Sobolev orthogonal polynomials on the unit circle and coherent pairs of measures of the second kind*

F. Marcellán[a] and A. Sri Ranga[b†]

Instituto de Ciencias Matemáticas (ICMAT) and Departamento de Matemáticas
Universidad Carlos III de Madrid, 28911, Leganés, Spain

[b]Departamento de Matemática Aplicada, IBILCE,
UNESP - Universidade Estadual Paulista
15054-000, São José do Rio Preto, SP, Brazil

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Abstract

We refer to a pair of non trivial probability measures (μ_0, μ_1) supported on the unit circle as a *coherent pair of measures of the second kind* on the unit circle if the corresponding sequences of monic orthogonal polynomials $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1; z)\}_{n\geq 0}$ satisfy $\frac{1}{n}\Phi_n'(\mu_0; z) = \Phi_{n-1}(\mu_1; z) - \chi_n \Phi_{n-2}(\mu_1; z)$, $n \geq 2$. It turns out that there are more interesting examples of pairs of measures on the unit circle with this latter coherency property than in the case of the standard coherence. The main objective in this contribution is to determine such pairs of measures. The polynomials orthogonal with respect to the Sobolev inner products associated with coherent pairs of measures of the second kind are also studied.

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1 Introduction

The concept of coherent pair of measures on the real line was introduced by A. Iserles et al. [15] in the framework of the theory of polynomials orthogonal with respect to a Sobolev inner product associated with a pair of nontrivial positive measures (ν_0, ν_1) supported on the real line. Specifically, the Sobolev inner product is

$$\langle f, g\rangle_S = \int_{\mathbb{R}} f(x)g(x)d\nu_0(x) + \lambda \int_{\mathbb{R}} f'(x)g'(x)d\nu_1(x), \tag{1.1}$$

where f and g are polynomials with real coefficients and λ is a nonnegative real number.

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†ranga@ibilce.unesp.br (corresponding author)

The pair of measures (ν_0, ν_1) is said to be coherent if the corresponding sequences of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n\geq 0}$ and $\{P_n(\nu_1; x)\}_{n\geq 0}$ satisfy

$$nP_{n-1}(\nu_1; x) = P_n'(\nu_0; x) + a_n P_{n-1}'(\nu_0; x), \quad n \geq 2, \tag{1.2}$$

with $a_n \neq 0$ for $n \geq 2$. We remark that since the polynomials involved are monic we have avoided the index $n = 1$ in (1.2) and also in other similar expressions such as (1.4), (1.5), (1.7) and (1.8).

Under (1.2) there exists a nice relation between the sequence of monic orthogonal polynomials $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n\geq 0}$ associated with the Sobolev inner product and the sequence of monic orthogonal polynomials $\{P_n(\nu_0; x)\}_{n\geq 0}$ with respect to the measure ν_0. Indeed,

$$S_n(\nu_0, \nu_1; \lambda; x) + b_n(\lambda) S_{n-1}(\nu_0, \nu_1; \lambda; x) = P_n(\nu_0; x) + a_n P_{n-1}(\nu_0; x), \quad n \geq 1. \tag{1.3}$$

H.G. Meijer in [20] proved that if (ν_0, ν_1) is a coherent pair of positive measures supported on the real line, that is for which (1.2) holds, then one of the measures has to be classical (Laguerre or Jacobi).

What was proved by Meijer [20] is slightly more general than what is stated above. He considers the orthogonal polynomials with respect to a pair of quasi-definite linear functionals on the set of polynomials with real coefficients and proves that one of the functionals has to be classical (i.e., associated with Laguerre, Jacobi or Bessel). Observe that the positive definite linear functionals are associated with nontrivial probability measures supported on the real line (see [11]). Thus, Meijer [20] also determines all the possible coherent pairs of positive measures supported within the real line.

The relation (1.3) proved to be very useful in order to study analytic properties of the respective Sobolev orthogonal polynomials. In particular, outer relative asymptotics have been deeply analyzed in the literature (see [18], [19] as well as the recent survey [17], where an updated list of references concerning this topic is presented).

In [8] the authors show that there are Sobolev inner products of the type (1.1) where the pair of measures (ν_0, ν_1) is not coherent, however the relation (1.3) still holds ([8, Thm. 4.1]) or a combination of Sobolev orthogonal polynomials of the form $S_n(\nu_0, \nu_1; \lambda; x) + b_n S_{n-1}(\nu_0, \nu_1; \lambda; x)$ has a simple two term (or even one term) combination of orthogonal polynomials $P_n(\nu; x)$, where the measure ν is closely related to the measures ν_0 and ν_1 ([8, Thm. 3.1]).

The results obtained in [8] can be covered by extending the concept of coherence to include semiclassical orthogonal polynomials (see [12]). It is important to observe that given the Sobolev inner product of the form (1.1) if the sequences $\{S_n(\nu_0, \nu_1; \lambda; x)\}_{n\geq 0}$ and $\{P_n(\nu_0; x)\}_{n\geq 0}$ satisfy (1.3) then

$$nP_{n-1}(\nu_1; x) + c_n P_{n-2}(\nu_1; x) = P_n'(\nu_0; x) + a_n P_{n-1}'(\nu_0; x), \quad n \geq 2, \tag{1.4}$$

with $a_n \neq 0$ for $n \geq 2$. When (1.4) holds (see [12]), the pair (ν_0, ν_1) is referred to as a $(1, 1)$-coherent pair. In this case, one of the measures must be semiclassical of class 1 and the other one is a rational perturbation of it.

The authors of [10] introduced the concept of coherent pair for Hermitian quasi-definite linear functionals (which can be represented by signed measures supported on the unit circle). In the positive definite case, the linear functional is associated with a nontrivial positive measure supported on the unit circle (see [21]).

Given a non trivial probability measure μ on the unit circle, let

$$\langle f, g\rangle_\mu = \int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) d\mu(\zeta) = \int_0^{2\pi} \overline{f(e^{i\theta})} g(e^{i\theta}) d\mu(e^{i\theta}).$$

We denote by $\{\Phi_n(\mu;z)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials (or monic OPUC) with respect to the inner product $\langle f,g\rangle_\mu$ and let $A_n^{(\mu)}$ and $\alpha_n^{(\mu)}$ be the quantities

$$A_n^{(\mu)} = \langle \Phi_n(\mu;.), \Phi_n(\mu;.)\rangle_\mu \quad \text{and} \quad \alpha_n^{(\mu)} = -\overline{\Phi_{n+1}(\mu;0)},$$

for $n \geq 0$. The constants $\alpha_n^{(\mu)}$ are the Verblunsky coefficients with respect to the measure μ and the constants $A_n^{(\mu)}$ represent the square of the norm of $\Phi_n(\mu;z)$ with respect to the measure μ. Clearly, $A_0^{(\mu)} = 1$ and, by convention, we take $A_{-1}^{(\mu)} = 0$.

Following [10], a pair (μ_0,μ_1) of positive measures supported on the unit circle is said to be a coherent pair of positive measures on the unit circle if the corresponding sequences of monic orthogonal polynomials $\{\Phi_n(\mu_0;z)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1;z)\}_{n\geq 0}$ satisfy the algebraic relation

$$n\Phi_{n-1}(\mu_1;z) = \Phi_n'(\mu_0;z) + \rho_n\Phi_{n-1}'(\mu_0;z), \quad n \geq 2. \tag{1.5}$$

Here, $\Phi_n'(\mu;z) = d\Phi_n(\mu;z)/dz$. As established in [10], if (μ_0,μ_1) is a coherent pair of positive measures on the unit circle then the following can be stated:

- If μ_0 is the Lebesgue measure then the companion measure μ_1 is such that $d\mu_1(z) = d\mu_0(z)/|z-\alpha|^2$, with $|\alpha| < 1$. That is, μ_1 is of the Bernstein-Szegő class.

- If μ_1 is the Lebesgue measure then the measure μ_0 is such that $d\mu_0(z) = |z-\alpha|^2 d\mu_1(z)$.

They also prove that the only Bernstein-Szegő measure μ_0 for which (μ_0,μ_1) is a coherent pair is the Lebesgue measure (i.e., μ_0 has to be the Lebesgue measure). Unfortunately, a full description of all coherent pairs of measures supported on the unit circle is not given and this remains as an open problem.

Given the pair of probability measures (μ_0,μ_1) supported on the unit circle and given the non negative real number s, we denote by $\{\Psi_n(\mu_0,\mu_1,s;z)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the Sobolev inner product

$$\langle f,g\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = \langle f,g\rangle_{\mu_0} + s\,\langle f',g'\rangle_{\mu_1}, \tag{1.6}$$

where $\langle f,g\rangle_{\mu_0} = \int_{\mathbb{T}} \overline{f(\zeta)}g(\zeta)d\mu_0(\zeta)$ and $\langle f,g\rangle_{\mu_1} = \int_{\mathbb{T}} \overline{f(\zeta)}g(\zeta)d\mu_1(\zeta)$.

