On Freud-Sobolev type orthogonal polynomials

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Abstract

In this contribution we deal with sequences of monic polynomials orthogonal with respect to the Freud Sobolev-type inner product

$$\langle p,q\rangle_s = \int_{\mathbb{R}} p(x)q(x)e^{-x^4}dx + M_0 p(0)q(0) + M_1 p'(0)q'(0),$$

where p,q are polynomials, M_0 and M_1 are nonnegative real numbers. Connection formulas between these polynomials and Freud polynomials are deduced and, as an application, an algorithm to compute their zeros is presented. The location of their zeros as well as their asymptotic behavior is studied. Finally, an electrostatic interpretation of them in terms of a logarithmic interaction in the presence of an external field is given.

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1. Introduction

Let $\mathbb{P}$ be the linear space of polynomials with real coefficients, and let us introduce the following inner product

$$\langle p,q\rangle = \int_{\mathbb{R}} p(x)q(x)e^{-x^4}dx, \quad p,q\in\mathbb{P}. \tag{1}$$

Let $\{F_n(x)\}_{n\geq 0}$ be the corresponding sequence of monic orthogonal polynomials (MOPS, in short). Since the linear functional u associated with $\omega(x)=e^{-x^4}$, i.e.

$$\langle u, p(x)\rangle = \int_{\mathbb{R}} p(x)\omega(x)dx,$$

satisfies the distributional (or Pearson) equation

$$[\sigma(x)\omega(x)]' = \tau(x)\omega(x),$$

where $\sigma(x) = 1$ and $\tau(x) = -4x^3$, $\{F_n\}_{n\geq 0}$ constitutes a family of semi–classical orthogonal polynomials (see [18], [21]). Indeed, this is a particular case of the so-called Freud type inner products [4].

Recent studies about this type of weights, including some parameters in the weight function $\omega(x)$, are [8], [9] and [11].

In this contribution, we consider the diagonal Freud Sobolev-type inner product

$$\langle p, q\rangle_s = \langle p, q\rangle + \mathbf{p}^T(0)\mathbf{M}\mathbf{q}(0), \tag{2}$$

where

$$\mathbf{p}(0) = [p(0), p'(0), \ldots, p^{(s)}(0)]^T$$

is a column vector of dimension $s+1$, the column vector $\mathbf{q}(0)$ is defined in an analogous way, and $\mathbf{M}$ is the diagonal and positive definite $(s+1)\times(s+1)$ matrix

$$\mathbf{M} = diag\,[M_0, M_1, \ldots, M_s], \quad M_k \in \mathbb{R}_+, k = 0, 1, \cdots, s.$$

Thus, (2) reads

$$\langle p, q\rangle_s = \langle p, q\rangle + \sum_{k=0}^{s} M_k p^{(k)}(0)q^{(k)}(0). \tag{3}$$

We will denote by $\{Q_n(x)\}_{n\geq 0}$ the MOPS with respect to the above inner product. This is the so called *diagonal case* for Sobolev-type inner products, see [2]. If there are no derivatives involved therein (i.e., $s = 0$), the polynomials orthogonal with respect to (3) are known in the literature as *Krall–type orthogonal polynomials*, and they are orthogonal with respect to a standard inner product, because the operator of multiplication by x is symmetric with respect to such an inner product, i.e. $\langle xp, q\rangle_{s=0} = \langle p, xq\rangle_{s=0}$, for every $p, q \in \mathbb{P}$. On the other hand, when $s > 0$ (2) becomes non–standard, and the corresponding polynomials are called *Sobolev–type orthogonal polynomials*. In this work we consider the Sobolev case, so we will refer $Q_n(x)$ as *Freud–Sobolev type* orthogonal polynomials.

The structure of the manuscript is as follows. In Section 2 we present the basic background regarding these polynomials, as well as some connection formulas between monic Freud-Sobolev type and monic Freud orthogonal polynomials. These results will be used in the sequel. In Section 3 we present an algorithm to numerically compute the zeros of the Freud-Sobolev type orthogonal polynomials, as the eigenvalues of a certain matrix. In Section 4 we study some analytic properties of zeros of Freud-Sobolev type orthogonal polynomials, in particular interlacing and asymptotic behavior. Section 5 is focused on the second order linear differential equation that such polynomials satisfy. As a direct consequence, the electrostatic interpretation of these polynomials in terms of a logarithmic potential interaction and an external potential is presented.

2. Background

Freud orthogonal polynomials are very well known in the literature. In the next Proposition, we summarize some of their properties that will be used in the sequel.

Proposition 1. *Let $\{F_n(x)\}_{n\geq 0}$ denote the sequence of monic polynomials orthogonal with respect to (1). Then, the following structural properties hold.*

(I) *Norm.*

$$||F_n||^2 = \int_{\mathbb{R}} [F_n(x)]^2 e^{-x^4} dx.$$

(II) *Three term recurrence relation (see [12]). Since $\omega(x)$ is an even weight function, the family $\{F_n(x)\}_{n\geq 0}$ is symmetric. For every $n \in \mathbb{N}$,*

$$xF_n(x) = F_{n+1}(x) + a_n^2 F_{n-1}(x), \quad n \geq 1, \tag{4}$$

with $F_{-1} := 0$, $F_0(x) = 1$, $F_1(x) = x$. Also, $a_n^2 = \frac{||F_n||^2}{||F_{n-1}||^2}$, $n \geq 1$, $a_0 = 0$, and

$$a_1^2 = \frac{\int_{\mathbb{R}} x^2\omega(x)dx}{\int_{\mathbb{R}} \omega(x)dx} = \frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}.$$

(III) *String equation (see [23, (2.12)]). An important feature of these polynomials is that the recurrence coefficients a_n in the above three term recurrence relation, satisfy the following nonlinear difference equation*

$$4a_n^2\left(a_{n+1}^2+a_n^2+a_{n-1}^2\right)=n, \quad n\geq 1.$$

This is known in the literature as the string equation or Freud equation (see [12], [14, (3.2.20)], among others).

(IV) *([19, Th. 5]). The polynomials $F_n(x)$ satisfy*

$$F_n'(x)=-4xa_n^2F_n(x)+4a_n^2\phi_n(x)F_{n-1}(x), \tag{5}$$

and

$$\begin{aligned} F_n''(x) &= [16a_n^4x^2-4a_n^2-16a_n^2\phi_n(x)\phi_{n-1}(x)]F_n(x) \\ &\quad +[8a_n^2x+16a_n^2x^3\phi_n(x)]F_{n-1}(x), \end{aligned} \tag{6}$$

where (see [19, eq. (14)])

$$\phi_n(x)=a_{n+1}^2+a_n^2+x^2.$$

(V) *Asymptotic behavior of the coefficients of the three term recurrence relation (see [15], [20])*

$$a_n^2=\left(\frac{n}{12}\right)^{\frac{1}{2}}\left[1+\frac{1}{24n^2}+\mathcal{O}(n^{-4})\right] \tag{7}$$

(VI) *Strong inner asymptotics (see [19, Th. 1], and eq.(8) in [20]). Let $\{f_n\}_{n\geq 0}$ denote the sequence of polynomials orthonormal with respect to (1). That is,*

$$f_n(x)=\gamma_nF_n(x)=\gamma_nx^n+\textit{lower degree terms},$$

where $\gamma_n=\left(||F_n||^2\right)^{-1/2}>0$. Then,

$$\begin{aligned} f_n\left(x\right) &= Ae^{x^4/2}n^{-1/8}\times \\ \sin&\left\{\left(\frac{64}{27}\right)^{1/4}n^{3/4}x+12^{-1/4}n^{1/4}x^3-\frac{n-1}{2}\pi\right\}+o(n^{-1/8}), \end{aligned} \tag{8}$$

where $A=\sqrt[8]{12}/\sqrt{\pi}$, uniformly for x in every compact subset $\Delta\subset\mathbb{R}$.

The kernel polynomials associated with the polynomial sequence $\{F_n\}_{n\geq 0}$ will play a key role in order to prove some of the results of the manuscript. The n-th degree reproducing kernel associated with $\{F_n\}_{n\geqslant 0}$ is (see [7, Ch. I-7], [22])

$$K_n(x,y)=\sum_{k=0}^{n}\frac{F_k(x)F_k(y)}{||F_k||^2}.$$

For $x\neq y$, the Christoffel-Darboux formula reads

$$K_n(x,y)=\frac{1}{||F_n||^2}\frac{F_{n+1}(x)F_n(y)-F_{n+1}(y)F_n(x)}{x-y}, \tag{9}$$

and its confluent expression becomes

$$K_n(x,x)=\sum_{k=0}^{n}\frac{[F_k(x)]^2}{||F_k||^2}=\frac{F_{n+1}'(x)F_n(x)-F_n'(x)F_{n+1}(x)}{||F_n||^2}. \tag{10}$$

We introduce the following standard notation for the partial derivatives of the n-th degree kernel $K_n(x,y)$

$$\frac{\partial^{j+k}K_n\left(x,y\right)}{\partial^jx\partial^ky}=:K_n^{(j,k)}\left(x,y\right), \quad 0\leq j,k\leq n. \tag{11}$$

Thus,

$$K_{n-1}^{(0,1)}(x,0) = K_{n-1}^{(1,0)}(0,x) = \frac{1}{||F_{n-1}||^2}\times \tag{12}$$

$$\left[\frac{F_n(x)F_{n-1}(0) - F_{n-1}(x)F_n(0)}{x^2} + \frac{F_n(x)F'_{n-1}(0) - F_{n-1}(x)F'_n(0)}{x}\right],$$

and, considering the coefficient of x in the above expression, we have

$$K_{n-1}^{(1,1)}(0,0) = \frac{1}{||F_{n-1}||^2}\times$$

$$\left[\frac{F'''_n(0)F_{n-1}(0) - F'''_{n-1}(0)F_n(0)}{6} + \frac{F''_n(0)F'_{n-1}(0) - F''_{n-1}(0)F'_n(0)}{2}\right].$$

