An inverse problem associated with $(1,1)$ symmetric coherent linear functionals

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ABSTRACT
In this paper we study an inverse problem associated with the non-coherent algebraic relation

$$\begin{aligned} & P_{n+1}^{[i]}(x) + a_n^{[1]} P_n^{[i]}(x) + a_n^{[2]} P_{n-1}^{[i]}(x) + b_n \left(Q_{n+1}(x) + c_n Q_n(x) \right) \\ = \; & \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), n \geq 0, \end{aligned}$$

where the sequences of monic polynomials $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are orthogonal with respect to quasi-definite linear functionals u, v and w, respectively, and $P_k^{[i]}(x) = \dfrac{P_{k+i}^{(i)}(x)}{(k+1)_i}$ for $i = 0, 1$. We assume that v is a monic polynomial perturbation of u. Moreover, in the case $i = 1$, we assume that u is a semiclassical linear functional of class s. In this way, we discuss the relation between the formal Stieltjes series associated with u and w. This inverse problem is motivated by the analysis of sequences of polynomials orthogonal with respect to $(1,1)$ symmetric coherent pairs of linear functionals when a symmetrization process is implemented.

KEYWORDS
Orthogonal polynomials; inverse problems; semiclassical linear functionals

1. Introduction

The inverse problems in the theory of orthogonal polynomials on the real line have been extensively studied in the recent literature. This kind of problems can be described as follows. Given an algebraic relation between two sequences of monic orthogonal polynomials how to find the relation between the respective quasi-definite

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linear functionals. For instance, if $\{P_n(x)\}_{n\geq 0}$ and $\{Q_n(x)\}_{n\geq 0}$ are sequences of monic polynomials, orthogonal with respect to u and v, respectively, and the polynomials in those sequences satisfy the algebraic relation

$$P_n(x) + \sum_{k=1}^{N} a_{n,k} P_{n-k}(x) = Q_n(x) + \sum_{j=1}^{M} b_{n,j} Q_{n-j}(x), \quad a_{n,N} b_{n,M} \neq 0,$$

for certain sequences of complex numbers $\{a_{n,k}\}_{n\geq 0}$ and $\{b_{n,k}\}_{n\geq 0}$, then, under certain conditions (to be more precise, the nonsingularity of some matrix, for example), it is possible to prove that the linear functionals u and v satisfy

$$\phi u = \psi v,$$

where ϕ and ψ are polynomials with $\deg \phi = M$ and $\deg \psi = N$, (see [1], [2]). The study of the above relation is an extension of some particular inverse problems studied for the cases $M = N = 1$ in [3] and [4], among others, or in [5], when $N = 0$ and $M = 3$. In [6], [7], [8] and [9], several extensions are studied when derivatives of the orthogonal polynomials are included in the algebraic relations. The above contributions are in the context of the so called *Generalized Coherent pairs of functionals*. Indeed, when the orthogonal polynomials associated with u and v satisfy

$$P_n^{[m]}(x) + \sum_{k=1}^{M} a_{n,k} P_{n-k}^{[m]}(x) = Q_n^{[p]}(x) + \sum_{k=1}^{N} b_{n,k} Q_{n-k}^{[p]}(x), \tag{1.1}$$

where $P_n^{[m]}(x) = \dfrac{P_{n+m}^{(m)}(x)}{(n+1)_m}$, (u, v) constitutes a $(M, N)-$*coherent pair of order* (m, p). The concepts of *coherent pair* and *symmetric coherent pair* were introduced in [10] in connection with Sobolev-type orthogonal polynomials associated with a pair of non-trivial positive measures supported on the real line. In this way, in [11] the description of all coherent pairs and symmetric coherent pairs is given. Therein is proved that one of the quasi-definite linear functionals must be classical. On the other hand, given a quasi-definite symmetric linear functional S and corresponding sequence of monic orthogonal polynomials (SMOP, in short), $\{S_n(x)\}_{n\geq 0}$, there exist quasi-definite linear functionals U and V such that their respective SMOP $\{A_n(x)\}_{n\geq 0}$ and $\{B_n(x)\}_{n\geq 0}$ satisfy