An extension of the concept of coherent pair on the unit circle has been introduced in [14] and the connection with Sobolev orthogonal polynomials has been discussed in [16]. Indeed, they deal with $(1,1)$-coherent pairs of measures such that the corresponding sequences of orthogonal polynomials satisfy

$$n\Phi_{n-1}(\mu_1;z) + \sigma_n\Phi_{n-2}(\mu_1;z) = \Phi_n'(\mu_0;z) + \rho_n\Phi_{n-1}'(\mu_0;z), \quad n \geq 2, \tag{1.7}$$

with $\rho_n \neq 0$ for $n \geq 2$. In such a contribution, the explicit expressions for σ_n and $\rho_n, n \geq 2$, are obtained.

The work in the present manuscript is focused on the study of sequences of orthogonal polynomials with respect to a pair of measures supported on the unit circle such that in the above relation we assume $\rho_n = 0$, $n \geq 2$. Indeed, we are in the situation neither of a (1,1)- coherent pair nor a coherent pair.

The motivation for our study is that very recently, in [23], a nice example of a family of pairs of measures (μ_0,μ_1) such that the relation

$$\frac{1}{n}\Phi_n'(\mu_0;z) = \Phi_{n-1}(\mu_1;z) - \chi_n\Phi_{n-2}(\mu_1;z), \quad n \geq 2, \tag{1.8}$$

holds has been introduced and an associated sequence of monic Sobolev orthogonal polynomials has also been studied.

We will refer to a pair of positive measures (μ_0, μ_1) on the unit circle for which the relation (1.8) holds as a *coherent pair of measures of the second kind.* We will also refer to the constants $\chi_n = \chi_n^{(\mu_0,\mu_1)}$ as the connection coefficients associated with the coherent pair of measures (μ_0, μ_1) of the second kind.

Few examples of pairs of measures supported on the unit circle and satisfying the original coherency property are analyzed in the literature and it seems that one of the measures must be the Lebesgue measure as we have discussed above. In the case of coherent pairs of measures of the second kind, it follows from [23] that there is a wide family of pairs of nontrivial probability measures supported on the unit circle. Moreover, [23] also shows that the study of the corresponding Sobolev orthogonal polynomials (as we have defined in Section 4) can be done in a natural way. Hence, the questions that we attempt to solve in this manuscript are the following:

Q.1.- Are there other examples of pairs of measures or families of pairs of measure for which the coherency property of the second kind holds?

Q.2.- How can one characterize pairs of measures with such a property?

Q.3.- What is the behavior of the corresponding sequences of Sobolev orthogonal polynomials?

Sections 2 and 3 deal with the first two questions which lead to three families of coherent pair of measures of the second kind on the unit circle. These three families of pairs of measures are presented, respectively, in Theorems 3.1, 3.2 and 3.3. In Section 4, we focus our attention on the properties of the sequences of polynomials orthogonal with respect to the Sobolev inner product involving coherent pairs of measures of the second kind on the unit circle. Finally, in Section 5 we provide an asymptotic result for Sobolev orthogonal polynomials when the associated coherent pair of measures is from the family of measures given by Theorem 3.3.

2 Some basic assumptions on the measures

From the definition (1.8) for a coherent pair of measures (μ_0, μ_1) of the second kind on the unit circle and from the orthogonality of the sequence $\{\Phi_n(\mu_1; z)\}_{n\geq 0}$ with respect to the measure μ_1, one must have

$$\int_{\mathbb{T}} \Phi_n'(\mu_0; \zeta)\zeta^{-k} d\mu_1(\zeta) = 0, \quad 0 \leq k \leq n-3, \tag{2.1}$$

for $n \geq 3$. Moreover, with the connection coefficients χ_n in (1.8) there hold

$$\int_{\mathbb{T}} \Phi_n'(\mu_0; \zeta)\zeta^{-n+2} d\mu_1(\zeta) = -n\chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2.$$

Hence, in order to extract information from (2.1), we impose certain differentiability conditions and other extra assumptions on the measures μ_0 and μ_1.

First assumption: The initial assumption that we make is that the measure μ_1 is such that $\mu_1(e^{i\theta}) \in C^1[0, 2\pi]$, $\mu_1(e^{i\theta}) \in C^2(0, 2\pi)$ and, further, if we write

$$d\mu_1(e^{i\theta}) = \omega_1(\theta)d\theta,$$

then $\omega_1(0) = \omega_1(2\pi) < \infty$.

Hence, with the observation

$$\Phi_n'(\mu_0;\zeta) = \frac{d\Phi_n(\mu_0;\zeta)}{d\zeta} = -ie^{-i\theta}\,\frac{d\Phi_n(\mu_0;e^{i\theta})}{d\theta},$$

we then have

$$\int_0^{2\pi} \frac{d\Phi_n(\mu_0;e^{i\theta})}{d\theta} e^{-ik\theta}\omega_1(\theta)d\theta = 0, \quad 1 \leq k \leq n-2.$$

These yield by integration by parts together with $\omega_1(0) = \omega_1(2\pi) < \infty$,

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta}) \frac{d\big(e^{-ik\theta}\omega_1(\theta)\big)}{d\theta} d\theta = 0, \quad 1 \leq k \leq n-2,$$

for $n \geq 3$, and

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta}) \frac{d\big(e^{-i(n-1)\theta}\omega_1(\theta)\big)}{d\theta} d\theta = i\, n \chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2.$$

Equivalently,

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta}) e^{-ik\theta}[-ik\omega_1(\theta) + \omega_1'(\theta)]d\theta = 0, \quad 1 \leq k \leq n-2, \tag{2.2}$$

for $n \geq 3$, and

$$\int_0^{2\pi} \Phi_n(\mu_0;e^{i\theta}) e^{-i(n-1)\theta}[-i(n-1)\omega_1(\theta) + \omega_1'(\theta)]d\theta = i\, n \chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2,$$

where $\omega_1'(\theta) = d\omega_1(\theta)/d\theta$.

Second assumption: The orthogonality property in (2.2) is in general not useful unless we use other important assumptions. In particular, if we consider μ_0 and μ_1 to be related as follows

$$d\mu_1(e^{i\theta}) = \omega_1(\theta)d\theta = \tau_1|\zeta-\alpha|^2 d\mu_0(e^{i\theta}) = \tau_1[1+|\alpha|^2 - \alpha e^{-i\theta} - \overline{\alpha} e^{i\theta}]\, d\mu_0(e^{i\theta}), \tag{2.3}$$

where α is a complex number, then the left hand side in (2.2) is 0 and we can take advantage of (2.2). The positive number τ_1 can be arbitrary, but throughout in this manuscript we choose its value to be such that μ_0 and μ_1 are non trivial probability measures.

Some of the consequences of (2.3) are the following:

- Since $\omega_1(\theta) \in C^1(0,2\pi)$ and $\omega_1(0) = \omega_1(2\pi)$,

  - if $|\alpha| \neq 1$ then the function $\mu_0(e^{i\theta})$ needs to be absolutely continuous in $[0,2\pi]$. Thus, if we set $\int_0^{2\pi}\omega_0(\theta)d\theta = 1$ and $d\mu_0(e^{i\theta}) = \omega_0(\theta)d\theta$ for $[0,2\pi]$, then $\omega_1(\theta) = \tau_1\,|e^{i\theta}-\alpha|^2\omega_0(\theta)$ and $\tau_1^{-1} = \int_0^{2\pi}|e^{i\theta}-\alpha|^2\omega_0(\theta)d\theta$. Moreover, $\omega_0(0) = \omega_0(2\pi)$;

  - if $\alpha = e^{i\varphi}$ then the measure could have a positive mass of size t ($0 \leq t < 1$) at $z = \alpha$. However, the function $\mu_0(e^{i\theta})$ needs to be absolutely continuous in $[0,\varphi)\cup(\varphi,2\pi]$. Thus, if we set $\int_0^{2\pi}\omega_0(\theta)d\theta = 1-t$ and

  $$\int_0^{2\pi} \ell(e^{i\theta})\, d\mu_0(e^{i\theta}) = \left[\int_0^{\varphi} \ell(e^{i\theta})\,\omega_0(\theta)d\theta + \int_{\varphi}^{2\pi} \ell(e^{i\theta})\,\omega_0(\theta)d\theta\right] + t\,\ell(e^{i\varphi}),$$

  where ℓ is any Laurent polynomial, then $\omega_1(\theta) = \tau_1\,|e^{i\theta}-\alpha|^2\omega_0(\theta)$ and $\tau_1^{-1} = \int_0^{2\pi}|e^{i\theta}-\alpha|^2\omega_0(\theta)d\theta$. Moreover, $\omega_0(0) = \omega_0(2\pi)$ if $\alpha \neq 1$.