From (10)

$$K_{n-1}(0,0) = \frac{F'_n(0)F_{n-1}(0) - F'_{n-1}(0)F_n(0)}{||F_{n-1}||^2},$$

and taking limit in (12) when $x \to 0$, we get

$$K_{n-1}^{(0,1)}(0,0) = K_{n-1}^{(1,0)}(0,0) = \frac{1}{||F_{n-1}||^2}\frac{F''_n(0)F_{n-1}(0) - F''_{n-1}(0)F_n(0)}{2}.$$

Taking a suitable index shifting in the last three expressions, we conclude

$$\begin{aligned} K_{2n-1}(0,0) &= \frac{-F'_{2n-1}(0)F_{2n}(0)}{||F_{2n-1}||^2}, \\ K_{2n-1}^{(0,1)}(0,0) &= K_{2n-1}^{(1,0)}(0,0) = 0, \\ K_{2n-1}^{(1,1)}(0,0) &= \frac{1}{||F_{2n-1}||^2}\left[\frac{F''_{2n}(0)F'_{2n-1}(0)}{2} - \frac{F'''_{2n-1}(0)F_{2n}(0)}{6}\right] \end{aligned} \tag{13}$$

as well as

$$\begin{aligned} K_{2n}(0,0) &= \frac{F'_{2n+1}(0)F_{2n}(0)}{||F_{2n}||^2}, \\ K_{2n}^{(0,1)}(0,0) &= K_{2n}^{(1,0)}(0,0) = 0, \\ K_{2n}^{(1,1)}(0,0) &= \frac{1}{||F_{2n}||^2}\left[\frac{F'''_{2n+1}(0)F_{2n}(0)}{6} - \frac{F''_{2n}(0)F'_{2n+1}(0)}{2}\right]. \end{aligned} \tag{14}$$

Another interesting property of the Freud kernels arises from the symmetry of $\{F_n(x)\}_{n\geq 0}$. From (10) and (11) we have

$$\begin{aligned} K_{2n+1}(x,0) &= \sum_{i=0}^{n-1} \frac{F_{2i}(0)}{\|F_{2i}\|^2} F_{2i}(x) = K_{2n}(x,0), \\ K_{2n}^{(0,1)}(x,0) &= K_{2n-1}^{(0,1)}(x,0), \\ K_{2n}^{(1,1)}(x,0) &= \frac{F'_{2n}(x)F'_{2n}(0)}{\|F_{2n}\|^2} + K_{2n-1}^{(1,1)}(x,0) = K_{2n-1}^{(1,1)}(x,0), \end{aligned}$$

This fact will be useful throughout the paper.

On the other hand, $\{F_n^{[2]}\}_{n\geq 0}$ will denote the sequence of 2-iterated monic Freud kernel polynomials, orthogonal with respect to the inner product

$$\langle p,q\rangle_{[2]} = \int_{\mathbb{R}} p(x)q(x)x^2e^{-x^4}dx, \tag{15}$$

which is the 2-*iterated Christoffel transformation of* μ (see [24]). We will denote by

$$||F_n^{[2]}||_{[2]}^2 = \langle F_n^{[2]}, F_n^{[2]}\rangle_{[2]} = \int_{\mathbb{R}} |F_n^{[2]}(x)|^2 x^2 e^{-x^4} dx$$

the corresponding norm. For convenience of the reader, we briefly reproduce here a useful Lemma already proved in ([3, Lemmas 1 and 2]), concerning sequences of 2-iterated monic Freud kernel polynomials.

Lemma 1 (Christoffel formula). *The 2-iterated Freud kernel polynomials and the Freud orthogonal polynomials satisfy the connection formulas,*

$$\begin{aligned} x^2 F_{2n-1}^{[2]}(x) &= F_{2n+1}(x) + d_{2n-1}\, F_{2n-1}(x), \quad n \geq 1, \\ x^2 F_{2n}^{[2]}(x) &= F_{2n+2}(x) + d_{2n}\, F_{2n}(x), \quad n \geq 1, \end{aligned}$$

where

$$d_{2n-1} = \frac{||F_{2n-1}^{[2]}||_{[2]}^2}{||F_{2n-1}||^2} = \frac{-F'_{2n+1}(0)}{F'_{2n-1}(0)}, \quad d_{2n} = \frac{||F_{2n}^{[2]}||_{[2]}^2}{||F_{2n}||^2} = \frac{-F_{2n+2}(0)}{F_{2n}(0)}.$$

Furthermore,

$$\begin{aligned} x^2 F_{2n-1}^{[2]}(x) &= xF_{2n}(x) + (d_{2n-1} - a_{2n}^2)\, F_{2n-1}(x), \\ xF_{2n}^{[2]}(x) &= F_{2n+1}(x), \end{aligned}$$

where $d_{2n-1} - a_{2n}^2 > 0$.

As a summary,

$$x^2 F_n^{[2]}(x) = F_{n+2}(x) + d_n\, F_n(x), \quad n \geq 0. \tag{16}$$

2.1. A general connection formula

Let us consider the aforementioned Sobolev-type inner product (3). In the sequel, we will denote by $\{Q_n(x)\}_{n\geq 0}$ the corresponding sequence of monic orthogonal polynomials and by

$$||Q_n||_s^2 = \langle Q_n, x^n \rangle_s$$

the norm of the n-th degree polynomial. The connection formula between $\{Q_n(x)\}_{n\geq 0}$ and $\{F_n(x)\}_{n\geq 0}$ is stated in the following lemma.

Lemma 2. *[1] For* $n \geq 1$, *we have*

$$Q_n(x) = F_n\,(x) - \sum_{k=0}^{s} M_k Q_n^{(k)}(0) K_{n-1}^{(0,k)}(x,0), \tag{17}$$

where, for $0 \leq k \leq s$,

$$Q_n^{(k)}(0) = (\det D)^{-1} \begin{vmatrix} 1 + M_0 K_{n-1}^{(0,0)}(0,0) & \cdots & F_n(0) & \cdots & M_s K_{n-1}^{(0,s)}(0,0) \\ M_0 K_{n-1}^{(1,0)}(0,0) & \cdots & F'_n(0) & \cdots & M_s K_{n-1}^{(1,s)}(0,0) \\ \vdots & & \vdots & \ddots & \vdots \\ M_0 K_{n-1}^{(s,0)}(0,0) & \cdots & F_n^{(s)}(0) & \cdots & 1 + M_s K_{n-1}^{(s,s)}(0,0) \end{vmatrix},$$

with

$$D = \begin{bmatrix} 1 + M_0 K_{n-1}^{(0,0)}(0,0) & M_1 K_{n-1}^{(0,1)}(0,0) & \cdots & M_s K_{n-1}^{(0,s)}(0,0) \\ M_0 K_{n-1}^{(1,0)}(0,0) & 1 + M_1 K_{n-1}^{(1,1)}(0,0) & \cdots & M_s K_{n-1}^{(1,s)}(0,0) \\ \vdots & \vdots & \ddots & \vdots \\ M_0 K_{n-1}^{(s,0)}(0,0) & M_1 K_{n-1}^{(s,1)}(0,0) & \cdots & 1 + M_s K_{n-1}^{(s,s)}(0,0) \end{bmatrix}.$$

Moreover, an easy computation shows that

$$K_{n-1}^{(0,k)}(x,0)=\frac{1}{||F_{n-1}||^2}\left(\sum_{\eta=0}^{k}\frac{k!}{\eta!}\frac{F_n(x)F_{n-1}^{(\eta)}(0)-F_{n-1}(x)F_n^{(\eta)}(0)}{x^{k-\eta+1}}\right),$$

and, as a consequence, we can write (17) as

$$x^{s+1}Q_n(x)=\mathcal{A}_s(n;x)F_n(x)+\mathcal{B}_s(n;x)F_{n-1}(x), \tag{18}$$

where

$$\begin{aligned}\mathcal{A}_s(n;x) &= \sum_{k=0}^{s}\left(x^{s+1}-\sum_{\eta=0}^{k}\frac{k!}{\eta!}\frac{M_kQ_n^{(k)}(0)F_{n-1}^{(\eta)}(0)}{||F_{n-1}||^2}x^{s-k+\eta}\right),\\ \mathcal{B}_s(n;x) &= \sum_{k=0}^{s}\left(\sum_{\eta=0}^{k}\frac{k!}{\eta!}\frac{M_kQ_n^{(k)}(0)F_n^{(\eta)}(0)}{||F_{n-1}||^2}x^{s-k+\eta}\right),\end{aligned}$$

are polynomials of degree $s+1$ and s, respectively.

2.2. Connection formula for the case $s=1$

In what follows, we restrict ourselves to study the case of only one mass point with derivative in the inner product (1), i.e., $s=1$, $M_0\geq 0$, and $M_1>0$,

$$\langle p,q\rangle_1=\langle p,q\rangle+M_0p(0)q(0)+M_1p'(0)q'(0). \tag{19}$$

In such a case, the connection formula (18) becomes

$$x^2Q_n(x)=\mathcal{A}_1(n;x)F_n(x)+\mathcal{B}_1(n;x)F_{n-1}(x), \tag{20}$$

where $\mathcal{A}_1(n;x)=x^2+\mathcal{A}_{10}(n)$, and $\mathcal{B}_1(n;x)=\mathcal{B}_{11}(n)x$ with

$$\mathcal{A}_{10}(n)=-\frac{M_1Q_n'(0)F_{n-1}(0)}{||F_{n-1}||^2},\quad \mathcal{B}_{11}(n)=\frac{M_0Q_n(0)F_n(0)+M_1Q_n'(0)F_n'(0)}{||F_{n-1}||^2}.$$

To obtain $Q_n(0)$ and $Q_n'(0)$ in the above expression, we evaluate (20) at $x=0$ and solve the corresponding linear system. Indeed,

$$\begin{aligned}Q_n(0) &= \frac{\begin{vmatrix}F_n(0) & M_1K_{n-1}^{(0,1)}(0,0)\\ F_n'(0) & 1+M_1K_{n-1}^{(1,1)}(0,0)\end{vmatrix}}{\begin{vmatrix}1+M_0K_{n-1}(0,0) & M_1K_{n-1}^{(0,1)}(0,0)\\ M_0K_{n-1}^{(1,0)}(0,0) & 1+M_1K_{n-1}^{(1,1)}(0,0)\end{vmatrix}},\\ Q_n'(0) &= \frac{\begin{vmatrix}1+M_0K_{n-1}(0,0) & F_n(0)\\ M_0K_{n-1}^{(1,0)}(0,0) & F_n'(0)\end{vmatrix}}{\begin{vmatrix}1+M_0K_{n-1}(0,0) & M_1K_{n-1}^{(0,1)}(0,0)\\ M_0K_{n-1}^{(1,0)}(0,0) & 1+M_1K_{n-1}^{(1,1)}(0,0)\end{vmatrix}}.\end{aligned}$$

As a consequence,

$$Q_n(0) = \frac{F_n(0)}{[1+M_0K_{n-1}(0,0)]}, \tag{21}$$

$$Q_n'(0) = \frac{F_n'(0)}{1+M_1K_{n-1}^{(1,1)}(0,0)}. \tag{22}$$

Thus,

$$\begin{aligned} Q_{2n}(0) &= \frac{F_{2n}(0)}{[1+M_0K_{2n-2}(0,0)]}, & Q_{2n+1}(0) &= 0, \\ Q'_{2n+1}(0) &= \frac{F'_{2n+1}(0)}{1+M_1K^{(1,1)}_{2n-1}(0,0)}, & Q'_{2n}(0) &= 0. \end{aligned} \tag{23}$$

Now, we obtain connection formulas that relate both families of monic orthogonal polynomials.