$$S_{2n}(x) = A_n(x^2), \qquad S_{2n+1}(x) = xB_n(x^2), n \geq 0,$$

and $V = xU$. This is the reverse process of the so called *symmetrization process*, (see [12]). As an extension of the concept of symmetric coherent pair, a pair of functionals is said to be a *symmetric* $(1,1)-$*coherent pair* if their respective SMOP $\{S_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy

$$R_{n+2}(x) + a_n R_n(x) = \frac{S'_{n+3}(x)}{n+3} + b_n \frac{S'_{n+1}(x)}{n+1}, \quad n \geq 0. \tag{1.2}$$

This kind of relation has been introduced in [13], where the associated inverse problem and the particular case when $\{S_n(x)\}_{n\geq 0}$ is classical and symmetric, i.e. Hermite or

Gegenbauer, is studied. Then, by dealing with the inverse process together with the expressions of the symmetrized polynomials, we can obtain

$$\frac{A'_{n+2}(x)}{n+2}+a_{2n+1}\frac{A'_{n+1}(x)}{n+1}=\widetilde{B}_{n+1}(x)+b_{2n+1}\widetilde{B}_{n}(x),\quad n\geq 0,\tag{1.3}$$

and

$$\frac{2x\widetilde{A}'_{n+1}(x)+\widetilde{A}_{n+1}(x)}{2n+3}+a_{2n}\frac{2x\widetilde{A}'_{n}(x)+\widetilde{A}_{n}(x)}{2n+1}=B_{n+1}(x)+b_{2n}B_{n}(x),n\geq 0,\tag{1.4}$$

where

$$\begin{aligned} S_{2n}(x) &= A_n(x^2), \qquad S_{2n+1}(x)=x\widetilde{A}_n(x^2),\\ R_{2n}(x) &= B_n(x^2), \qquad S_{2n+1}(x)=x\widetilde{B}_n(x^2). \end{aligned}$$

Relation (1.3) means that you have a $(1,1)-$*coherent pair of functionals*. Of course, this kind of coherence is also a particular case of (1.1) and it has been studied in [14], where the associated inverse problem is discussed and a complete classification of all $(1,1)-$coherent pairs is given. However, the second one is not a coherence relation with respect to (1.1) and, as far as we know, this kind of algebraic relation has not been still studied. Then, in order to give an equivalent expression of (1.4), it is well known that (see [12])

$$x\widetilde{A}_n(x)=A_{n+1}(x)-\frac{A_{n+1}(0)}{A_n(0)}A_n(x),\qquad n\geq 0.$$

Taking derivatives

$$x\widetilde{A}'_n(x)+\widetilde{A}_n(x)=A'_{n+1}(x)-\frac{A_{n+1}(0)}{A_n(0)}A'_n(x),n\geq 0,\tag{1.5}$$

and replacing in (1.4) we obtain

$$\begin{aligned} &\frac{2A'_{n+2}(x)-2\frac{A_{n+2}(0)}{A_{n+1}(0)}A'_{n+1}(x)-\widetilde{A}_{n+1}(x)}{2n+3}\\ &+a_{2n}\frac{2A'_{n+1}(x)-2\frac{A_{n+1}(0)}{A_n(0)}A'_n(x)-\widetilde{A}_n(x)}{2n+1}\\ =\ &B_{n+1}(x)+b_{2n}B_n(x),n\geq 0, \end{aligned}$$

or, equivalently,

$$\begin{aligned} &A'_{n+2}(x)+\left(\frac{a_{2n}\,(2n+3)}{(2n+1)}-\frac{A_{n+2}(0)}{A_{n+1}(0)}\right)A'_{n+1}(x)\\ &-\frac{a_{2n}\,(2n+3)}{(2n+1)}\frac{A_{n+1}(0)}{A_n(0)}A'_n(x)-\frac{1}{2}\widetilde{A}_{n+1}(x)-\frac{a_{2n}\,(2n+3)}{2\,(2n+1)}\widetilde{A}_n(x)\\ =\ &\frac{2n+3}{2}\left(B_{n+1}(x)+b_{2n}B_n(x)\right),n\geq 0. \end{aligned}$$