- From the orthogonality of $\Phi_n(\mu_0; z)$ with respect to μ_0,

$$\begin{aligned}\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \omega_1(\theta) d\theta &= \tau_1 \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} |e^{i\theta} - \alpha|^2 \omega_0(\theta) d\theta \\ &= \tau_1 \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} |e^{i\theta} - \alpha|^2 d\mu_0(\theta) = 0, \quad 1 \leq k \leq n-2,\end{aligned}$$

  for $n \geq 3$.

- Since $\frac{d\,|e^{i\theta}-\alpha|^2}{d\theta} = i(\alpha e^{-i\theta} - \overline{\alpha} e^{i\theta})$, again from the orthogonality of $\Phi_n(\mu_0; z)$ with respect to μ_0,

$$\begin{aligned}&\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \frac{d\,|e^{i\theta}-\alpha|^2}{d\theta} \omega_0(\theta) d\theta \\ &\qquad = \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} \frac{d\,|e^{i\theta}-\alpha|^2}{d\theta} d\mu_0(\theta) = 0, \quad 1 \leq k \leq n-2.\end{aligned}$$

  for $n \geq 3$.

With the above observations, we obtain from (2.2)

$$\int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-ik\theta} |e^{i\theta} - \alpha|^2 \omega_0'(\theta) d\theta = 0, \quad 1 \leq k \leq n-2, \tag{2.4}$$

for $n \geq 3$. For the associated connection coefficients we get

$$i\, n\alpha A_n^{(\mu_0)} + \int_0^{2\pi} \Phi_n(\mu_0; e^{i\theta}) e^{-i(n-1)\theta} |e^{i\theta} - \alpha|^2 \omega_0'(\theta) d\theta = i\, \frac{n}{\tau_1} \chi_n A_{n-2}^{(\mu_1)}, \quad n \geq 2. \tag{2.5}$$

Third assumption: For the orthogonality property in (2.4) to hold one must have

$$|\zeta - \alpha|^2 \omega_0'(\theta) = \big[R(\zeta) + \overline{R(\zeta)}\,\big] \omega_0(\theta), \tag{2.6}$$

where $R(\zeta) = R(e^{i\theta}) = \mathfrak{u} e^{i\theta} + \mathfrak{v}$, with $\mathfrak{u}, \mathfrak{v} \in \mathbb{C}$. In the case when $\alpha = e^{i\varphi}$ if the measure μ_0 has a positive mass t $(0 < t < 1)$ at $z = \alpha$ then $\mathfrak{u}$ and $\mathfrak{v}$ are also such that $R(\alpha) + \overline{R(\alpha)} = 2\,\mathrm{Re}(\mathfrak{u}\alpha) + 2\,\mathrm{Re}(\mathfrak{v}) = 0$.

When ω_0 satisfies (2.6) then (2.5) yields

$$\chi_n = \tau_1 \big[\alpha - i\frac{\overline{\mathfrak{u}}}{n}\big] \frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_1)}}, \quad n \geq 2.$$

Observing that $\omega_0'(\theta) = i\zeta\, d\omega_0(\theta)/d\zeta$, we thus have from (2.6) the following interesting requirement on the weight function ω_0.

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = \frac{\mathfrak{u}\zeta^2 + (\mathfrak{v} + \overline{\mathfrak{v}})\zeta + \overline{\mathfrak{u}}}{\zeta(\zeta - \alpha)(1 - \overline{\alpha}\zeta)}. \tag{2.7}$$

3 Determining the measures

To obtain information about the weight function ω_0 from (2.7) we consider the following three situations that need to be analyzed separately.

1.- If $\alpha = 0$, then

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = \mathfrak{u} + (\mathfrak{v} + \overline{\mathfrak{v}})\frac{1}{\zeta} + \overline{\mathfrak{u}}\frac{1}{\zeta^2}. \tag{3.1}$$

2.- If $\alpha \neq 0$ but $|\alpha| \neq 1$, then

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = -\frac{1}{\overline{\alpha}}\frac{\mathfrak{u}\zeta^2 + (\mathfrak{v}+\overline{\mathfrak{v}})\zeta + \overline{\mathfrak{u}}}{\zeta(\zeta-\alpha)(\zeta-1/\overline{\alpha})}. \tag{3.2}$$

3.- If $\alpha = e^{i\varphi}$, then

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = -\frac{1}{\overline{\alpha}}\frac{\mathfrak{u}\zeta^2 + (\mathfrak{v}+\overline{\mathfrak{v}})\zeta + \overline{\mathfrak{u}}}{\zeta(\zeta-\alpha)^2}. \tag{3.3}$$

3.1 The case $\alpha = 0$

Integrating (3.1) with respect to ζ we have

$$i\ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = \mathfrak{u}[\zeta - e^{i\theta_0}] - \overline{\mathfrak{u}}\big[\frac{1}{\zeta} - \frac{1}{e^{i\theta_0}}\big] + i(\mathfrak{v}+\overline{\mathfrak{v}})(\theta-\theta_0),$$

where θ_0 is assumed to be such that $\omega_0(\theta_0) > 0$. Thus,

$$\ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = 2\sin\frac{1}{2}(\theta-\theta_0)\big[\mathfrak{u}e^{i(\theta+\theta_0)/2} + \overline{\mathfrak{u}}e^{-i(\theta+\theta_0)/2}\big] + (\mathfrak{v}+\overline{\mathfrak{v}})(\theta-\theta_0).$$

Thus, with the requirement $\omega_0(0) = \omega_0(2\pi)$ we must have $\mathfrak{v}+\overline{\mathfrak{v}} = 0$. Hence,

$$\omega_0(\theta) = \omega_0(\theta_0)e^{2\sin\frac{1}{2}(\theta-\theta_0)[\mathfrak{u}e^{i(\theta+\theta_0)/2} + \overline{\mathfrak{u}}e^{-i(\theta+\theta_0)/2}]}$$

and

$$\omega_0'(\theta) = (\mathfrak{u}e^{i\theta} + \overline{\mathfrak{u}}e^{-i\theta})\,\omega_0(\theta).$$

Observe that

$$\begin{aligned}\sin\tfrac{1}{2}(\theta-\theta_0)\,[\mathfrak{u}e^{i(\theta+\theta_0)/2} + \overline{\mathfrak{u}}e^{-i(\theta+\theta_0)/2}] &= 2|\mathfrak{u}|\sin\tfrac{1}{2}(\theta-\theta_0)\cos\tfrac{1}{2}(\theta+\theta_0+2\arg\mathfrak{u}),\\ &= |\mathfrak{u}|[\sin(\theta+\arg\mathfrak{u}) - \sin(\theta_0+\arg\mathfrak{u})].\end{aligned}$$

Hence, we can state the following theorem.

Theorem 3.1 *With $\mathfrak{u} \in \mathbb{C}$, let*

$$\omega_0(\mathfrak{u};\theta) = \tau_0(\mathfrak{u})\,e^{2|\mathfrak{u}|\sin(\theta+\arg\mathfrak{u})},$$

where the positive constant $\tau_0(\mathfrak{u})$ be such that $\int_0^{2\pi}\omega_0(\mathfrak{u};\theta)d\theta = 1$. Let the pair of probability measures (μ_0,μ_0) on the unit circle be given by $d\mu_0(\mathfrak{u};e^{i\theta}) = \omega_0(\mathfrak{u};\theta)d\theta$. Then (μ_0,μ_0) is a coherent pair of probability measures of the second kind on the unit circle. For the associated connection coefficients $\chi_n = \chi_n^{(\mu_0,\mu_0)}$ we have

$$\chi_n = -i\frac{\overline{\mathfrak{u}}}{n}\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_0)}}, \quad n \geq 2.$$

3.2 The case $\alpha \neq 0$ and $|\alpha| \neq 1$

In this case, from (3.2) we easily arrive at the partial decomposition

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = \frac{A}{\zeta} + \frac{B}{\zeta - \alpha} - \frac{\overline{B}}{\zeta - 1/\overline{\alpha}},$$

where

$$A = -\frac{\overline{\mathfrak{u}}}{\alpha} \quad \text{and} \quad B = \frac{\mathfrak{u}\alpha + (\mathfrak{v} + \overline{\mathfrak{v}}) + \overline{\mathfrak{u}}/\alpha}{1 - |\alpha|^2}.$$

Thus, by integrating with respect to ζ

$$i \ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = iA\,(\theta - \theta_0) + B\,\ln\Big(\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\Big) - \overline{B}\,\ln\Big(\frac{\zeta - 1/\overline{\alpha}}{e^{i\theta_0} - 1/\overline{\alpha}}\Big),$$

where θ_0 is assumed to be such that $\omega_0(\theta_0) > 0$.

Since

$$\frac{\zeta - 1/\overline{\alpha}}{e^{i\theta_0} - 1/\overline{\alpha}} = \frac{\zeta}{e^{i\theta_0}}\frac{\overline{\zeta} - \overline{\alpha}}{e^{-i\theta_0} - \overline{\alpha}}$$

we then have

$$\ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = C\,(\theta - \theta_0) - iB\,\ln\Big(\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\Big) + i\overline{B}\,\ln\Big(\frac{\overline{\zeta} - \overline{\alpha}}{e^{-i\theta_0} - \overline{\alpha}}\Big),$$

where $C = A - \overline{B}$.