Proposition 2. *The Freud-Sobolev type orthogonal polynomials satisfy*

$$x^2Q_n(x) = \left[x^2 - \frac{r_n\kappa_n^{[1]}}{4\phi_n(0)}\right]F_n(x) + a_n^2\left(\kappa_n^{[0]}+\kappa_n^{[1]}\right)xF_{n-1}(x), \quad n \geq 1, \tag{24}$$

where

$$\kappa_n^{[0]} = \frac{1+M_0K_n(0,0)}{1+M_0K_{n-1}(0,0)} - 1, \quad \kappa_n^{[1]} = \frac{1+M_1K_n^{(1,1)}(0,0)}{1+M_1K_{n-1}^{(1,1)}(0,0)} - 1, \quad r_n = \frac{1-(-1)^n}{2}.$$

Moreover, for the even and odd degrees, respectively, we have

$$Q_{2n}(x) = F_{2n}(x) - M_0\frac{F_{2n}(0)}{[1+M_0K_{2n-2}(0,0)]}K_{2n-2}(x,0), \quad n \geq 1, \tag{25}$$

$$Q_{2n+1}(x) = F_{2n+1}(x) - M_1\frac{F'_{2n+1}(0)}{1+M_1K^{(1,1)}_{2n-1}(0,0)}K^{(0,1)}_{2n-1}(x,0), \quad n \geq 1. \tag{26}$$

In other words, Q_{2n} (resp. Q_{2n+1}) is an even (resp. odd) polynomial.

Proof. Setting $s=1$ in (17) we get

$$Q_n(x) = F_n(x) - M_0Q_n(0)K_{n-1}(x,0) - M_1Q'_n(0)K^{(0,1)}_{n-1}(x,0). \tag{27}$$

From (12) we have

$$K^{(0,1)}_{n-1}(x,0) = \left(\frac{F_{n-1}(0)+xF'_{n-1}(0)}{x^2||F_{n-1}||^2}\right)F_n(x) - \left(\frac{F_n(0)+xF'_n(0)}{x^2||F_{n-1}||^2}\right)F_{n-1}(x),$$

and taking into account (9), (21), (22), and the symmetry of the Freud polynomials, we get

$$\begin{aligned} Q_n(x) &= \left[1 - \frac{F_{n-1}(0)}{F'_n(0)}\frac{||F_n||^2}{x^2||F_{n-1}||^2}\left(\frac{1+M_1K_n^{(1,1)}(0,0)}{1+M_1K_{n-1}^{(1,1)}(0,0)} - 1\right)\right]F_n(x) \\ &+ \frac{a_n^2}{x}\left[\left(\frac{1+M_0K_n(0,0)}{1+M_0K_{n-1}(0,0)} - 1\right) + \left(\frac{1+M_1K_n^{(1,1)}(0,0)}{1+M_1K_{n-1}^{(1,1)}(0,0)} - 1\right)\right]F_{n-1}(x). \end{aligned}$$

Therefore, noticing that from (5) we have $F_{n-1}(0)/F'_n(0) = 1/4a_n^2\phi_n(0)$, we obtain

$$Q_n(x) = \left[1 - \frac{r_n}{4x^2\phi_n(0)}\kappa_n^{[1]}\right]F_n(x) + \frac{a_n^2}{x}\left(\kappa_n^{[0]}+\kappa_n^{[1]}\right)F_{n-1}(x), \tag{28}$$

which is (24). On the other hand, shifting the index $n \to 2n$, and taking into account (23) we obtain (25). For the odd case, (26) follows similarly by using (23). ■

Remark 1. *Notice that, from the symmetry of the Freud polynomials, we have $\kappa_{2n+1}^{[0]} = 0$ and $\kappa_{2n}^{[1]} = 0$ for $n \geq 1$. As a consequence, (24) becomes*

$$\begin{aligned} xQ_{2n}(x) &= xF_{2n}(x) + a_{2n}^2\kappa_{2n}^{[0]}F_{2n-1}(x), \quad n \geq 1, \\ x^2Q_{2n+1}(x) &= \left[x^2 - \frac{r_{2n+1}\kappa_{2n+1}^{[1]}}{4\phi_{2n+1}(0)}\right]F_{2n+1}(x) + a_{2n+1}^2\kappa_{2n+1}^{[1]}xF_{2n}(x), \quad n \geq 1. \end{aligned}$$

At this point, observe the remarkable fact, which is a straightforward consequence of (19), that the multiplication operator by x^2 is a symmetric operator with respect to such a discrete Sobolev inner product. Indeed, for polynomials $h(x), g(x) \in \mathbb{P}$

$$\langle x^2 h(x), g(x)\rangle_1 = \langle h(x), x^2 g(x)\rangle_1, \tag{29}$$

and also notice that

$$\langle x^2 h(x), g(x)\rangle_1 = \langle h(x), g(x)\rangle_{[2]}. \tag{30}$$

An equivalent formulation of (30) is

$$\langle x^2 h(x), g(x)\rangle_1 = \langle x^2 h(x), g(x)\rangle. \tag{31}$$

The relation between the norms of $Q_n(x)$ and $F_n(x)$ is given in the following result.

Proposition 3. *For $n \geq 1$, we have*

$$\begin{aligned}\frac{\|F_{2n}\|^2}{\|Q_{2n}\|_1^2} &= \frac{1 + M_0 K_{2n-2}(0,0)}{1 + M_0 K_{2n}(0,0)},\\ \frac{\|F_{2n+1}\|^2}{\|Q_{2n+1}\|_1^2} &= \frac{1 + M_1 K_{2n-1}^{(1,1)}(0,0)}{1 + M_1 K_{2n+1}^{(1,1)}(0,0)}.\end{aligned} \tag{32}$$

Proof. Shifting the index $n \to 2n$ in (24) yields

$$x^2 Q_{2n}(x) = x^2 F_{2n}(x) + a_{2n}^2 \kappa_{2n}^{[0]} x F_{2n-1}(x),$$

because $\kappa_{2n}^{[1]} = 0$. Next, we multiply all the above equation by $F_{2n-2}(x)$ and we apply the inner product $\langle \cdot, \cdot \rangle_1$. Thus

$$\langle x^2 Q_{2n}(x), F_{2n-2}(x)\rangle_1 = \langle x^2 F_{2n}(x), F_{2n-2}(x)\rangle_1 + a_{2n}^2 \kappa_{2n}^{[0]} \langle x F_{2n-1}(x), F_{2n-2}(x)\rangle_1.$$

Using (29), (30) and (31) we deduce

$$||Q_{2n}||_1^2 = ||F_{2n}||^2 + a_{2n}^2 \kappa_{2n}^{[0]} ||F_{2n-1}||^2.$$

From Proposition 1-(2) we know $a_n^2 = ||F_n||^2 \big/ ||F_{n-1}||^2$, and therefore

$$||Q_{2n}||_1^2 = ||F_{2n}||^2 \left(1 + \kappa_{2n}^{[0]}\right)$$

which is the first equation of (32). Similar considerations lead to prove the second equation of (32), after shifting the index $n \to 2n+1$ in (24). ■

Remark 2. *Notice that, by defining $Q_{2n}(x) := P_n(x^2)$ and $Q_{2n+1}(x) := x R_n(x^2)$, $n \geq 0$, and introducing the change of variable $x = \sqrt{y}$, we obtain the following orthogonality relations*

$$\begin{aligned} 0 &= \langle Q_{2n}, Q_{2m}\rangle_1 = \int_0^\infty P_n(y) P_m(y) y^{-1/2} e^{-y^2} dy + M_0 P_n(0) P_m(0), \quad n \neq m \\ 0 &= \langle Q_{2n+1}, Q_{2m+1}\rangle_1 = \int_0^\infty R_n(y) R_m(y) y^{1/2} e^{-y^2} dy + M_1 R_n(0) R_m(0), \quad n \neq m, \end{aligned}$$

i.e. $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are MOPS with respect to standard inner products associated with the measures $d\sigma(x) = x^{-1/2} e^{-x^2} dx + M_0 \delta(x)$ and $x d\sigma(x) + M_1 \delta(x)$, respectively, supported on the positive real semiaxis.

To conclude this Section, we point out that the results in Propositions 3 and 2, as well as in Remark 1, appear also in [5], in a slightly different form. We have included our proofs here for the sake of completeness.

3. Computation of zeros

In this section, we propose an algorithm to compute in an efficient way the zeros of Freud-type orthogonal polynomials. It is based on the relation between the Jacobi matrix associated with such polynomials, and the Jacobi matrix associated with the 2-iterated monic Freud kernel polynomials defined previously.

Now, we obtain connection formulas that relate $\{Q_n\}_{n\geq 0}$ with $\{F_n^{[2]}\}_{n\geq 0}$.