The above relation is the reason why in this paper we will study the inverse problem associated with the algebraic relation

$$\begin{aligned} & P_{n+1}^{[i]}(x)+a_n^{[1]}P_n^{[i]}(x)+a_n^{[2]}P_{n-1}^{[i]}(x)+b_n\left(Q_{n+1}(x)+c_nQ_n(x)\right) \\ = & \left(1+b_n\right)R_{n+1}(x)+d_nR_n(x), n\geq 0, \end{aligned} \tag{1.6}$$

where the sequences $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w, respectively, with $P_k^{[i]}(x) := \dfrac{P_{k+i}^{(i)}(x)}{(k+1)_i}$, $i=0,1$, and $a_n^{[i]}b_nc_nd_n(1+b_n)\neq 0$, $n\geq 0$. Besides, the linear functionals u and v are related through the rational relation

$$\rho(x)u=v, \tag{1.7}$$

where ρ is a monic polynomial of degree m.

Notice that, when $i=0$, you can apply the standard Christoffel formula, ([15]), but you must multiply (1.6) by ρ and using the iteration of the three term recurrence relation for each sequence $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ you get a relation between both sequences in the terms discussed in [1]. But in such a case, the number of terms is not optimal in order to determine the degrees of the polynomials involved in the relation between u and w. Another choice is to apply the standard Geronimus formula ([15]) and you have a relation between $\{Q_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ as studied in [1] but, again, the number of terms is not optimal in order to determine the degrees of the polynomials involved in the relation between v and w.
When $i=1$, you can multiply both hand sides in (1.6) by the polynomial ϕ. Then you can apply the structure relation for the sequence $\{P_n\}_{n\geq 0}$ in the left hand side and the iteration of the three term recurrence relation for the sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$. Thus, you have a relation as in [1] but the number of terms involved therein is not optimal in order to determine the degrees of the polynomials appearing in the relation between u and w.

The structure of the paper is as follows. In Section 2 we deal with some basic facts about quasi-definite linear functionals, orthogonal polynomials, semiclassical linear functionals and their relation with formal Stieltjes series. In Section 3 we study the inverse problem associated with (1.6) when $i=0$. In Section 4, such an inverse problem is discussed for $i=1$. Furthermore, a differential relation for u and v is obtained. Finally, the connection between the respective Stieltjes series is analyzed.

2. Preliminaries

2.1. Quasi-definite linear functionals and Orthogonal Polynomials

Let u be a linear functional defined on the space of polynomials with complex coefficients $\mathcal{P}$. The linear space of linear functionals on $\mathcal{P}$, i.e. the algebraic dual space of $\mathcal{P}$, will be denoted by $\mathcal{P}'$. $\langle u,p\rangle$ means the action of u on any polynomial p and, as usual, $\|p\|_u^2 := \left\langle u,p^2(x)\right\rangle$. Also, let $\{u_n\}_{n\geq 0}$ be the sequence of moments associated with u, where $u_n := \langle u,x^n\rangle$. The infinite matrix $H := (u_{i+j})_{i,j=0}^{\infty}$ is called *Hankel* *matrix* associated with u.

The $p-th$ derivative of the functional u, $p \in \mathbb{Z}^+ \cup \{0\}$, denoted by $D^p u$, is a linear functional on $\mathcal{P}$ defined by

$$\langle D^p u, q(x)\rangle := (-1)^p \left\langle u, q^{(p)}(x)\right\rangle, \quad q \in \mathcal{P}.$$

Besides, given $q \in \mathcal{P}$, we define $qu \in \mathcal{P}'$, the *left-multiplication*, as follows

$$\langle q(x)u, p(x)\rangle := \langle u, q(x)p(x)\rangle, \quad p \in \mathcal{P}.$$