Thus,

$$\frac{\omega_0(\theta)}{\omega_0(\theta_0)} = e^{C\,(\theta - \theta_0)}\Big(\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\Big)^{-iB}\Big(\frac{\overline{\zeta} - \overline{\alpha}}{e^{-i\theta_0} - \overline{\alpha}}\Big)^{i\overline{B}},$$

which yields

$$\omega_0(\theta) = \omega_0(\theta_0)\,e^{C\,(\theta - \theta_0)}e^{2\,\mathrm{Re}(B)\,\arg\left(\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\right)}\left|\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\right|^{2\,\mathrm{Im}(B)}.$$

It is easy to verify that

$$C = A - \overline{B} = \frac{-1}{1 - |\alpha|^2}\big[\mathfrak{v} + \overline{\mathfrak{v}} + \frac{\mathfrak{u}}{\overline{\alpha}} + \frac{\overline{\mathfrak{u}}}{\alpha}\big] = \frac{2}{1 - |\alpha|^2}\big[-\mathrm{Re}(\mathfrak{v}) + \mathrm{Re}(A)\big].$$

Since C is real,

$$\mathrm{Im}(B) = \mathrm{Im}(\overline{A}) = -\,\mathrm{Im}(\mathfrak{u}/\overline{\alpha}),$$

and

$$\mathrm{Re}(B) = \mathrm{Re}(\overline{A}) - C = \frac{2}{1 - |\alpha|^2}\,\mathrm{Re}(\mathfrak{v}) - \frac{1 + |\alpha|^2}{1 - |\alpha|^2}\,\mathrm{Re}(A).$$

Observe that $\omega_0(\theta)$ can also be given as

$$\omega_0(\theta) = \omega_0(\theta_0)\,e^{(C + 2\,\mathrm{Re}(B))(\theta - \theta_0)}e^{2\,\mathrm{Re}(B)\,\arg\left(\frac{1 - \alpha e^{-i\theta}}{1 - \alpha e^{-i\theta_0}}\right)}\left|\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\right|^{2\,\mathrm{Im}(B)}.$$

Thus, to satisfy the requirement $\omega_0(0) = \omega_0(2\pi)$, one must have $C + 2\,\mathrm{Re}(B) = 0$, which is equivalent to choosing

$$\mathrm{Re}(\mathfrak{v}) - |\alpha|^2\,\mathrm{Re}(A) = 0.$$

This also means

$$\mathrm{Re}(B) = -\,\mathrm{Re}(A) = \mathrm{Re}(\mathfrak{u}/\overline{\alpha}),$$

$$\omega_0(\theta) = \tau_0(\theta_0, \mathfrak{u}, \alpha)\, e^{2\,\mathrm{Re}(B)\arg(1-\alpha e^{-i\theta})}|e^{i\theta} - \alpha|^{2\,\mathrm{Im}(B)}$$

and

$$|e^{i\theta} - \alpha|^2\, \omega_0'(\theta) = \big[\mathfrak{u}(e^{i\theta} - \alpha) + \overline{\mathfrak{u}}(e^{-i\theta} - \overline{\alpha})\big]\omega_0(\theta),$$

where $\tau_0(\theta_0, \mathfrak{u}, \alpha) = \omega_0(\theta_0)\, e^{-2\,\mathrm{Re}(B)\arg(1-\alpha e^{-i\theta_0})}|e^{i\theta_0} - \alpha|^{-2\,\mathrm{Im}(B)}$. Hence, we can state the following theorem.

Theorem 3.2 *With $\mathfrak{u}, \alpha \in \mathbb{C}$ and $|\alpha| \neq 1$, let*

$$\omega_0(\mathfrak{u}, \alpha; \theta) = \tau_0(\mathfrak{u}, \alpha)\, e^{2\,\mathrm{Re}(\mathfrak{u}/\overline{\alpha})\arg(1-\alpha e^{-i\theta})}|e^{i\theta} - \alpha|^{-2\,\mathrm{Im}(\mathfrak{u}/\overline{\alpha})},$$

and

$$\omega_1(\mathfrak{u}, \alpha; \theta) = \tau_1(\mathfrak{u}, \alpha)\, e^{2\,\mathrm{Re}(\mathfrak{u}/\overline{\alpha})\arg(1-\alpha e^{-i\theta})}|e^{i\theta} - \alpha|^{2[1-\mathrm{Im}(\mathfrak{u}/\overline{\alpha})]},$$

where the positive constants $\tau_0(\mathfrak{u}, \alpha)$ and $\tau_1(\mathfrak{u}, \alpha)$ are such that $\int_0^{2\pi} \omega_0(\mathfrak{u}, \alpha; \theta)d\theta = 1$ and $\int_0^{2\pi} \omega_1(\mathfrak{u}, \alpha; \theta)d\theta = 1$, respectively. Let the pair of probability measures on the unit circle (μ_0, μ_1) be such that $d\mu_0(\mathfrak{u}, \alpha; e^{i\theta}) = \omega_0(\mathfrak{u}, \alpha; \theta)d\theta$ and $d\mu_1(\mathfrak{u}, \alpha; e^{i\theta}) = \omega_1(\mathfrak{u}, \alpha; \theta)d\theta$. Then (μ_0, μ_1) is a coherent pair of probability measures of the second kind on the unit circle. Moreover, for the associated connection coefficients $\chi_n = \chi_n^{(\mu_0,\mu_1)}$ we have

$$\chi_n^{(\mu_0,\mu_1)} = \tau_1\big[\alpha - i\frac{\overline{\mathfrak{u}}}{n}\big]\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_1)}}, \quad n \geq 2.$$

3.3 The case $\alpha = e^{i\varphi}$

We have from (3.3),

$$i\frac{d\omega_0(\theta)/d\zeta}{\omega_0(\theta)} = \frac{A}{\zeta} + \frac{B}{\zeta - \alpha} - \frac{\alpha C}{(\zeta - \alpha)^2},$$

where

$$A = -\overline{\mathfrak{u}}\,\overline{\alpha}, \quad B = \overline{\mathfrak{u}}\,\overline{\alpha} - \mathfrak{u}\alpha \quad \text{and} \quad C = [\mathfrak{u}\alpha + (\mathfrak{v} + \overline{\mathfrak{v}}) + \overline{\mathfrak{u}}\,\overline{\alpha}].$$

By integration with respect to ζ yields

$$i\ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = iA\,(\theta - \theta_0) + B\,\ln\Big(\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha}\Big) + \frac{\alpha\, C}{\zeta - \alpha} - \frac{\alpha\, C}{e^{i\theta_0} - \alpha},$$

where θ_0 is assumed to be such that $\omega_0(\theta_0) > 0$. With $\alpha = e^{i\varphi}$, since

$$\frac{\zeta - \alpha}{e^{i\theta_0} - \alpha} = \frac{e^{i\theta/2}}{e^{i\theta_0/2}}\frac{\sin\frac{1}{2}(\theta - \varphi)}{\sin\frac{1}{2}(\theta_0 - \varphi)} \quad \text{and} \quad \frac{i2\alpha}{\zeta - \alpha} - \frac{i2\alpha}{e^{i\theta_0} - \alpha} = \frac{\sin\frac{1}{2}(\theta_0 - \theta)}{\sin\frac{1}{2}(\theta - \varphi)\,\sin\frac{1}{2}(\theta_0 - \varphi)},$$

we then have

$$\ln\Big(\frac{\omega_0(\theta)}{\omega_0(\theta_0)}\Big) = (A + B/2)\,(\theta - \theta_0) - iB\,\ln\Big(\frac{\sin\frac{1}{2}(\theta - \varphi)}{\sin\frac{1}{2}(\theta_0 - \varphi)}\Big) - \frac{(C/2)\,\sin\frac{1}{2}(\theta_0 - \theta)}{\sin\frac{1}{2}(\theta - \varphi)\,\sin\frac{1}{2}(\theta_0 - \varphi)},$$

where $A + B/2 = -\,\mathrm{Re}(\mathfrak{u}\alpha)$, $iB = 2\,\mathrm{Im}(\mathfrak{u}\alpha)$ and $C = 2\,\mathrm{Re}(\mathfrak{u}\alpha) + 2\,\mathrm{Re}(\mathfrak{v})$.

