmatrix}.$$

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