Proposition 4. *Let $\{Q_n\}_{n\geq 0}$ be the MOPS associated with (19), and let $\{F_n^{[2]}\}_{n\geq 0}$ denote the 2-iterated monic Freud kernel polynomials defined by (15). Then, we have*

$$Q_n(x) = F_n^{[2]}(x) + c_n F_{n-2}^{[2]}(x), \tag{33}$$

where

$$c_n = \frac{\|F_n\|^2 + d_{n-2}[s_n F_{n-2}(0) + t_n F'_{n-2}(0)]}{d_{n-2}\|F_{n-2}\|^2}, \tag{34}$$

with $s_n = -\frac{M_0 F_n(0)}{[1+M_0 K_{n-1}(0,0)]}$ *and* $t_n = -\frac{M_1 F'_n(0)}{[1+M_1 K_{n-1}^{(1,0)}(0,0)]}$.

Proof. We can expand $Q_n(x)$ in terms of the SMOP $\{F_n^{[2]}(x)\}_{n\geq 0}$ as

$$Q_n(x) = F_n^{[2]}(x) + \sum_{k=0}^{n-1} \alpha_{n,k} F_k^{[2]}(x),$$

with

$$\alpha_{n,k} = \frac{\langle Q_n, F_k^{[2]}\rangle_{[2]}}{||F_k^{[2]}||^2_{[2]}}.$$

From (30), the above coefficient becomes

$$\alpha_{n,k} = \frac{\langle x^2 Q_n, F_k^{[2]}\rangle_1}{||F_k^{[2]}||^2_{[2]}},$$

which, by orthogonality, it yields $\alpha_{n,k} = 0$ for $0 \leq k \leq n-3$ and, after some computations, it is not difficult to deduce that

$$\alpha_{n,n-1} = \frac{\langle Q_n(x), x^2 F_{n-1}^{[2]}(x)\rangle}{\|F_{n-1}^{[2]}\|^2_{[2]}} = 0.$$

Thus, since both families are monic, we have (33) with

$$c_n = \frac{\langle Q_n(x), x^2 F_{n-2}^{[2]}(x)\rangle}{\|F_{n-2}^{[2]}\|^2_{[2]}}.$$

Now, taking into account (21), (22) and (27) we have

$$Q_n(x) = F_n(x) + s_n K_{n-1}(x,0) + t_n K_{n-1}^{(0,1)}(x,0), \tag{35}$$

where

$$s_n = -M_0 Q_n(0) = -M_0 \frac{F_n(0)}{[1+M_0 K_{n-1}(0,0)]}, \quad t_n = -M_1 Q'_n(0) = -M_1 \frac{F'_n(0)}{1+M_1 K_{n-1}^{(1,1)}(0,0)}.$$

Next, combining (35) with (16) we deduce

$$\begin{aligned}\langle Q_n(x), x^2 F_{n-2}^{[2]}(x)\rangle &= \langle F_n(x) + s_n K_{n-1}(x,0) + t_n K_{n-1}^{(0,1)}(x,0), F_n(x) + d_{n-2}F_{n-2}(x)\rangle \\ &= \|F_n\|^2 + d_{n-2}[s_n F_{n-2}(0) + t_n F'_{n-2}(0)].\end{aligned}$$

Finally, since

$$\begin{aligned}
\|F_n^{[2]}\|_{[2]}^2 &= \int_0^\infty \left(F_n^{[2]}\right)^2 x^2 e^{x^4} dx \\
&= \int_0^\infty F_n^{[2]}[F_{n+2}(x) + d_n F_n(x)] e^{x^4} dx \\
&= d_n \|F_n\|^2,
\end{aligned}$$

the result follows. ∎

Notice that defining $\mathbf{Q}_n = [Q_0(x), Q_1(x), \ldots, Q_n(x)]^T$ and $\mathbf{F}_n^{[2]} = [F_0^{[2]}(x), F_1^{[2]}(x), \ldots, F_n^{[2]}(x)]^T$, (33) can be written in matrix form as

$$\mathbf{Q}_n = \mathbf{H}_n \mathbf{F}_n^{[2]}, \tag{36}$$

where $\mathbf{H}_n$ is the $(n+1)\times(n+1)$ lower tridiagonal matrix

$$\mathbf{H}_n = \begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ c_2 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & c_n & 0 & 1 \end{pmatrix}.$$

On the other hand, let $\mathbf{S}_n$ be the truncated Jacobi matrix associated with $\{F_n^{[2]}\}_{n\geq 0}$, and let $\tilde{\mathbf{J}}_n$ the Hessenberg matrix associated with the multiplication operator with respect to the basis $\{Q_n\}_{n\geq 0}$, then

$$\begin{aligned}
x\mathbf{Q}_n &= \tilde{\mathbf{J}}_n \mathbf{Q}_n + Q_{n+1}\mathbf{e}_{n+1}\,, \\
x\mathbf{F}_n^{[2]} &= \mathbf{S}_n \mathbf{F}_n^{[2]} + F_{n+1}^{[2]}\mathbf{e}_{n+1}\,,
\end{aligned}$$

where $\mathbf{e}_{n+1} = [0, \ldots, 0, 1]^T$. Using (36) and (33), we have

$$\begin{aligned}
x\mathbf{Q}_n &= \tilde{\mathbf{J}}_n \mathbf{Q}_n + Q_{n+1}\mathbf{e}_{n+1} \\
x\mathbf{H}_n\mathbf{F}_n^{[2]} &= \tilde{\mathbf{J}}_n \mathbf{H}_n \mathbf{F}_n^{[2]} + [F_{n+1}^{[2]}(x) + c_{n+1}F_{n-1}^{[2]}(x)]\mathbf{e}_{n+1} \\
&= \tilde{\mathbf{J}}_n \mathbf{H}_n \mathbf{F}_n^{[2]} + \mathbf{A}_n \mathbf{F}_n^{[2]} + F_{n+1}^{[2]}(x)\mathbf{e}_{n+1}\,,
\end{aligned}$$

where $\mathbf{A}_n$ is the $(n+1)\times(n+1)$ matrix

$$\mathbf{A}_n = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & & \cdots & 0 \\ 0 & \cdots & 0 & c_{n+1} & 0 \end{pmatrix}.$$

Furthermore, since $\mathbf{H}_n^{-1}$ is a lower triangular matrix with ones on the diagonal, we clearly have $\mathbf{H}_n^{-1}\mathbf{A}_n = \mathbf{A}_n$ and $\mathbf{H}_n^{-1}\mathbf{e}_{n+1} = \mathbf{e}_{n+1}$ and, as a consequence,

$$x\mathbf{F}_n^{[2]} = \mathbf{H}_n^{-1}\tilde{\mathbf{J}}_n \mathbf{H}_n \mathbf{F}_n^{[2]} + \mathbf{A}_n \mathbf{F}_n^{[2]} + F_{n+1}^{[2]}(x)\mathbf{e}_{n+1},$$

so that

$$\mathbf{S}_n = \mathbf{H}_n^{-1}\tilde{\mathbf{J}}_n \mathbf{H}_n + \mathbf{A}_n.$$

Therefore, we have proved the following proposition.

Proposition 5. *The following expression holds*

$$\tilde{\mathbf{J}}_n = \mathbf{H}_n[\mathbf{S}_n - \mathbf{A}_n]\mathbf{H}_n^{-1}.$$

In other words, $\tilde{\mathbf{J}}_n$ is similar to a rank one perturbation of $\mathbf{S}_n$.

Now, since the zeros of $Q_n(x)$ are the eigenvalues of $\tilde{\mathbf{J}}_n$, and since similar matrices have the same eigenvalues, the zeros of $Q_n(x)$ can be computed as describes Algorithm 1.

Algorithm 1: Algorithm for computing the zeros of the Freud-type orthogonal polynomials $\{Q_k\}_{k=1}^{n}$

```
Input: Matrices A_k, S_k, k = 1, ..., n.
Output: Zeros of Q_k(x), k = 1, ..., n.
initialization;
for k = 1, 2, ... n do
    Compute the truncated Jacobi matrix S_k
    Find the matrix A_k by using (34)
    Compute numerically the eigenvalues of S_k − A_k
    return Zeros of Q_k(x) are the eigenvalues of S_k − A_k
```

Notice that only information related to $\{F_n\}_{n\geq 0}$ and $\{F_n^{[2]}\}_{n\geq 0}$ is required. On the other hand, to compute the Jacobi matrix associated with $\{Q_n\}_{n\geq 0}$, we also need to compute the matrix $\mathbf{H}_n$, using (34).

4. Analytic properties of zeros

In this Section we analyze some properties of the zeros of the polynomials $\{Q_n(x)\}_{n\geq 0}$. Notice that inn our case the mass point is located in the support of the measure, while in the literature the mass point is located either in the boundary or outside the support of the measure (see [1], [16], among others).

4.1. Interlacing rupture

From (25) and (26), it is clear that the zeros of even $Q_{2n}(x)$ and odd $Q_{2n+1}(x)$ Freud-Sobolev type polynomials act in an independent way. From those expressions, we observe that the variation of M_0 (respectively M_1) exclusively influences the position of the zeros of $Q_{2n}(x)$ (respectively $Q_{2n+1}(x)$) without affecting the zeros of $Q_{2n+1}(x)$ (respectively $Q_{2n}(x)$). This interesting phenomena leads to the destruction of the zero interlacing for two consecutive polynomials of the sequence $\{Q_n(x)\}_{n\geq 0}$ for certain values of M_0 and M_1. Notice that the zeros of $Q_n(x), n \geq 1$, are real and simple (see [16], Proposition 3.2). In the next two tables we provide numerical evidence that supports this fact. In the sequel, let $\{\eta_{n,k}\}_{k=0}^{n} \equiv \eta_{n,1} < \eta_{n,2} < \ldots < \eta_{n,n}$ be the zeros of $Q_n(x)$ and $\{x_{n,k}\}_{k=0}^{n}$ be the zeros of $F_n(x)$ arranged in an increasing order. Next we show the position of the second zero of the Freud-Sobolev-type polynomial of degree $n = 4$ (namely $Q_4(x)$) and the second and third zeros of $Q_5(x)$ for some choices of the masses M_0 and M_1. For $M_0 = M_1 = 0$ we obviously recover the corresponding zeros of the Freud polynomials. The first table shows the position of the aforementioned zeros for $M_0 = 0$ and several values for M_1. The cases when between the second and third (resp. third and fourth) zeros of $Q_5(x)$ there are no zeros of $Q_4(x)$, i.e. the zero interlacing for the sequence $\{Q_n(x)\}_{n\geq 0}$ fails, are shown in bold.