Also, $uq \in \mathcal{P}$ denotes the *right-multiplication* of u by q, and it is the polynomial defined by

$$(uq)(t) := \left\langle u, \frac{tq(t) - xq(x)}{t - x}\right\rangle,$$

where u acts on the variable x. Given $a \in \mathbb{C}$, the linear operator $\theta_a : \mathcal{P} \to \mathcal{P}$ is defined by

$$\theta_a(q(x)) := \frac{q(x) - q(a)}{x - a}. \tag{2.1}$$

In this way, given $p \in \mathcal{P}$, with $p(x) = \sum_{n=0}^{h} g_n x^n$, and $u \in \mathcal{P}'$, after straightforward computations, it can be proved that

$$(u\theta_0 p)(z) := (u(\theta_0 p))(z) = \sum_{n=0}^{h-1} g_{n+1} \sum_{p=0}^{n} u_p z^{n-p} = \sum_{n=0}^{h-1}\sum_{p=n}^{h-1} (g_{p+1}u_{p-n}) z^n, \tag{2.2}$$

and thus $u\theta_0 p$ is a polynomial in z.

Given a sequence of polynomials $\{Q_n(x)\}_{n\geq 0}$ with $\deg Q_n = n$, the sequence $\{\mathbf{q}_n\}_{n\geq 0}$, in the dual space $\mathcal{P}'$, such that $\langle \mathbf{q}_n, Q_m(x)\rangle := \delta_{n,m}$, where $\delta_{n,m}$ is the Kronecker Delta, is called the *dual basis* associated with $\{Q_n(x)\}_{n\geq 0}$.

A linear functional u is called *quasi-definite* if and only all principal leading submatrices of the corresponding Hankel matrix of its moments are nonsingular. Therefore, if u is quasi-definite, then there exists a sequence of polynomials $\{P_n(x)\}_{n\geq 0}$, where $\deg P_n = n$, such that the terms of that sequence satisfy $\langle u, P_n(x)P_m(x)\rangle = K_n\delta_{m,n}$, where $K_n \neq 0$. The sequence $\{P_n(x)\}_{n\geq 0}$ is called **a** *sequence of orthogonal polynomials* associated with u. In particular, if every P_n is monic, then $\{P_n(x)\}_{n\geq 0}$ is called **the** SMOP associated with u. Thus, the next result follows.

Theorem 2.1. *Let $\{P_n(x)\}_{n\geq 0}$ be the SMOP with respect to the quasi-definite linear functional u and let $\{\mathbf{P}_n\}_{n\geq 0}$ be the corresponding dual basis. If $\left\{\mathbf{P}_n^{[1]}\right\}_{n\geq 0}$ is the dual basis associated with the sequence $\left\{\frac{P'_{n+1}(x)}{n+1}\right\}_{n\geq 0}$, then the (distributional) derivative* *of* $\mathbf{P}_n^{[1]}$ *satifies*

$$D\mathbf{P}_n^{[1]} = -\frac{(n+1)P_{n+1}(x)}{\|P_{n+1}\|_u^2}u. \tag{2.3}$$

On the other hand, for every n, $\mathbf{P}_n$ *can be written as*

$$\mathbf{P}_n = \frac{P_n(x)}{\|P_n\|_u^2}u. \tag{2.4}$$

2.2. Semiclassical linear functional and formal Stieltjes Series

Definition 2.2. A quasi-definite linear functional u is called semiclassical if there exist a monic polynomial ϕ and a polynomial ψ, with $\deg\psi \geq 1$, such that

$$D(\phi u) = \psi u. \tag{2.5}$$

In this way, if $\mathcal{A}$ is the set of all pairs of polynomials such that (2.5) holds, then the **class** of u is a non-negative integer number s defined by

$$s = \min_{(\phi,\psi)\in\mathcal{A}} \max\left\{\deg(\phi)-2, \deg(\psi)-1\right\}.$$

Next, we will show two very useful characterizations of semiclassical linear functionals.