Clearly, for $\ln(\omega_0(\theta))$ to be bounded from above in $(0, 2\pi)$, first we must take $C = 0$. That is, $\mathfrak{v}$ and $\mathfrak{u}$ must satisfy $2\,\mathrm{Re}(\mathfrak{u}\alpha) = -2\,\mathrm{Re}(\mathfrak{v})$. Consequently,

$$\frac{\omega_0(\theta)}{\omega_0(\theta_0)} = \Big(\frac{\sin\frac{1}{2}(\theta-\varphi)}{\sin\frac{1}{2}(\theta_0-\varphi)}\Big)^{-iB} e^{(A+B/2)\,(\theta-\theta_0)},$$

Finally, in order to the following conditions hold

- $\omega_0(\theta)$ is a non negative weight function in $[0, 2\pi]$,
- $\omega_1(0) = \omega_1(2\pi)$ and
- the integral $\int_0^{2\pi} \omega_0(\theta) d\theta$ exists,

one must also have $\alpha = e^{i\varphi} = 1$, $\theta_0 \neq \varphi$ and $-iB = -2\,\mathrm{Im}(\mathfrak{u}\alpha) > 1$. Consequently, the weight function ω_0 is such that

$$\omega_0(\theta) = \omega_0(\theta_0)\, e^{(A+B/2)\,(\theta-\theta_0)} \Big(\frac{\sin(\theta/2)}{\sin(\theta_0/2)}\Big)^{-iB}.$$

Clearly, changing the value of θ_0 is equivalent to multiplying the weight function ω_0 by a constant factor. Thus, we chose $\theta_0 = \pi$. Hence, also by substituting $\mathfrak{u}$ by $-ib$, we can state the following theorem.

Theorem 3.3 *With $b \in \mathbb{C}$ is such that $\mathrm{Re}(b) > -1/2$, let*

$$\omega_0(b;\theta) = \tau_0(b,t)\,(e^{\pi-\theta})^{\mathrm{Im}(b)} \big(\sin^2(\theta/2)\big)^{\mathrm{Re}(b)},$$

where the positive constant $\tau_0(b,t)$ is such that $\int_0^{2\pi} \omega_0(b;\theta)d\theta = 1-t$, with $0 \leq t < 1$. Let the probability measure μ_0 on the unit circle be given by $d\mu_0(b,t;e^{i\theta}) = \omega_0(b;\theta)d\theta + t\delta(1)$. Moreover, let the probability measure μ_1 be such that $d\mu_1(b;e^{i\theta}) = \omega_1(b;\theta)d\theta = \tau_1(b)$ $|e^{i\theta}-1|^2\omega_0(b;\theta)d\theta$, where $\tau_1(b)$ is such that $\int_0^{2\pi} d\mu_1(b;e^{i\theta}) = 1$. Then (μ_0,μ_1) is a coherent pair of probability measures of the second kind on the unit circle. Furthermore, the associated connection coefficients satisfy

$$\chi_n^{(\mu_0,\mu_1)} = \tau_1\big[1+\frac{\bar{b}}{n}\big]\frac{A_n^{(\mu_0)}}{A_{n-2}^{(\mu_1)}}, \quad n \geq 2.$$

The family of coherent pairs of measures of the second kind on the unit circle given by Theorem 3.3 is what was studied in [23]. This large and interesting family of examples was our motivation to look for other examples, successfully leading to the results stated in Theorems 3.1 and 3.2.

3.4 Comments with respect to the example of Theorem 3.1

The measure μ_0 given in Theorem 3.1 when $\mathfrak{u} = 0$ is the Lebesgue measure. Hence, when $\mathfrak{u} \neq 0$ this measure can be seen as an interesting extension of the Lebesgue measure. It would be nice to know more about the properties of this measure and the properties of the associated OPUC. For example, for its moments

$$\mathfrak{c}_n^{(\mu_0)} = \int_0^{2\pi} e^{-in\theta} d\mu_0(\mathfrak{u};e^{i\theta}) = \tau_0(\mathfrak{u}) \int_0^{2\pi} e^{-in\theta} e^{2|\mathfrak{u}|\sin(\theta+\arg\mathfrak{u})} d\theta, \quad n = 0, \pm1, \pm2, \ldots,$$

we can state the following.

Theorem 3.4 *If* $\mathfrak{u} \neq 0$ *then*

$$\overline{\mathfrak{u}}\, \mathfrak{c}_{n+1}^{(\mu_0)} = i\, n\, \mathfrak{c}_{n}^{(\mu_0)} - \mathfrak{u}\, \mathfrak{c}_{n-1}^{(\mu_0)}, \quad n \geq 1.$$

The moments with negative index follow from $\mathfrak{c}_{-n}^{(\mu_0)} = \overline{\mathfrak{c}}_{n}^{(\mu_0)}$, $n \geq 1$.

Proof. First we remind that $\tau_0(\mathfrak{u})$ is such that $\mathfrak{c}_{0}^{(\mu_0)} = 1$. Application of integration by parts to $\mathfrak{c}_{n}^{(\mu_0)} = \tau_0(\mathfrak{u}) \int_0^{2\pi} e^{-in\theta} e^{2|\mathfrak{u}|\, \sin(\theta+\arg \mathfrak{u})} d\theta$ yields

$$\mathfrak{c}_{n}^{(\mu_0)} = -\tau_0(\mathfrak{u}) \int_0^{2\pi} \frac{e^{-in\theta}}{-i\, n}\, e^{2|\mathfrak{u}|\sin(\theta+\arg \mathfrak{u})} 2|u| \cos(\theta + \arg \mathfrak{u}) d\theta.$$

Thus, the theorem follows from $2|\mathfrak{u}|\, \cos(\theta + \arg \mathfrak{u}) = \mathfrak{u} e^{i\theta} + \overline{\mathfrak{u}} e^{-i\theta}$. ■

Now observe that if we know information (especially, the Verblunsky coefficients) about the orthogonal polynomials on the unit circle with respect to the measure μ_0 for one particular value of $\arg \mathfrak{u}$ then we can also easily obtain information about the Verblunsky coefficients for other values of $\arg \mathfrak{u}$ by applying rotations in the measure. Clearly, a good choice of a particular value of $\arg \mathfrak{u}$ that we can consider is $\pi/2$ (that is, $\mathfrak{u} = i|\mathfrak{u}|$). This choice makes all the moments $\mathfrak{c}_{n}^{(\mu_0)}$ real and hence also the associated Verblunsky coefficients real. We also have as the particular measure

$$d\mu_0(i|\mathfrak{u}|; \theta) = \tau(i|\mathfrak{u}|)\, e^{2|\mathfrak{u}|\cos(\theta)} d\theta.$$

The Delsarte and Genin transformation $x = \cos(\theta/2)$ (see, for example, [9], [13] and [24]) connects real orthogonal polynomials on the unit circle with symmetric orthogonal polynomials on the interval $[-1, 1]$. Using this transformation the problem of finding information about the real orthogonal polynomials on the unit circle with respect to $\mu_0(i|\mathfrak{u}|; \theta)$ becomes equivalent to finding information about symmetric orthogonal polynomials on the interval $[-1, 1]$ with respect to the weight function $e^{4|\mathfrak{u}|x^2}/\sqrt{1-x^2}$.

4 The Sobolev OPUC

Given the pair of probability measures (μ_0, μ_1) supported on the unit circle and given the non negative real number s, we denote by $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ the sequence of monic orthogonal polynomials with respect to the Sobolev inner product

$$\langle f, g\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = \langle f, g\rangle_{\mu_0} + s\, \langle f', g'\rangle_{\mu_1}, \tag{4.1}$$

where $\langle f, g\rangle_{\mu_0} = \int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) d\mu_0(\zeta)$ and $\langle f, g\rangle_{\mu_1} = \int_{\mathbb{T}} \overline{f(\zeta)} g(\zeta) d\mu_1(\zeta)$. Clearly, the positive definiteness of the inner product assures the existence of the sequences of orthogonal polynomials $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$.

Studies of such Sobolev orthogonal polynomials (or SOPUC) $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ with respect to particular examples of pair of measures (μ_0, μ_1) on the unit circle already appear in the literature.

For example, in [4] and [5], the authors study the Sobolev orthogonal polynomials with respect to the pair of measures (μ_0, μ_1), where $d\mu_1 = \frac{dz}{2\pi iz}$ is the Lebesgue measure supported on the unit circle and $d\mu_0 = \frac{d\mu_1}{|z-\alpha|^2}$. In [2], the same authors deal with asymptotic properties of Sobolev orthogonal polynomials with respect to a pair of measures (μ_0, μ_1), where $d\mu_0$ belongs to the Szegő class and, again, $d\mu_1$ is the Lebesgue measure supported on the unit circle.

In [3], two cases of Sobolev orthogonal polynomials with respect to a pair of measures (μ_0, μ_1), where $d\mu_0$ is either $|z-\alpha|^2 d\mu_1$ or $d\mu_1$ perturbed by a Dirac measure supported on a point a of the unit circle and, again, $d\mu_1$ is the Lebesgue measure, are studied.

Finally, in [6], [7] and [1], strong asymptotic properties of Sobolev orthogonal polynomials under different assumptions on the measures involved are deduced.