	$M_0 = 0.0$				
	$\eta_{5,2}$	$\eta_{4,2}$	$\eta_{5,3}$	$\eta_{4,3}$	$\eta_{5,4}$
$M_1 = 0.0$	-0.655248	-0.39615	0.0	0.39615	0.655248
$M_1 = 0.2$	-0.458455	-0.39615	0.0	0.39615	0.458455
$M_1 = 0.4$	$-$**0.371898**	-0.39615	0.0	0.39615	**0.371898**
$M_1 = 1.0$	$-$**0.261023**	-0.39615	0.0	0.39615	**0.261023**

Table 1: Zeros of $Q_5(x)$ and $Q_4(x)$ for fixed $M_0 = 0.0$ and some values of M_1.

Observe that, as expected, the variation of M_1 only affects the position of $\eta_{5,2}$ and $\eta_{5,4}$ and the variation of M_0 only affects the position of $\eta_{4,2}$ and $\eta_{4,4}$. This numerical example is also reflected in Figure 1.

	$M_0 = 1.0$				
	$\eta_{5,2}$	$\eta_{4,2}$	$\eta_{5,3}$	$\eta_{4,3}$	$\eta_{5,4}$
$M_1 = 0.0$	-0.655248	-0.284325	0.0	0.284325	0.655248
$M_1 = 0.4$	-0.371898	-0.284325	0.0	0.284325	0.371898
$M_1 = 0.9$	$-$**0.272822**	-0.284325	0.0	0.284325	**0.272822**
$M_1 = 2.0$	$-$**0.192081**	-0.284325	0.0	0.284325	**0.192081**

Table 2: Zeros of $Q_5(x)$ and $Q_4(x)$ for fixed $M_0 = 1.0$ and some values of M_1.

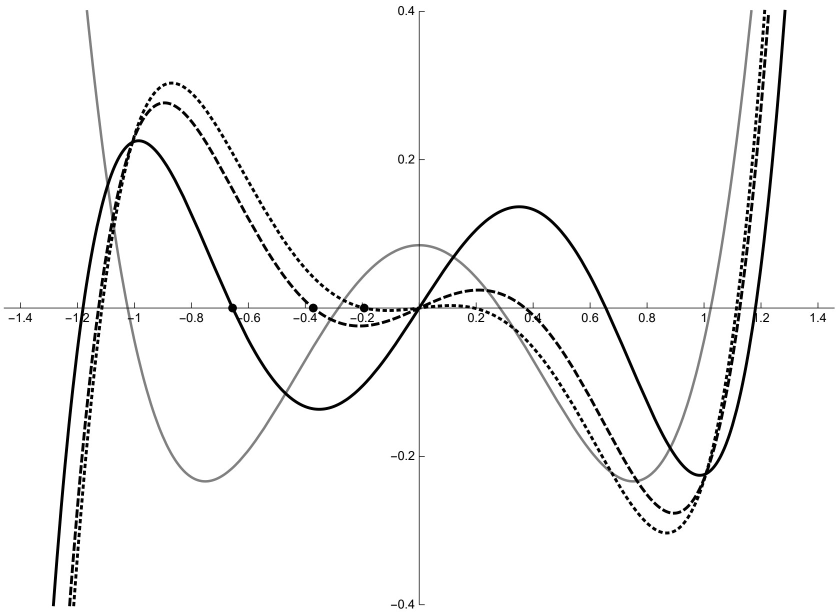

Figure 1: The figure shows, for a fixed value $M_0 = 1$, the evolution of the second zero of the Freud-Sobolev type polynomial $Q_5(x)$ for three different values of the mass M_1. The curve in gray color represents the Freud-Sobolev type $Q_4(x)$, which is not affected by the variation of M_1. The zero of $Q_5(x; M_1 = 0) = F_5(x)$ (continuous black graph) occurs at $\eta_{5,2}(M_1 = 0) = -0.655248$. For $Q_5(x; M_1 = 0.2)$ (dashed line) we have $\eta_{5,2}(M_1 = 0.2) = -0.371898$ and for $Q_5(x; M_1 = 2)$ (dotted line) occurs at $\eta_{5,2}(M_1 = 2) = -0.19208$. Notice that for $M_0 = 1$ and $M_1 = 2$ there is no zero of the polynomial $Q_4(x)$ between the second ($\eta_{5,2}(M_1 = 2) = -0.19208$) and third ($\eta_{5,3}(M_1 = 2) = 0$) roots of $Q_5(x; M_1 = 2)$, so the interlacing of the complete Freud-Sobolev type orthogonal polynomial sequence $\{Q_n(x)\}_{n\geq 0}$ has been broken.

4.2. Asymptotic behavior

We are interested in the dynamics of the zeros of the Freud-Sobolev type when M_0 and M_1 tend, respectively, to infinity. To that end, let us introduce the following the limit polynomials

$$\begin{aligned} G_{2n}(x) &= \lim_{M_0\to\infty} Q_{2n}(x) = F_{2n}(x) - \frac{F_{2n}(0)}{K_{2n-2}(0,0)} K_{2n-2}(x,0), \\ J_{2n+1}(x) &= \lim_{M_1\to\infty} Q_{2n+1}(x) = F_{2n+1}(x) - \frac{F'_{2n+1}(0)}{K^{(1,1)}_{2n-1}(0,0)} K^{(0,1)}_{2n-1}(x,0). \end{aligned} \tag{37}$$

Similar polynomials have been previously studied in [16], when the discrete mass points are located outside the support of the perturbed measure. Here, we find a slightly different situation because the support of the measure is the whole real line and the discrete masses M_0 and M_1 are both located at $x = 0 \in \mathbb{R}$. As stated before, M_0 only affects the even degree polynomials, and the dynamics for the zeros of $\{Q_{2n}(x)\}_{n\geq 0}$ has been already obtained in [3]. Next, we extend those results for the odd sequence $\{Q_{2n+1}(x)\}_{n\geq 0}$.

Our goal is to obtain results concerning the monotonicity and speed of convergence of the zeros of $Q_{2n+1}(x)$. For this purpose we need the following lemma concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials with interlacing zeros, whose proof we omit (see [6, Lemma 1] or [10, Lemma 3]).

Lemma 3. *Let* $\mathfrak{f}_n(x) = a(x-x_1)\cdots(x-x_n)$ *and* $\mathfrak{j}_n(x) = b(x-y_1)\cdots(x-y_n)$ *be polynomials with real and simple zeros, where* a *and* b *are positive real constants.*

If

$$y_1 < x_1 < \cdots < y_n < x_n,$$

then, for any real constant $c > 0$*, the polynomial*

$$\mathfrak{q}_n(x) = \mathfrak{f}_n(x) + c\mathfrak{j}_n(x)$$

has n *real zeros* $\eta_1 < \cdots < \eta_n$ *which interlace with the zeros of* $\mathfrak{f}_n(x)$ *and* $\mathfrak{j}_n(x)$ *as follows*

$$y_1 < \eta_1 < x_1 < \cdots < y_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ *is a decreasing function of* c *and, for each* $k = 1, \ldots, n$*,*

$$\lim_{c\to\infty} \eta_k = y_k \quad \text{and} \quad \lim_{c\to\infty} c[\eta_k - y_k] = \frac{-\mathfrak{f}_n(y_k)}{\mathfrak{j}_n'(y_k)}.$$

Before stating the main result of this Section, we will prove some auxiliary results concerning the interlacing properties of $\{F_{2n+1}\}_{n\geq 0}$, $\{K_{2n-1}^{(0,1)}(x,0)\}_{n\geq 0}$, and $\{J_{2n+1}\}_{n\geq 0}$.

Lemma 4. *The zeros of* $\{K_{2n+1}^{(0,1)}(x,0)\}_{n\geq 0}$*, are real and simple. Moreover, for every* $n \geq 1$*, the non vanishing zeros of* $K_{2n+1}^{(0,1)}(x,0)$ *and* $K_{2n-1}^{(0,1)}(x,0)$ *interlace.*

Proof. First, since $K_{2n-1}^{(0,1)}(x,0)$ is an odd polynomial, we can write $K_{2n+1}^{(0,1)}(x,0) = xs_n(x^2)$, where s_n is a polynomial of degree n. We will prove that $\{s_n(y)\}_{n\geq 0}$, with $y = x^2$, is an orthogonal polynomial sequence with respect to the measure $d\sigma(y) = y^{3/2}e^{-y^2}dy$, which is positive in the positive real line. Indeed, for $n \neq m$, we have

$$\begin{aligned}
\int_0^\infty s_n(y)s_m(y)d\sigma(y) &= \int_{-\infty}^\infty \frac{K_{2n+1}^{(0,1)}(x,0)}{x}\frac{K_{2m+1}^{(0,1)}(x,0)}{x}x^3e^{-x^4}(2xdx) \\
&= 2\int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0)K_{2m+1}^{(0,1)}(x,0)x^2e^{-x^4}dx \\
&= 0,
\end{aligned}$$

by using the reproducing property of $K_{2n-1}^{(0,1)}(x,0)$. On the other hand, for $n = m$, and taking into account (12) and the symmetry of the Freud polynomials, we get

$$\begin{aligned}
\int_0^\infty s_n^2(y)d\sigma(y) &= \int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0)K_{2n+1}^{(0,1)}(x,0)x^2e^{-x^4}dx \\
&= \int_{-\infty}^\infty K_{2n+1}^{(0,1)}(x,0)\frac{xF_{2n+2}(x)F_{2n+1}'(0) - F_{2n+1}(x)F_{2n+2}(0)}{\|F_{2n+1}\|^2}e^{-x^4}dx \\
&= \frac{1}{\|F_{2n+1}\|^2}\left((F_{2n-1}'(0))^2\|F_{2n+2}\|^2 - F_{2n+1}'(0)F_{2n+2}(0)\right) > 0,
\end{aligned}$$

since $F_{2n+1}'(0)F_{2n+2}(0) < 0$. As a consequence, the zeros of $s_n(x)$ are real, simple, and they are located in the positive real semiaxis. Moreover, the zeros of $s_n(x)$ and $s_{n-1}(x)$ interlace. Now, because of the symmetry, all polynomials of the sequence $\{K_{2n+1}^{(0,1)}(x,0)\}_{n\geq 0}$ have a zero at the origin, and the remaining zeros are located symmetrically at both sides of the origin. Furthermore, if we denote by $s_{n,k}$ the kth zero of $s_n(x)$, then it is clear from the definition that $\pm\sqrt{s_{n,k}}$ are zeros of $K_{2n+1}^{(0,1)}(x,0)$. As a consequence, the (non vanishing) zeros of $K_{2n+1}^{(0,1)}(x,0)$ and $K_{2n-1}^{(0,1)}(x,0)$ interlace. ∎

The next Lemma shows that the non vanishing zeros of F_{2n+1} and $K_{2n-1}^{(0,1)}(x,0)$ also interlace.