Theorem 2.3. *([16]). If two quasi-definite linear functionals* U *and* V *are related by an expression of rational type as*

$$p(x)U = r(x)V,$$

where p *and* r *are nonzero polynomials, then* U *is a semiclassical linear functional if and only if so is* V*. Moreover, if the class of* U *is* s*, then the class of* V *is at most* $s+\deg(p)+\deg(r)$*.*

Theorem 2.4. *([16]). Let* U *be a quasi-definite linear functional and* $\{P_n(x)\}_{n\geq 0}$ *the corresponding SMOP.* U *is semiclassical of class* s *if and only if there exists a polynomial* σ*, with* $\deg(\sigma) = t \leq s+2$*, such that* $\{P_n(x)\}_{n\geq 0}$ *satisfies*

$$\sigma(x)\frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} a_{n,k}P_k(x), \quad n \geq s, \tag{2.6}$$

with $a_{n,n-s} \neq 0$ *and* $n \geq s+1$*.*

Definition 2.5. Given a quasi-definite linear functional u, the formal Stieltjes series S_u associated with u is defined by

$$S_u(z) := -\sum_{n\geq 0}\frac{u_n}{z^{n+1}}.$$

Theorem 2.6. *([16]). A regular linear functional U is semiclassical if and only if there exist nonzero polynomials σ, C, and D such that its formal Stieltjes series S_U is a (formal) solution of the following non-homogeneous first order linear differential equation*

$$\sigma(z)S_U'(z) = C(z)S_U(z) + D(z).$$

Moreover, if the polynomials σ, C, and D are mutually coprime, then the class of U is given by $s = max\{\deg(C) - 1, \deg(D)\}$.

Notice that, according to (2.5), you get

$$\sigma(z) = \phi(z), C(z) = -\phi'(z) + \psi(z),$$

and

$$D(z) = -(u\theta_0\phi)'(z) + (u\theta_0\Psi)(z).$$

By using standard techniques, (see [6] or [7]), it is not difficult to prove the next

Theorem 2.7. *Let u and v be quasi-definite linear functionals and let Φ_{M+n} and Ψ_{N+p+n} be polynomials such that $\deg \Phi_{M+n} = M + n$ and $\deg \Psi_{N+p+n} = N + p + n$, with $M, N, p \in \mathbb{Z}^+ \cup \{0\}$ and $n \geq 0$. Assume that*

$$D^p(\Phi_{M+n}(x)u) = \Psi_{N+p+n}(x)v$$

holds. Then the formal Stieltjes series associated with u and v satisfy

$$\Psi_{N+p+n}(z)S_v(z) - (\Phi_{M+n}(z)S_u(z))^{(p)} = A_n(z),$$

where, according to (2.2),

$$A_n(z) = (u\theta_0\Phi_{M+n})^{(p)}(z) - (v\theta_0\Psi_{N+p+n})(z),$$

being θ_0 the linear operator defined in (2.1). Besides, S_u satisfies the following non-homogeneous linear ordinary differential equation of order p

$$\sum_{j=0}^{p} B_j(z)S_u^{(j)}(z) = C(z),$$

where

$$B_j(z) = \binom{p}{j}\left(\Psi_{N+p}(z)\Phi_{M+1}^{(p-j)}(z) - \Psi_{N+p+1}(z)\Phi_M^{(p-j)}(z)\right)$$

and

$$C(z) = \Psi_{N+p+1}(z)A_0(z) - \Psi_{N+p}(z)A_1(z).$$

3. Case $i = 0$

In this section we consider (1.6) with $i = 0$, and therefore we study the algebraic relation

$$\begin{aligned} & P_{n+1}(x) + a_n^{[1]} P_n(x) + a_n^{[2]} P_{n-1}(x) + b_n \left(Q_{n+1}(x) + c_n Q_n(x) \right) \\ = \; & \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \end{aligned} \tag{3.1}$$

where $1 + b_n \neq 0, n \geq 0$. Here the sequences $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are SMOP with respect to the quasi-definite linear functionals u, v and w, respectively, as well as the linear functionals u and v are related by the rational relation

$$\rho(x) u = v,$$

where $\deg \rho \geq 0$. Let define the sequence $\{T_n(x)\}_{n\geq 0}$ as

$$T_{n+1}(x) := \left(1 + b_n\right) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \; T_0(x) = 1. \tag{3.2}$$