Our objective here is to study the sequence of monic orthogonal polynomials $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ when the pair of measures (μ_0, μ_1) is a coherent pair of measures of the second kind on the unit circle. That is, when the respective sequences of monic orthogonal polynomials $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ and $\{\Phi_n(\mu_1; z)\}_{n\geq 0}$ satisfy (1.8) for some constants $\chi_n = \chi_n^{(\mu_0,\mu_1)}$, $n \geq 2$.

Observe that the pairs of measures studied in [4] and [5] are coherent pairs of measures of the second kind. They are particular cases of pairs of measures given by Theorem 3.1 with $\mathfrak{u} = 0$ and by Theorem 3.1 with $\mathfrak{u} = 0$ and $|\alpha| < 1$, respectively.

Theorem 4.1 *If (μ_0, μ_1) is a coherent pair of measures of the second kind on the unit circle, then the sequence of monic orthogonal polynomials, $\{\Psi_n(z)\}_{n\geq 0}$, with respect to the Sobolev inner product $\langle f, g\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}$, where $\Psi_n(z) = \Psi_n(\mu_0, \mu_1, s; z)$, $n \geq 1$, satisfies*

$$\Psi_n(z) - \beta_n \Psi_{n-1}(z) = \Phi_n(\mu_0; z), \quad n \geq 1,$$

where the coefficients $\beta_n = \beta_n^{(\mu_0,\mu_1,s)}$, $n \geq 1$, are such that $\beta_1 = 0$ and

$$\beta_{n+1} = \frac{q_{n+1}}{p_n - \overline{q}_n\, \beta_n}, \quad n \geq 1.$$

Here,

$$\left.\begin{aligned} q_n &= q_n^{(\mu_0,\mu_1,s)} = s\, n(n-1) A_{n-2}^{(\mu_1)}\, \chi_n, \\ p_n &= p_n^{(\mu_0,\mu_1,s)} = A_n^{(\mu_0)} + s\, n^2 \big[A_{n-1}^{(\mu_1)} + A_{n-2}^{(\mu_1)}\, |\chi_n|^2\big], \end{aligned}\right\} \quad n \geq 1,$$

where we choose $A_{-1}^{(\mu_1)} = 0$ and $\chi_1 = 0$.

Proof. For notational convenience in the proof, we first adopt the notations $\Phi_{0,n}(z)$ and $\Phi_{1,n}(z)$ for $\Phi_n(\mu_0; z)$ and $\Phi_n(\mu_1; z)$, respectively.

Let

$$\Phi_{0,n}(z) = \sum_{j=0}^{n-2} h_{n,j}\, \Psi_j(z) - \beta_n\, \Psi_{n-1}(z) + \Psi_n(z), \quad n \geq 2,$$

be the Fourier expansion of $\Phi_{0,n}(z)$ in terms of the sequence of Sobolev orthogonal polynomials $\{\Psi_j(z)\}_{j\geq 0}$. For $0 \leq j \leq n-2$ we get

$$\begin{aligned} \overline{h}_{n,j} &= \frac{\big\langle \Phi_{0,n}, \Psi_j \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}}{\big\langle \Psi_j, \Psi_j \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}} \\ &= \frac{\big\langle \Phi_{0,n}, \Psi_j \big\rangle_{\mu_0} + s \big\langle \Phi'_{0,n}, \Psi'_j \big\rangle_{\mu_1}}{\big\langle \Psi_j, \Psi_j \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}}. \end{aligned}$$

Thus, from (1.8) it is easy to see that $h_{n,j} = 0$, $j = 0, 1, \ldots, n-2$, and $\Phi_{0,n}(z) = -\beta_n\, \Psi_{n-1}(z) + \Psi_n(z)$, $n \geq 2$.

Now we establish the required recurrence formula for the coefficients β_n. We have

$$\left\langle \Phi_{0,1}, \Psi_0 \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = \left\langle \Phi_{0,1}, 1 \right\rangle_{\mu_0} + s \left\langle 1, 0 \right\rangle_{\mu_1} = 0$$

and, for $n \geq 1$,

$$\begin{aligned}\left\langle \Phi_{0,n+1}, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} &= \left\langle \Phi_{0,n+1}, \Psi_n \right\rangle_{\mu_0} + s \left\langle \Phi'_{0,n+1}, \Psi'_n \right\rangle_{\mu_1} \\ &= s \left\langle \Phi'_{0,n+1}, \Psi'_n \right\rangle_{\mu_1}.\end{aligned}$$

Hence, from (1.8)

$$\left\langle \Phi_{0,n+1}, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = s \left\langle (n+1)\Phi_{1,n} - (n+1)\chi_{n+1}\Phi_{1,n-1}, \Psi'_n \right\rangle_{\mu_1},$$

from which

$$\left\langle \Phi_{0,n+1}, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = -s\, n(n+1)\, A^{(\mu_1)}_{n-1}\, \overline{\chi}_{n+1} = -\overline{q}_{n+1}, \quad n \geq 1.$$

Therefore, from

$$\overline{\beta}_{n+1} = -\frac{\left\langle \Phi_{0,n+1}, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}}{\left\langle \Psi_n, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)}}, \quad n \geq 0,$$

we have $\beta_1 = 0$ and also that $\overline{\chi}_n\, \beta_n > 0$ for $n \geq 2$.

Now from Ψ_n being monic, we have $\left\langle \Psi_0, \Psi_0 \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = 1$ and, for $n \geq 1$,

$$\begin{aligned}\left\langle \Psi_n, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} &= \left\langle \Phi_{0,n}, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\ &= A^{(\mu_0)}_n + s \left\langle \Phi'_{0,n}, \Psi'_n \right\rangle_{\mu_1}.\end{aligned}$$

Hence,

$$\left\langle \Psi_1, \Psi_1 \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = A^{(\mu_0)}_1 + s \left\langle 1, 1 \right\rangle_{\mu_1} = A^{(\mu_0)}_1 + s$$

and, for $n \geq 2$,

$$\left\langle \Psi_n, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = A^{(\mu_0)}_n + s \left\langle \Phi'_{0,n}, \Psi'_n \right\rangle_{\mu_1}.$$

From $\Psi'_n(z) = \Phi'_{0,n}(z) + \beta_n\, \Psi'_{n-1}(z)$ and from (1.8) the last formula can be written as

$$\begin{aligned}&\left\langle \Psi_n, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\ &\quad = A^{(\mu_0)}_n + s \left\langle n\Phi_{1,n-1} - n\chi_n\, \Phi_{1,n-2},\ n\Phi_{1,n-1} - n\chi_n\, \Phi_{1,n-2} + \beta_n\, \Psi'_{n-1} \right\rangle_{\mu_1},\end{aligned}$$

for $n \geq 2$. Thus,

$$\begin{aligned}&\left\langle \Psi_n, \Psi_n \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\ &\quad = A^{(\mu_0)}_n + s\, n^2 \big[A^{(\mu_1)}_{n-1} + |\chi_n|^2\, A^{(\mu_1)}_{n-2}\big] - s\, n(n-1)\, A^{(\mu_1)}_{n-2}\, \overline{\chi}_n\, \beta_n, \qquad (4.2) \\ &\quad = p_n - \overline{q}_n\, \beta_n,\end{aligned}$$

for $n \geq 2$. Thus, we get the requested recurrence formula for β_n. ■

Theorem 4.2 *The recurrence coefficients β_n in Theorem 4.1 satisfy*

$$0 < \overline{\chi}_n\, \beta_n < \frac{n}{n-1}\, |\chi_n|^2, \quad n \geq 2.$$

Proof. The positiveness of $\overline{\chi}_n \beta_n$ for $n \geq 2$ was established in the proof of Theorem 4.1. Now from

$$\big\langle \Psi_n, \Psi_n \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} = \big\langle \Psi_n, \Psi_n \big\rangle_{\mu_0} + s \big\langle \Psi_n', \Psi_n' \big\rangle_{\mu_1}, \quad n \geq 2,$$

and the minimal norm property of monic orthogonal polynomials,

$$\big\langle \Psi_n, \Psi_n \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} > \big\langle \Phi_{0,n}, \Phi_{0,n} \big\rangle_{\mu_0} + s\, n^2 \big\langle \Phi_{1,n-1}, \Phi_{1,n-1} \big\rangle_{\mu_1}, \quad n \geq 2.$$

Thus,

$$\big\langle \Psi_n, \Psi_n \big\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} > A_n^{(\mu_0)} + s\, n^2 A_{n-1}^{(\mu_1)}, \quad n \geq 2.$$

Hence, from (4.2) we have

$$s\, n(n-1)\, A_{n-2}^{(\mu_1)} \overline{\chi}_n \beta_n < s\, n^2 |\chi_n|^2 \, A_{n-2}^{(\mu_1)}, \quad n \geq 2.$$

This gives the inequality of the statement of our Theorem. ∎

We now consider the expression for $\Psi_n(\mu_0, \mu_1, s; z)$ as a linear combination of the polynomials $\{\Phi_j(\mu_0; z)\}_{j \geq 0}$.