Lemma 5. *Let $\{x_{2n+1,k}\}_{k=1}^{2n+1}$ and $\{z_{2n-1,k}\}_{k=1}^{2n-1}$ be the set of zeros of F_{2n+1} and $K_{2n-1}^{(0,1)}(x,0)$, respectively, arranged in increasing order. Then, we have*

$$\begin{aligned} x_{2n+1,k} &< z_{2n-1,k} < x_{2n+1,k+1}, \quad 1 \leq k \leq n-1, \\ x_{2n+1,k+1} &< z_{2n-1,k} < x_{2n+1,k+2} \quad n+1 \leq k \leq 2n-1. \end{aligned}$$

Proof. Due to the symmetry of both polynomials, it suffices to prove the interlacing for the positive zeros. Since $x_{2n+1,n+1} = z_{2n-1,n} = 0$, we consider the case when $n+1 \leq k \leq 2n-1$. From (12) and the symmetry of the Freud polynomials, we have

$$\begin{aligned} x^2 K_{2n-1}^{(0,1)}(x,0) &= \frac{1}{\|F_{2n-1}\|^2} \left(x F_{2n}(x) F'_{2n-1}(0) - F_{2n-1}(x) F_{2n}(0) \right) \\ &= x F_{2n}(x) F'_{2n-1}(0) - \left(\frac{x F_{2n}(x) - F_{2n+1}(x)}{a_{2n}^2} \right) F_{2n}(0), \end{aligned}$$

where we have used (4) on the second equality. As a consequence, evaluating the previous equation in $x_{2n+1,k+1}$ and $x_{2n+1,k+2}$ we obtain, respectively,

$$\begin{aligned} x_{2n+1,k+1}^2 K_{2n-1}^{(0,1)}(x_{2n+1,k+1},0) &= x_{2n+1,k+1} F_{2n}(x_{2n+1,k+1}) \left(F'_{2n-1}(0) - \frac{F_{2n}(0)}{a_{2n}^2} \right), \\ x_{2n+1,k+2}^2 K_{2n-1}^{(0,1)}(x_{2n+1,k+2},0) &= x_{2n+1,k+2} F_{2n}(x_{2n+1,k+2}) \left(F'_{2n-1}(0) - \frac{F_{2n}(0)}{a_{2n}^2} \right). \end{aligned}$$

Since $x_{2n+1,k+1}$ and $x_{2n+1,k+2}$ are positive and the zeros of the Freud polynomials interlace, $F_{2n}(x_{2n+1,k+1})$ and $F_{2n}(x_{2n+1,k+2})$ have distinct sign. As a consequence, $K_{2n-1}^{(0,1)}(x_{2n+1,k+1},0)$ and $K_{2n-1}^{(0,1)}(x_{2n+1,k+2},0)$ differ in sign, which means that $K_{2n-1}^{(0,1)}(x,0)$ has a zero between the zeros $x_{2n+1,k+1}$ and $x_{2n+1,k+2}$. ■

Remark 3. *Notice that F_{2n+1} and $K_{2n-1}^{(0,1)}(x,0)$ differ in two degrees. This causes that the zeros interlacing between them is not complete. Indeed, $K_{2n-1}^{(0,1)}(x,0)$ has not zeros in the interval $[x_{2n+1,n}, x_{2n+1,n+2}]$, i.e. between the origin and the first zeros of $F_{2n+1}(x)$ at both sides.*

We will need some results concerning the interlacing properties of the zeros of $F_{2n+1}(x)$, $J_{2n+1}(x)$ and $Q_{2n+1}(x)$. By symmetry, for the zeros of $F_{2n+1}(x)$, we have $x_{2n+1,n+1} = 0$ and $x_{2n+1,k} = -x_{2n+1,2n+2-k}$ for $1 \leq k \leq n$. As a consequence, it suffices to analyze the behavior of the positive zeros. In order to simplify the notation, we denote $x_k := x_{2n+1,n+1+k}$, $1 \leq k \leq n$, i.e. $\{x_k\}_{k=1}^n$ are the n positive zeros of F_{2n+1} arranged in increasing order. A similar notation will be used for the zeros of Q_{2n+1} and F_{2n+1}. The following result is a straightforward corollary of Lemma 5.

Corollary 1. *Let us denote by $\{y_{n,k}\}_{k=1}^n$ the set of positive zeros of $J_{2n+1}(x)$ arranged in increasing order. Then, for $1 \leq k \leq n-1$, we have*

$$x_k < y_{k+1} < x_{k+1}, \tag{38}$$

i.e., positive zeros of $J_{2n+1}(x)$ and $F_{2n+1}(x)$ interlace.

Proof. Taking into account the symmetry and the fact that $J'_{2n+1}(0) = 0$, we deduce that $J_{2n+1}(x)$ has a zero of multiplicity 3 at the origin. This is, $y_1 = 0$. The result follows by evaluating (37) at two consecutive zeros x_k and x_{k+1} of F_{2n+1}, for $1 \leq k \leq n-1$, and noticing that by Lemma 5, $J_{2n+1}(x_k)$ and $J_{2n+1}(x_{k+1})$ have different signs. ■

Remark 4. *Observe that due to the triple zero at the origin, $J_{2n+1}(x)$ does not have a zero in the interval $(0, x_1)$, i.e. between the origin and the first positive zero of $F_{2n+1}(x)$. Since $F_{2n+1}(x)$ only has $n-1$ positive zeros, we have $y_1 = 0$.*

Now, we are ready to enunciate the main result of this Section.

Theorem 1. *On the positive real line, the following interlacing property holds*

$$0 = y_1 < \eta_1 < x_1 < y_2 < \eta_2 < x_2 \cdots < y_k < \eta_n < x_n.$$

Moreover, each $\eta_k := \eta_k(M_1)$ is a decreasing function of M_1 and, for each $k = 1, \ldots, n$,

$$\lim_{M_1 \to \infty} \eta_k(M_1) = y_k \,, \tag{39}$$

as well as

$$\lim_{M_1 \to \infty} M_1[\eta_k(M_1) - y_k] = \frac{-F_{2n+1}(y_k)}{K^{(1,1)}_{2n-1}(0,0)[J'_{2n+1}(y_k)]}. \tag{40}$$

Proof. Notice that the polynomials $\{\tilde{Q}_{2n+1}(x)\}_{n\geq 0}$ with $\tilde{Q}_{2n+1}(x) = \rho_{2n+1}Q_{2n+1}(x)$, can be represented as

$$\tilde{Q}_{2n+1}(x) = F_{2n+1}(x) + M_1 K^{(1,1)}_{2n-1}(0,0)\, J_{2n+1}(x)\,,$$

where

$$\rho_{2n+1} = 1 + M_1 K^{(1,1)}_{2n-1}(0,0)\,.$$

Thus, the interlacing follows at once from (38) and Lemma 3. On the other hand, we can write

$$x\hat{q}_n(x^2) = x\hat{f}_n(x^2) + M_1 K^{(1,1)}_{2n-1}(0,0) x \hat{j}_n(x^2),$$

with

$$\begin{aligned} \hat{f}_n &= (x - x_1^2)\cdots(x - x_n^2), \\ \hat{q}_n &= (x - \eta_1^2)\cdots(x - \eta_n^2), \\ \hat{j}_n &= (x - y_1^2)\cdots(x - y_n^2), \end{aligned}$$

and by the previous results, their zeros are real, simple and interlace, so they satisfy the conditions on Lemma 3, and therefore

$$\lim_{M_1 \to \infty} \eta_k^2 = y_k^2,$$

and

$$\lim_{M_1 \to \infty} = M_1 K^{(1,1)}_{2n-1}(0,0)[\eta_k^2 - y_k^2] = -\frac{\hat{f}_n(y_k^2)}{\hat{j}'_n(y_k^2)} = -\frac{2y_k F_{2n+1}(y_k)}{J'_{2n+1}(y_k)},$$

and since $\eta_k^2 - y_k^2 = (\eta_k + y_k)(\eta_k - y_k)$ and $\lim_{M_1 \to \infty} \eta_k = y_k$, the result follows. ■

Remark 5. *Because of the symmetry, the limits* (39) *and* (40) *also hold for the negative zeros. The only difference is that those zeros are increasing functions of M_1.*