Let $\{\mathbf{t}_n\}_{n\geq 0}$, $\{\mathbf{P}_n\}_{n\geq 0}$, $\{\mathbf{q}_n\}_{n\geq 0}$, and $\{\mathbf{r}_n\}_{n\geq 0}$ be the corresponding dual bases associated with the sequences $\{T_n(x)\}_{n\geq 0}$, $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$, respectively. First, from (1.7), if $\mathbf{q}_n = \sum \langle \mathbf{q}_n, P_k(x) \rangle \mathbf{P}_k$, we get

$$\langle \mathbf{q}_n, P_k(x) \rangle = \left\langle \frac{Q_n(x)}{\|Q_n\|_v^2} v, P_k(x) \right\rangle = \frac{1}{\|Q_n\|_v^2} \langle v, Q_n(x) P_k(x) \rangle .$$

Thus $\langle \mathbf{q}_n, P_k(x) \rangle = 0$ for $k < n$. Moreover, if $k \geq n$, then

$$\langle \mathbf{q}_n, P_k(x) \rangle = \frac{1}{\|Q_n\|_v^2} \langle u, P_k(x) Q_n(x) \rho(x) \rangle ,$$

and, if $k > n + m$, then using orthogonality, we get $\langle \mathbf{q}_n, P_k(x) \rangle = 0$. In this way we get

$$\mathbf{q}_n = \sum_{k=n}^{n+m} \eta_{n,k} \mathbf{P}_k, \tag{3.3}$$

where we have defined $\eta_{n,k} := \langle \mathbf{q}_n, P_k(x) \rangle$. In particular, for $n \geq 0$,

$$\eta_{n,n+m} = \langle \mathbf{q}_n, P_{n+m}(x) \rangle = \frac{1}{\|Q_n\|_v^2} \langle u, P_{n+m}(x) Q_n(x) \rho(x) \rangle = \frac{\|P_{n+m}\|_u^2}{\|Q_n\|_v^2} > 0.$$

On the other hand, expanding $\mathbf{q}_n$ in terms of the basis $\{\mathbf{t}_k\}_{k\geq 0}$, $\langle \mathbf{q}_n, T_k(x) \rangle = 0$

holds for $n > k$. Now, if $k \geq n$, by using (3.3) we get

$$\begin{aligned} & \langle \mathbf{q}_n, T_k(x) \rangle \\ = & \left\langle \mathbf{q}_n, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x) \right\rangle \\ & + b_{k-1} \langle \mathbf{q}_n, Q_k(x) + c_{k-1} Q_{k-1}(x) \rangle \\ = & \left\langle \sum_{j=n}^{n+m} \eta_{n,j} \mathbf{P}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x) \right\rangle \\ & + b_{k-1} \langle \mathbf{q}_n, Q_k(x) + c_{k-1} Q_{k-1}(x) \rangle . \end{aligned}$$

If $k > n+m+2$, then

$$\left\langle \sum_{j=n}^{n+m} \eta_{n,j} \mathbf{P}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x) \right\rangle = 0.$$

Besides, $\langle \mathbf{q}_n, Q_k(x) + c_{k-1} Q_{k-1}(x) \rangle = 0$ if $k \neq n, n+1$. In this way

$$\mathbf{q}_n = \sum_{k=n}^{n+m+2} \xi_{n,k} \mathbf{t}_k. \tag{3.4}$$

Here $\xi_{n,k} := \left\langle \sum_{j=n}^{n+m} \eta_{n,j} \mathbf{P}_j, P_k(x) + a_{k-1}^{[1]} P_{k-1}(x) + a_{k-1}^{[2]} P_{k-2}(x) \right\rangle$, for $n+2 \leq k \leq n+m+2$. On the other hand, $\xi_{n,n+1} = \eta_{n,n} a_n^{[1]} + \eta_{n,n+1} + b_n c_n$ and $\xi_{n,n} = \eta_{n,n} + b_{n-1}$.