Theorem 4.3 *If (μ_0, μ_1) is a coherent pair of measures of the second kind on the unit circle, then the monic orthogonal polynomials $\Psi_n(z) = \Psi_n(\mu_0, \mu_1, s; z)$, $n \geq 1$, with respect to the Sobolev inner product $\langle f, g \rangle_{\mathcal{S}(\mu_0,\mu_1,s)}$, satisfy*

$$\begin{aligned} &\Psi_1(z) = \Phi_1(\mu_0; z) \quad \textit{and} \\ &\Psi_{n+1}(z) = \sum_{j=1}^{n} a_j^{(n+1)} \Phi_j(\mu_0; z) + \Phi_{n+1}(\mu_0; z), \quad n \geq 1, \end{aligned}$$

where $a_1^{(2)} = 2s\chi_2/[A_1^{(\mu_0)} + s]$ and if $\mathbf{a}_n = [a_1^{(n+1)}, a_2^{(n+1)}, \ldots, a_n^{(n+1)}]^T$, then

$$\mathbf{T}_n\, \mathbf{a}_n = q_{n+1}\, \mathbf{e}_n, \quad n \geq 2.$$

Here, $\mathbf{e}_n$ is the n-th column of the $n \times n$ identity matrix and $\mathbf{T}_n = \mathbf{T}_n^{(\mu_0,\mu_1,s)}$ is the $n \times n$ Hermitian tridiagonal matrix

$$\mathbf{T}_n = \begin{bmatrix} p_1 & -q_2 & & & & \\ -\overline{q}_2 & p_2 & -q_3 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & -\overline{q}_{n-1} & p_{n-1} & -q_n \\ & & & & -\overline{q}_n & p_n \end{bmatrix}.$$

The numbers $p_k = p_k^{(\mu_0,\mu_1,s)}$ and $q_k = q_k^{(\mu_0,\mu_1,s)}$ are as in Theorem 4.1.

Proof. Again with the notational convention $\Phi_{0,n}(z)$ and $\Phi_{1,n}(z)$ for $\Phi_n(\mu_0; z)$ and $\Phi_n(\mu_1; z)$, respectively, let $\Psi_{n+1}(z) = \sum_{j=0}^{n+1} a_j^{(n+1)} \Phi_{0,j}(z)$. Clearly, $a_j^{(n+1)}$ are the Fourier coefficients of Ψ_{n+1} with respect to the orthogonal system $\{\Phi_{0,j}(z)\}_{n \geq 0}$.

Taking into account $\{\Psi_{n+1}(z)\}_{n \geq 0}$ is a sequence of monic polynomials orthogonal with respect to the inner product $\langle f, g \rangle_{\mathcal{S}(\mu_0,\mu_1,s)}$ we have the following.

With $n \geq 0$, from

$$
\begin{aligned}
0 &= \left\langle \Phi_{0,0},\, \Psi_{n+1} \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\
&= \left\langle \Phi_{0,0},\, \sum_{j=0}^{n+1} a_j^{(n+1)} \Phi_{0,j}(z) \right\rangle_{\mu_0} + s \left\langle 0,\, \sum_{j=0}^{n+1} a_j^{(n+1)} \Phi'_{0,j}(z) \right\rangle_{\mu_1},
\end{aligned}
$$

we have $a_0^{(n+1)} = 0$ for $n \geq 0$.

With $n \geq 1$, from

$$
\begin{aligned}
0 &= \left\langle \Phi_{0,1},\, \Psi_{n+1} \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\
&= \left\langle \Phi_{0,1},\, \sum_{j=1}^{n+1} a_j^{(n+1)} \Phi_{0,j}(z) \right\rangle_{\mu_0} \\
&\quad + s \left\langle \Phi_{1,0},\, a_1^{(n+1)} \Phi_{1,0}(z) + \sum_{j=2}^{n+1} j a_j^{(n+1)} [\Phi_{1,j-1}(z) - \chi_j \Phi_{1,j-2}(z)] \right\rangle_{\mu_1},
\end{aligned}
$$

there follows

$$
[A_1^{(\mu_0)} + s A_0^{(\mu_1)}] a_1^{(n+1)} - 2s A_0^{(\mu_1)} \chi_2\, a_2^{(n+1)} = 0.
$$

Thus, with the observation $a_2^{(2)} = 1$, we obtain the formula for $a_1^{(2)}$.

Now with $n \geq k \geq 2$, from

$$
\begin{aligned}
0 &= \left\langle \Phi_{0,k},\, \Psi_{n+1} \right\rangle_{\mathcal{S}(\mu_0,\mu_1,s)} \\
&= \left\langle \Phi_{0,k},\, \sum_{j=1}^{n+1} a_j^{(n+1)} \Phi_{0,j}(z) \right\rangle_{\mu_0} \\
&\quad + s \left\langle k[\Phi_{1,k-1} - \chi_k \Phi_{1,k-2}],\, a_1^{(n+1)} \Phi_{1,0}(z) \right\rangle_{\mu_1} \\
&\quad\quad + s \left\langle k[\Phi_{1,k-1} - \chi_k \Phi_{1,k-2}],\, \sum_{j=2}^{n+1} j a_j^{(n+1)} [\Phi_{1,j-1}(z) - \chi_j \Phi_{1,j-2}(z)] \right\rangle_{\mu_1},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
-s(k-1)k A_{k-2}^{(\mu_1)} \overline{\chi}_k\, a_{k-1}^{(n+1)} + \big[A_k^{(\mu_0)} + s\, k^2 \big(A_{k-1}^{(\mu_1)} + A_{k-2}^{(\mu_1)} |\chi_k|^2 \big) \big] a_k^{(n+1)} \\
- sk(k+1) A_{k-1}^{(\mu_1)} \overline{\chi}_{k+1}\, a_{k+1}^{(n+1)} = 0,
\end{aligned}
$$

for $n \geq k \geq 2$. We can write this as

$$
-\overline{q}_k a_{k-1}^{(n+1)} + p_k a_k^{(n+1)} - q_{k+1} a_{k+1}^{(n+1)} = 0, \quad 2 \leq k \leq n.
$$

Hence, with the observation $a_{n+1}^{(n+1)} = 1$, we obtain the system of linear equations $\mathbf{T}_n\, \mathbf{a}_n = q_{n+1}\, \mathbf{e}_n$. This completes the proof of the theorem. ■

4.1 Some further interesting results

Given the coherent pair of measures of the second kind (μ_0, μ_1), let

$$
d_n = d_n^{(\mu_0,\mu_1,s)} = \frac{|q_{n+1}|^2}{p_n\, p_{n+1}}, \quad n \geq 1,
$$

where $p_n = p_n^{(\mu_0,\mu_1,s)}$ and $q_n = q_n^{(\mu_0,\mu_1,s)}$ are as in Theorem 4.1.

Theorem 4.4 *The sequence* $\{d_n\}_{n\geq 1}$ *is a positive chain sequence.*

Proof. The recurrence formula for β_n in Theorem 4.1 yields

$$(1 - g_{n-1})g_n = d_n, \quad n \geq 1,$$

where

$$g_0 = 0, \quad g_n = \frac{\overline{q}_{n+1}\,\beta_{n+1}}{p_{n+1}} = \frac{n(n+1)sA_{n-1}^{(\mu_1)}\,\overline{\chi}_{n+1}\,\beta_{n+1}}{A_{n+1}^{(\mu_0)} + (n+1)^2 s\big[A_n^{(\mu_1)} + A_{n-1}^{(\mu_1)}\,|\chi_{n+1}|^2\big]}, \quad n \geq 1.$$

Observing that $g_0 = 0$, $g_n > 0$, $n \geq 1$, and $d_n > 0$, $n \geq 1$, we easily verify that $\{d_n\}_{n\geq 1}$ is a positive chain sequence and that $\{g_n\}_{n\geq 0}$ is its minimal parameter sequence. ■

Since the matrix $\mathbf{T}_n = \mathbf{T}_n^{(\mu_0,\mu_1,s)}$ given in Theorem 4.3 is Hermitian and tridiagonal, it is easily seen that

$$det(\mathbf{T}_n) = p_n\, det(\mathbf{T}_{n-1}) - |q_n|^2\, det(\mathbf{T}_{n-2}), \quad n \geq 2,$$

with $det(\mathbf{T}_0) = 1$ and $det(\mathbf{T}_1) = p_1$. Thus,

$$\frac{det(\mathbf{T}_n)}{p_n\, det(\mathbf{T}_{n-1})}\left[1 - \frac{det(\mathbf{T}_{n+1})}{p_{n+1}\, det(\mathbf{T}_n)}\right] = d_n, \quad n \geq 1,$$

where $\{d_n\}_{n\geq 1}$ is the positive chain sequence given by Theorem 4.4. This means the sequence $\{m_n\}_{n\geq 0}$, where

$$m_{n-1} = 1 - \frac{det(\mathbf{T}_n)}{p_n\, det(\mathbf{T}_{n-1})}, \quad n \geq 1, \tag{4.3}$$

is the minimal parameter sequence of $\{d_n\}_{n\geq 1}$. Therefore, we can state the following.