5. Holonomic equation and electrostatic interpretation

In this section, we deduce a second order linear differential equation satisfied by $\{Q_n(x)\}_{n\geq 0}$ and, as an application, an electrostatic interpretation of its zeros is presented. We will use the connection formula between Q_n and F_n, which for convenience will take the form (28). We will also use the structure formula (5) (for the monic normalization) and the three term recurrence relation (4). Let us rewrite these formulas as

$$Q_n(x) = A_n(x)F_n(x) + B_n(x)F_{n-1}(x), \tag{41}$$

$$F'_n(x) = \varphi_n(x)F_n(x) + \psi_n(x)F_{n-1}(x), \tag{42}$$

$$F_{n+1}(x) = \Omega_n(x)F_n(x) + \Upsilon_n(x)F_{n-1}(x), \tag{43}$$

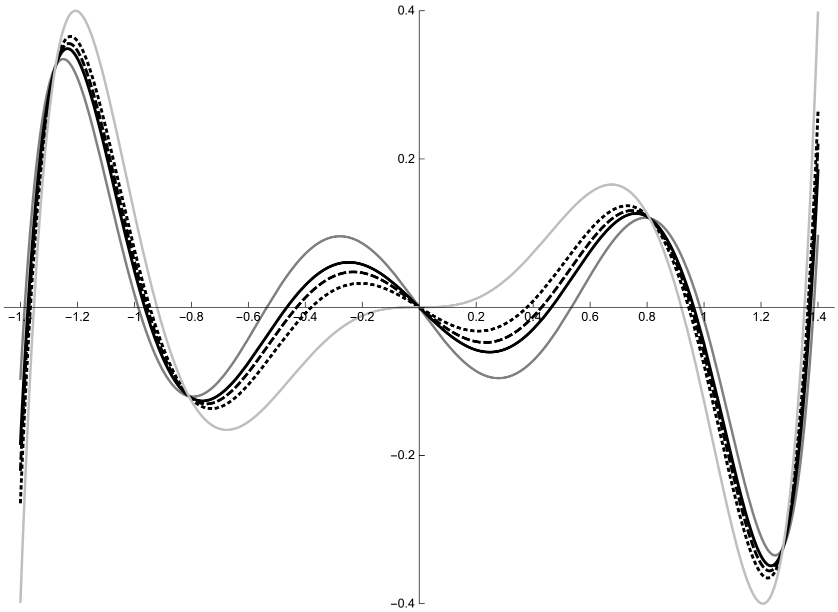

Figure 2: It illustrates the variation of the zeros of an odd degree Freud-Sobolev type polynomials when M_1 varies as described in Theorem 1. The graphs of $Q_7(x)$ for three different values of M_1 are plotted. The black continuous, dashed, and dotted lines correspond to $M_1 = 0.03$, $M_1 = 0.05$, and $M_1 = 0.09$, respectively. We also include the graphs of $F_7(x)$ (medium gray color) and $J_7(x)$ (light gray color), showing that the zeros of $Q_7(x)$ are increasing functions of M_1 in the negative real semiaxis, traveling from the negative zeros of $F_7(x)$ to the corresponding zeros of $J_7(x)$ as M_1 increases. Likewise, the positive zeros of $Q_7(x)$ are decreasing functions of M_1, traveling from the positive zero of $F_7(x)$ to the corresponding zero of $J_7(x)$ according with Theorem 1. Observe that in this picture, the value of M_0 is irrelevant.

where the coefficients above are given according to (28), (5) and (4), respectively, i.e.,

$$\begin{aligned} A_n(x) &= 1 - \frac{r_n}{4x^2\phi_n(0)}\kappa_n^{[1]}, \quad B_n(x) = \frac{a_n^2}{x}\left(\kappa_n^{[0]} + \kappa_n^{[1]}\right), \\ \varphi_n(x) &= -4xa_n^2, \quad \psi_n(x) = 4a_n^2\phi_n(x), \\ \Omega_n(x) &= x, \quad \Upsilon_n(x) = -a_n^2. \end{aligned}$$

Before stating our main result in this section, we need the following Lemmas.

Lemma 6. *The monic sequences* $\{Q_n(x)\}_{n\geq 0}$ *and* $\{F_n(x)\}_{n\geq 0}$ *satisfy*

$$Q_n'(x) = C_1(x;n)F_n(x) + D_1(x;n)F_{n-1}(x) \tag{44}$$

where

$$\begin{aligned} C_1(x;n) &= A_n'(x) + A_n(x)\varphi_n(x) + B_n(x)\frac{\psi_{n-1}(x)}{\Upsilon_{n-1}(x)}, \\ D_1(x;n) &= B_n'(x) + A_n(x)\psi_n(x) + B_n(x)\left(\varphi_{n-1}(x) - \frac{\Omega_{n-1}(x)}{\Upsilon_{n-1}(x)}\right). \end{aligned} \tag{45}$$

Proof. Combining (42) and (43) we have

$$F_{n-1}'(x) = \frac{\psi_{n-1}(x)}{\Upsilon_{n-1}(x)}F_n(x) + \left(\varphi_{n-1}(x) - \frac{\Omega_{n-1}(x)}{\Upsilon_{n-1}(x)}\right)F_{n-1}(x).$$

The result follows by replacing the last equation and (42) into the derivative with respect to x of (41). ■

Lemma 7. *The sequences of monic polynomials* $\{Q_n(x)\}_{n\geq 0}$ *and* $\{F_n(x)\}_{n\geq 0}$ *are also related by*

$$Q_{n-1}(x) = A_2(x;n)F_n(x) + B_2(x;n)F_{n-1}(x), \tag{46}$$

$$Q_{n-1}'(x) = C_2(x;n)F_n(x) + D_2(x;n)F_{n-1}(x), \tag{47}$$

where

$$
\begin{aligned}
A_2(x;n) &= \frac{B_{n-1}(x)}{\Upsilon_{n-1}(x)}, \quad B_2(x;n) = A_{n-1}(x) - B_{n-1}(x)\frac{\Omega_{n-1}(x)}{\Upsilon_{n-1}(x)}, \\
C_2(x;n) &= \frac{D_1(x;n-1)}{\Upsilon_{n-1}(x)}, \quad D_2(x;n) = C_1(x;n-1) - D_1(x;n-1)\frac{\Omega_{n-1}(x)}{\Upsilon_{n-1}(x)}.
\end{aligned}
$$

The coefficients $C_1(x;n-1)$ and $D_1(x;n-1)$ are given in (45).

Proof. The expressions follow from (41) and (44), respectively, after a shift in the degree, and using (43) to express both of them in terms of F_n and F_{n-1}. ■

Lemma 8. *The following "inverse connection" formulas hold.*

$$
\begin{aligned}
F_n(x) &= \frac{B_2(x;n)}{\Lambda(x;n)}Q_n(x) - \frac{B_n(x)}{\Lambda(x;n)}Q_{n-1}(x), &(48)\\
F_{n-1}(x) &= \frac{-A_2(x;n)}{\Lambda(x;n)}Q_n(x) + \frac{A_n(x)}{\Lambda(x;n)}Q_{n-1}(x), &(49)
\end{aligned}
$$

where

$$\Lambda(x;n) = A_n(x)B_2(x;n) - A_2(x;n)B_n(x).$$

Proof. The result follows by solving the linear system defined by (41) and (46). ■

Now, we replace (48) and (49) in (44) and (47), respectively, to obtain the ladder equations

$$
\begin{aligned}
Q_n'(x) &= \left[\frac{C_1(x;n)B_2(x;n)}{\Lambda(x;n)} - \frac{D_1(x;n)A_2(x;n)}{\Lambda(x;n)}\right]Q_n(x) \\
&\quad + \left[\frac{A_n(x)D_1(x;n)}{\Lambda(x;n)} - \frac{C_1(x;n)B_n(x)}{\Lambda(x;n)}\right]Q_{n-1}(x),
\end{aligned}
$$

$$
\begin{aligned}
Q_{n-1}'(x) &= \left[\frac{C_2(x;n)B_2(x;n)}{\Lambda(x;n)} - \frac{A_2(x;n)D_2(x;n)}{\Lambda(x;n)}\right]Q_n(x) \\
&\quad + \left[\frac{A_n(x)D_2(x;n)}{\Lambda(x;n)} - \frac{C_2(x;n)B_n(x)}{\Lambda(x;n)}\right]Q_{n-1}(x),
\end{aligned}
$$

which can be written in the more compact way

$$
\begin{aligned}
(\Xi(x;n,2)I - D_x)Q_n(x) &= \Xi(x;n,1)Q_{n-1}(x),\\
(\Theta(x;n,1)I + D_x)Q_{n-1}(x) &= \Theta(x;n,2)Q_n(x),
\end{aligned}
$$

where I and D_x are the identity and x-derivative operators, respectively, by defining the determinants

$$
\begin{aligned}
\Xi(x;n,k) &= \frac{1}{\Lambda(x;n)}\begin{vmatrix} C_1(x;n) & A_k(x;n) \\ D_1(x;n) & B_k(x;n) \end{vmatrix}, &(50)\\
\Theta(x;n,k) &= \frac{1}{\Lambda(x;n)}\begin{vmatrix} C_2(x;n) & A_k(x;n) \\ D_2(x;n) & B_k(x;n) \end{vmatrix}, &(51)
\end{aligned}
$$

for $k = 1,2$, where $A_1(x;n) := A_n(x)$ and $B_1(x;n) := B_n(x)$. As a consequence, we have the following result.

Theorem 2. *Let $\mathfrak{b}_n$ and $\mathfrak{b}_n^\dagger$ be the differential operators*

$$
\begin{aligned}
\mathfrak{b}_n &= \Xi(x;n,2)I - D_x,\\
\mathfrak{b}_n^\dagger &= \Theta(x;n,1)I + D_x.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\mathfrak{b}_n[Q_n(x)] &= \Xi(x;n,1)Q_{n-1}(x),\\
\mathfrak{b}_n^\dagger[Q_{n-1}(x)] &= \Theta(x;n,2)Q_n(x),
\end{aligned}
$$

where $\Xi(x;n,k)$ and $\Theta(x;n,k)$ are given in (50) and (51), respectively.

Finally, we state the main result of this section.