Notice that $\xi_{n,k}$ depends on m. In particular, for $n \geq 0$

$$\begin{aligned} & \xi_{n,n+m+2} \\ = & \left\langle \sum_{j=n}^{n+m} \eta_{n,j} \mathbf{P}_j, P_{n+m+2}(x) + a_{n+m+1}^{[1]} P_{n+m+1}(x) + a_{n+m+1}^{[2]} P_{n+m}(x) \right\rangle \\ = & \eta_{n,n+m} a_{n+m+1}^{[2]} \neq 0. \end{aligned}$$

If we assume that $\mathbf{r}_n = \sum \langle \mathbf{r}_n, T_k(x) \rangle \mathbf{t}_k$, then

$$\langle \mathbf{r}_n, T_k(x) \rangle = \langle \mathbf{r}_n, (1 + b_{k-1}) R_k(x) + d_{k-1} R_{k-1}(x) \rangle .$$

If $k \neq n, n+1$, then $\langle \mathbf{r}_n, T_k(x) \rangle = 0$ and, as a consequence,

$$\mathbf{r}_n = (1 + b_{n-1}) \mathbf{t}_n + d_n \mathbf{t}_{n+1}, \quad n \geq 0. \tag{3.5}$$

If we consider the above relation for $n = 0, .., m+1$, and (3.4) for $n = 0$, we obtain a system of $m+3$ linear equations with $m+3$ unknowns $\{\mathbf{t}_k\}_{k=0}^{m+2}$, namely

$$D_m \begin{pmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{m+2} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{m+1} \\ \mathbf{q}_0 \end{pmatrix},$$

where the entries of $D_m := \left(h_{i,j}^{[m]}\right)_{i,j=1}^{m+3}$ are

$$h_{i,j}^{[m]} = \begin{cases} 0, & if \;\; m+3 > i > j \text{ and } i+1 < j, \\ 1 + b_{i-2}, & if \;\; i = j \neq m+3, \\ d_{j-2}, & if \;\; i+1 = j, \\ \xi_{0,j-1}, & if \;\; i = m+3, \end{cases}$$

with $b_{-1} := 0$. If we assume that D_m is a nonsingular matrix, then we can find a matrix $(\mu_{i,j})_{i,j=0}^{m+2}$ such that if $k = 0, ..., m+2$, then each $\mathbf{t}_{n+k}$ can be written as

$$\mathbf{t}_k = \mu_{k,0}\mathbf{q}_0 + \mu_{k,1}\mathbf{r}_0 + \mu_{k,2}\mathbf{r}_1 + \cdots + \mu_{k,m+1}\mathbf{r}_m + \mu_{k,m+2}\mathbf{r}_{m+1}. \tag{3.6}$$

Now, multiplying (3.4), for $n = 1$, and (3.5), for $n = m+2$, by d_{m+2} and $\xi_{m+3} := \xi_{1,m+3}$, respectively, and subtracting both expressions, we obtain

$$d_{m+2}\mathbf{q}_1 - \delta_{m+3}\mathbf{r}_{m+2} = \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mathbf{t}_k,$$

where $\widetilde{\xi}_{m,k} := d_{m+2}\xi_{1,k}$, $1 \leq k \leq m+1$, and $\widetilde{\xi}_{m,m+2} := d_{m+2}\xi_{1,m+2} - \xi_{m+3}\left(1 + b_{m+1}\right)$. Now, replacing (3.6) in the above relation, we get

$$d_{m+2}\mathbf{q}_1 - \xi_{m+3}\mathbf{r}_{m+2} = \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k} \left(\mu_{k,0}\mathbf{q}_0 + \sum_{p=0}^{m+1} \mu_{k,p+1}\mathbf{r}_p \right),$$

or, equivalently,

$$d_{m+2}\mathbf{q}_1 - \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,0}\mathbf{q}_0 = \xi_{m+3}\mathbf{r}_{m+2} + \sum_{p=0}^{m+1} \left(\sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,p+1} \right) \mathbf{r}_p.$$