Theorem 4.5 *For the Hermitian tridiagonal matrices* $\mathbf{T}_n$ *given in Theorem 4.3 we get*

$$1 - \frac{det(\mathbf{T}_n)}{p_n\, det(\mathbf{T}_{n-1})} = \frac{\overline{q}_n\,\beta_n}{p_n}, \quad n \geq 1$$

and

$$0 < det(\mathbf{T}_n) < p_n\, det(\mathbf{T}_{n-1}), \quad n \geq 2.$$

5 An asymptotic result

As we have mentioned earlier, some asymptotic properties of Sobolev orthogonal polynomials on the unit circle have already been considered in [1], [2], [3], [6] and [7]. Here, we deal with an asymptotic property of monic Sobolev orthogonal polynomials $\Psi_n(\mu_0, \mu_1, s; z)$ which follow as a direct consequence of the recurrence relation given by Theorem 4.1. They hold when (μ_0, μ_1) is a coherent pair of measures of the second kind. For example, under the coherency property of the second kind, if we know something about the asymptotics of the sequence of polynomials $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ as well as the behavior of the sequence of coefficients $\{\beta_n\}_{n\geq 1}$, we may be able to use the recurrence relation in Theorem 4.1 to perform such a study. It turns out that this is the case for the pair of measures given in Theorem 3.3 and, as given below, we use the respective recurrence relation to find the outer relative asymptotics, i.e. the behavior of the ratios $\Psi_n(\mu_0,\mu_1,s;z)/\Phi_n(\mu_0;z)$ outside the unit disc.

First we recall from [22] and [23] the following explicit results that hold with respect to this pair of probability measures given by Theorem 3.3.

$$\tau_0(b,t)=(1-t)\,\frac{2^{b+\bar{b}}\,|\Gamma(b+1)|^2}{2\pi\,\Gamma(b+\bar{b}+1)},\quad \tau_1(b)=\frac{1}{1-t}\frac{|b+1|^2}{(b+\bar{b}+1)(b+\bar{b}+2)},$$

$$\alpha_{n-1}^{(\mu_1)}=-\frac{(b+1)_n}{(\bar{b}+2)_n},\qquad \alpha_{n-1}^{(\mu_0)}=\frac{(b+1)_n}{(\bar{b}+1)_n}\frac{1-2m_n^{(b,t)}-ic_n}{1+ic_n},$$

$$A_n^{(\mu_1)}=\frac{(b+\bar{b}+3)_n\,n!}{|(b+2)_n|^2},\qquad A_n^{(\mu_0)}=(1-t)\frac{1-m_n^{(b,t)}}{1-M_n}\frac{(b+\bar{b}+1)_n\,n!}{|(b+1)_n|^2},$$

and

$$\chi_{n+1}=2[1-m_{n+1}^{(b,t)}]\frac{n}{n+1}\frac{\lambda+n+1}{b+n+1},$$

for $n\geq 1$, where $c_n=\eta/(n+\lambda)$, $n\geq 1$, and $\{m_n^{(b,t)}\}_{n\geq 0}$ and $\{M_n\}_{n\geq 0}$ are, respectively, the minimal and maximal parameter sequences of the positive chain sequence $\{\hat{d}_n\}_{n\geq 1}$ given by

$$\hat{d}_1=\frac{1-t}{2}\frac{2\lambda+n}{\lambda+n}\qquad \hat{d}_{n+1}=\frac{1}{4}\frac{n(2\lambda+n+1)}{(\lambda+n)(\lambda+n+1)},\quad n\geq 1.$$

For the maximal parameter sequence also holds

$$M_0=t,\quad M_n=\frac{1}{2}\frac{2\lambda+n}{\lambda+n},\quad n\geq 1.$$

Here $\lambda=\mathrm{Re}(b)$ and $\eta=\mathrm{Im}(b)$ and the notation $(\,.\,)_n$ represents the Pochhammer symbol.

Moreover, for the connection coefficients χ_n and the recurrence coefficients β_n in Theorem 4.1, we get

$$\lim_{n\to\infty}\chi_n=1\quad\text{and}\quad \lim_{n\to\infty}\beta_n=1.$$

The convergence for $\{\chi_n\}_{n\geq 2}$ follows from the convergence of the above positive chain sequences $\{\hat{d}_n\}_{n\geq 1}$ to the limit 1/4 and, consequently, the convergence of its minimal $\{m_n^{(b,t)}\}_{n\geq 0}$ and maximal $\{M_n\}_{n\geq 0}$ parameter sequences to the limit 1/2 (see [11, p.102]). With the convergence result for χ_n, to obtain the convergence for $\{\beta_n\}_{n\geq 2}$ we use the convergence of the positive chain sequence $\{d_n\}_{n\geq 1}$ given by Theorem 4.4 to the limit 1/4 and, as a consequence, the convergence of its minimal parameter sequence $\{g_n\}_{n=0}^{\infty}$ to the limit 1/2.

It is also easy to verify that $\lim_{n\to\infty}\alpha_n^{(\mu_0)}=0$ and $\lim_{n\to\infty}\alpha_n^{(\mu_1)}=0$. Hence both measures given by Theorem 3.3 belong to the Nevai class (see [21]). Hence, in this case, we have the outer ratio asymptotics

$$\lim_{n\to\infty}\frac{\Phi_n(\mu_0;z)}{\Phi_{n-1}(\mu_0;z)}=z\quad\text{for}\quad |z|>1.$$

The convergence is uniform in every compact subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$. Here, $\overline{\mathbb{D}}$ is the closed unit circle.

From the recurrence relation in Theorem 4.1 we obtain

$$U_n(z)-\gamma_n(z)U_{n-1}(z)=1,\quad n\geq 1, \tag{5.1}$$

where

$$\gamma_n(z) = \beta_n \frac{\Phi_{n-1}(\mu_0; z)}{\Phi_n(\mu_0; z)} \quad \text{and} \quad U_n(z) = \frac{\Psi_n(\mu_0, \mu_1, s; z)}{\Phi_n(\mu_0; z)}$$

are analytic functions in $\mathbb{C} \setminus \overline{\mathbb{D}}$. Moreover, $\{\gamma_n(z)\}_{n\geq 1}$ converges to $1/z$ in every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$. Thus, in every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$, there exist a real number R, where $0 < R < 1$, and a positive integer n_0 such that

$$|\gamma_n(z)| \leq R, \quad \text{for every} \quad n \geq n_0.$$

Consequently, from (5.1)

$$|U_n(z)| \leq R\,|U_{n-1}(z)| + 1,$$

for every $n \geq n_0$.

Now, let us consider the sequence $\{V_n(z)\}_{n\geq 0}$ given by

$$V_n(z) = \begin{cases} |U_n(z)|, & 0 \leq n < n_0, \\ R\,V_{n-1}(z) + 1, & n \geq n_0. \end{cases}$$

For every $m \geq n_0 - 1$ and $r \geq 1$, we have

$$V_{m+r}(z) = R^r\, V_m(z) + \frac{1 - R^r}{1 - R}.$$

Thus, by taking the limit as $r \to \infty$, we obtain

$$\lim_{n\to\infty} V_n(z) = \frac{1}{1 - R}.$$

This means that the functions $V_n(z)$ are uniformly bounded for all n large enough. Since $0 \leq |U_n(z)| \leq V_n(z)$ for every positive integer n, the functions $U_n(z)$ are also uniformly bounded for n large enough in every compact subset of $\mathbb{C}\setminus\overline{\mathbb{D}}$. In other words, the sequence $\{U_n(z)\}_{n\geq 0}$ is a normal family of analytic functions and, thus, there exists a function $U(z)$ analytic in $\mathbb{C} \setminus \overline{\mathbb{D}}$ such that

$$\lim_{n\to\infty} U_n(z) = U(z).$$

Taking limit in (5.1) we have

$$U(z) - \frac{1}{z}\,U(z) = 1,$$

from which $U(z) = z/(z - 1)$. Hence, we can state the following theorem

Theorem 5.1 *Let (μ_0, μ_1) be the coherent pair of probability measures of the second kind on the unit circle given in Theorem 3.3. Let $\{\Phi_n(\mu_0; z)\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to μ_0 and let $\{\Psi_n(\mu_0, \mu_1, s; z)\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to the Sobolev inner product* (4.1). *Then*

$$\lim_{n\to\infty} \frac{\Psi_n(\mu_0, \mu_1, s; z)}{\Phi_n(\mu_0; z)} = \frac{z}{z - 1},$$

in every compact subset of $\mathbb{C} \setminus \overline{\mathbb{D}}$.

As a direct consequence of the Hurwitz Theorem we deduce that, for n large enough, the zeros of the polynomial $\Psi_n(\mu_0, \mu_1, s; z)$ are located inside the unit circle.

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