Theorem 3. *The Sobolev-Freud type polynomials $\{Q_n(x)\}_{n\geq 0}$ satisfy the second order linear differential equation*

$$Q_n''(x) + \mathcal{R}(x;n)Q_n'(x) + \mathcal{S}(x;n)Q_n(x) = 0, \tag{52}$$

where

$$\begin{aligned}\mathcal{R}(x;n) &= \Theta(x;n,1) - \Xi(x;n,2) - \frac{[\Xi(x;n,1)]'}{\Xi(x;n,1)},\\ \mathcal{S}(x;n) &= \Xi(x;n,2)\left[\frac{[\Xi(x;n,1)]'}{\Xi(x;n,1)} - \Theta(x;n,1)\right] - [\Xi(x;n,2)]'.\end{aligned}$$

Proof. The result follows in a straightforward way from the ladder operators provided in Theorem 2. The usual technique (see, for example [14, Th. 3.2.3]) consists in applying the raising operator to both sides of the equation satisfied by the lowering operator, i.e.

$$\mathfrak{b}_n^\dagger\left[\frac{1}{\Xi(x;n,1)}\mathfrak{b}_n[Q_n(x)]\right] = \mathfrak{b}_n^\dagger[Q_{n-1}(x)] = \Theta(x;n,2)Q_n(x),$$

and then using the definition $\mathfrak{b}_n^\dagger$ to compute the left hand side. After some computations, (52) follows. ■

We point out that we have obtained a second order linear differential equation for the complete sequence $\{Q_n(x)\}_{n\geq 0}$. However, as we have mentioned in the previous sections, the even and odd degree polynomials behave differently. Indeed, they have another connection formula, and the previous results hold in either case just by taking the coefficients of the connection formula (41) accordingly. Using Mathematica®, the expression for $\mathcal{R}(x;n)$ was obtained according to Theorem 3. In the sequel, we provide the expressions for the odd case ($\kappa_n^{[0]} = 0, \kappa_{2n}^{[1]} = 0, r_{2n+1} = 1, r_{2n} = 0$), together with an electrostatic interpretation of the zeros of $\{Q_n(x)\}_{n\geq 0}$. The even case was analyzed in [3] and [13]. We found

$$\mathcal{R}(x;2n+1) = \frac{2}{x} - 4x^3 - \frac{u'(x;2n+1)}{u(x;2n+1)},$$

where $u(x;2n+1)$ is the biquartic polynomial

$$u(x;2n+1) = u_4(n)\,x^4 + u_2(n)\,x^2 + u_0(n) \tag{53}$$

with

$$\begin{aligned}u_4(n) &= 16\phi_{2n+1}^2(0)[1+\kappa_{2n+1}^{[1]}],\\ u_2(n) &= 4\phi_{2n+1}(0)\left[4\phi_{2n+1}^2(0) + \kappa_{2n+1}^{[1]}(2+\kappa_{2n+1}^{[1]})(4a_{2n+1}^2\phi_{2n+1}(0)-1)\right],\\ u_0(n) &= \kappa_{2n+1}^{[1]}\left[-12\phi_{2n+1}^2(0) + \kappa_{2n+1}^{[1]}\left\{1+8a_{2n+1}^2\phi_{2n+1}(0)\left[-1+2\phi_{2n}(0)\phi_{2n+1}(0)\right]\right\}\right].\end{aligned}$$

Now, the evaluation of (52) at the zeros $\{y_{2n+1,k}\}_{k=1}^{2n+1}$ of $Q_{2n+1}(x)$ which are different from zero , i.e. if they are listed in an increasing order we will have $y_{2n+1,n+1} = 0$, yields

$$\frac{Q_{2n+1}''(y_{2n+1,k})}{Q_{2n+1}'(y_{2n+1,k})} = -\mathcal{R}(y_{2n+1,k};2n+1) = -\frac{2}{y_{2n+1,k}} + 4(y_{2n+1,k})^3 + \frac{u'(y_{2n+1,k};2n+1)}{u(y_{2n+1,k};2n+1)}.$$

The above equation represents the electrostatic equilibrium condition for the $2n$ zeros $\{y_{2n+1,k}\}_{k=1}^{2n+1}$ of $Q_{2n+1}, k\neq n+1$, and can be rewritten as (see [14] and [17])

$$\sum_{j=1, j\neq k, n+1}^{2n+1}\frac{1}{y_{2n+1,j}-y_{2n+1,k}} + \frac{u'(y_{2n+1,k};2n+1)}{2u(y_{2n+1,k};2n+1)} - \frac{2}{y_{2n+1,k}} + 2(y_{2n+1,k})^3 = 0,$$

for $k = 1,\ldots,2n+1$, $k\neq n+1$. Therefore, the zeros of $Q_{2n+1}(x)$ are critical points of the total energy. Thus, the electrostatic interpretation of the distribution of the $2n$ zeros different from the $y_{2n+1,n+1} = 0$, since it remains fixed, means that we have an equilibrium position under the action of the external potential

$$V_{ext}(x, 2n+1) = \frac{1}{2} \ln u(x; 2n+1) - \frac{1}{2} \ln x^4 e^{-x^4},$$

where the first term represents a short range potential which corresponds to unit charges located at the four zeros of $u(x; 2n+1)$, and the second term is a long range potential associated with a Christoffel perturbation of the Freud weight function.

If $z_+(n)$ and $z_-(n)$ are the solutions of the associated quadratic equation

$$u_4(n)\, z^2 + u_2(n)\, z + u_0(n) = 0,$$

then the zeros of (53) are

$$x_1(n) = +\sqrt{z_+}(n),\ x_2(n) = -\sqrt{z_+}(n),\ x_3(n) = +\sqrt{z_-}(n),\ x_4(n) = -\sqrt{z_-}(n).$$

	$M = 0.1$		$M = 1$		$M = 10$	
	$\pm\sqrt{z_1}$	$\pm\sqrt{z_2}$	$\pm\sqrt{z_1}$	$\pm\sqrt{z_2}$	$\pm\sqrt{z_1}$	$\pm\sqrt{z_2}$
$n=1$	± 0.369164	$\pm 0.878731\, i$	± 0.745497	$\pm 0.914759\, i$	± 0.905303	$\pm 0.928589\, i$
$n=3$	± 0.397067	$\pm 1.059517\, i$	± 0.387740	$\pm 1.089036\, i$	± 0.159258	$\pm 1.106825\, i$
$n=5$	± 0.329766	$\pm 1.181451\, i$	± 0.197206	$\pm 1.197172\, i$	± 0.068685	$\pm 1.201241\, i$
$n=7$	± 0.251172	$\pm 1.272375\, i$	± 0.116257	$\pm 1.279623\, i$	± 0.038576	$\pm 1.280856\, i$
$n=9$	± 0.189032	$\pm 1.345977\, i$	± 0.076318	$\pm 1.349456\, i$	± 0.024825	$\pm 1.349937\, i$
$n=11$	± 0.144418	$\pm 1.408813\, i$	± 0.053943	$\pm 1.410616\, i$	± 0.017374	$\pm 1.410839\, i$
$n=13$	± 0.112816	$\pm 1.464184\, i$	± 0.040222	$\pm 1.465192\, i$	± 0.012969	$\pm 1.465308\, i$
$n=15$	± 0.089745	$\pm 1.513969\, i$	± 0.029902	$\pm 1.514571\, i$	± 0.004691	$\pm 1.514637\, i$
$n=17$	± 0.073204	$\pm 1.559345\, i$	± 0.024134	$\pm 1.559723\, i$	± 0.005169	$\pm 1.559764\, i$
$n=19$	± 0.060787	$\pm 1.601140\, i$	± 0.019950	$\pm 1.601389\, i$	± 0.005144	$\pm 1.601416\, i$

Table 3: Zeros of $u(x; 2n+1)$ for several values of M_1 and odd values of n.

Table 3 shows the zeros of $u(x; 2n+1)$ for some fixed values of M_1 and several values of n. With just a little more effort, we can describe the asymptotic behavior with n of these four zeros. Combining (8) with (5) and (6) it is possible to obtain the asymptotic behavior for n large enough of the kernels (13)-(14). Using this information, and (7), after some tedious but straightforward computations, the asymptotic behavior of the three coefficients yields

$$\begin{aligned} u_4(n) &= \frac{32}{3} n \left(1 + \frac{15}{8n} + \mathcal{O}(n^{-2})\right), \\ u_2(n) &= 8\sqrt{6} n^{1/2} \left(1 + \frac{4n}{9} + \mathcal{O}(n^{-1})\right), \\ u_0(n) &= \frac{-9}{2} \left(1 + \frac{9}{8n} + \mathcal{O}(n^{-2})\right). \end{aligned}$$

Then, the asymptotic behavior of the aforementioned z_+ and z_- is

$$\begin{aligned} z_+(n) &= \frac{27}{64}\sqrt{\frac{3}{2}}\frac{1}{n^{3/2}} - \frac{243}{512}\sqrt{\frac{3}{2}}\frac{1}{n^{5/2}} + \mathcal{O}(n^{-7/2}), \\ z_-(n) &= -\sqrt{\frac{2}{3}}\frac{1}{n^{1/2}} - \frac{1}{4}\sqrt{\frac{3}{2}}\frac{1}{n^{5/2}} + \mathcal{O}(n^{-7/2}) \end{aligned}$$

The above shows that, as n goes to infinity, $z_+(n)$ remains positive and $z_-(n)$ negative, so $u(x; 2n+1)$ will always have two symmetric real zeros $x_1, x_2 = \pm\sqrt{z_+}(n)$, and two extra simple conjugate pure imaginary zeros $x_3, x_4 = \pm\sqrt{z_-}(n)$.

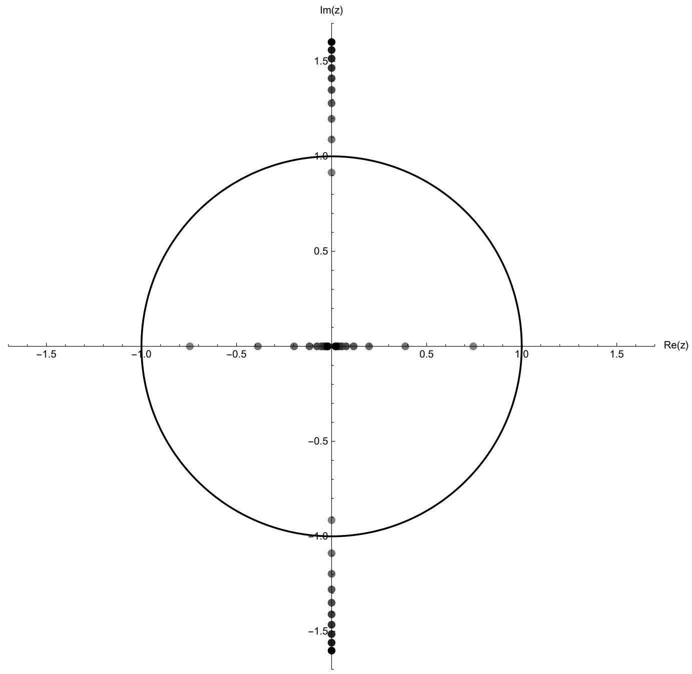

Figure 3: Zeros of $u(x; 2n+1)$ for $M_1 = 1$ and odd values of n, from 1 to 19.

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