From (2.4) and (3.3) we get the next

Theorem 3.1. *Let u, v and w be quasi-definite linear functionals and let $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ be, respectively, the corresponding SMOP. Assume that the linear functionals u and v are related by*

$$\rho(x)u = v,$$

where ρ is a monic polynomial with $\deg\rho = m \geq 0$. *Also, assume that (3.1) holds. If the matrix* $D_m = \left(h_{i,j}^{[m]}\right)_{i,j=1}^{m+3}$, *with entries*

$$h_{i,j}^{[m]} = \begin{cases} 0, & if \quad m+3 > i > j \text{ and } i+1 < j, \\ 1+b_{i-2}, & if \; i = j \neq m+3, \\ d_{j-2}, & if \; i+1 = j, \\ \xi_{0,j-1}, & if \; i = m+3, \end{cases}$$

is nonsingular, then there exist polynomials ϕ_{m+2} and φ_1, with $\deg(\phi_{m+2}) = m+2$ *and* $\deg(\varphi_1) = 1$, *such that*

$$\phi_{m+2}(x)w = \varphi_1(x)v = \varphi_1(x)\rho(x)u.$$

Here

$$\varphi_1(x) = \frac{d_{m+2}}{\|Q_1(x)\|_v} Q_1(x) - \sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,0},$$

$$\varphi_{m+2}(x) = \frac{\xi_{m+3}}{\|R_{m+2}(x)\|_w} R_{m+2}(x) + \sum_{p=0}^{m+1} \frac{\left(\sum_{k=1}^{m+2} \widetilde{\xi}_{m,k}\mu_{k,p+1}\right)}{\|R_p(x)\|_w} R_p(x).$$

For instance, when $\rho(x) = 1$, i. e. $u = v$, (3.1) becomes

$$P_{n+1}(x) + a_n^{[1]} P_n(x) + a_n^{[2]} P_{n-1}(x) = R_{n+1}(x) + d_n R_n(x).$$

Of course, this is a $(2,1)-$coherence relation. According to the above theorem, since

$$D_0 = \begin{bmatrix} 1 & d_0 & 0 \\ 0 & 1+b_0 & d_1 \\ \xi_{0,0} & \xi_{0,1} & \xi_{0,2} \end{bmatrix},$$

where $\xi_{0,0} := 1$, $\xi_{0,1} := a_0^{[1]}$, and $\xi_{0,2} := a_1^{[2]}$, if $|D_0| = d_0 d_1 - a_0^{[1]} d_1 + a_1^{[2]}(1+b_0) \neq 0$, then the pair (u,v) satisfies

$$\phi_2(x)w = \varphi_1(x)u.$$

This is the same result obtained in [1] for the particular case $M = 1$ and $N = 2$. Notice that this case has been deeply studied in [17].

Now, we assume $\rho(x) = x$ and u is supported on $[a,b]$, with $a > 0$. We also suppose that the SMOP $\{P_n(x)\}_{n\geq 0}$ satisfies the three term recurrence relation (TTRR in short)

$$xP_n(x) = P_{n+1}(x) + \alpha_n P_n(x) + \beta_n P_{n-1}(x), \quad n \geq 0,$$

and $\{Q_n(x)\}_{n\geq 0}$ satisfies $xQ_n(x) = P_{n+1}(x) - \sigma_n P_n(x)$, $n \geq 0$, where $\sigma_n = \frac{P_{n+1}(0)}{P_n(0)}$. The TTRR associated with $\{R_n(x)\}_{n\geq 0}$ is $xR_n(x) = R_{n+1}(x) + \lambda_n R_n(x) + \gamma_n R_{n-1}(x, n \geq$

0). From (3.1) we get the algebraic relation

$$
\begin{aligned}
& (1+b_n)\, P_{n+2}(x) + \left[\alpha_{n+1} + a_n^{[1]} + b_n\,(c_n - \sigma_{n+1})\right] P_{n+1}(x) \\
& + \left[\beta_{n+1} + a_n^{[1]}\alpha_n + a_n^{[2]} - b_n c_n \sigma_n\right] P_n(x) \\
& + \left[\beta_n a_n^{[1]} + a_n^{[2]}\alpha_{n-1}\right] P_{n-1}(x) + \beta_{n-1} a_n^{[2]} P_{n-2}(x) \\
= \;& (1+b_n)\, R_{n+2}(x) + \left[\lambda_{n+1}\,(1+b_n) + d_n\right] R_{n+1}(x) + \left[\gamma_{n+1}\,(1+b_n) + \lambda_n d_n\right] R_n(x) \\
& + \gamma_n d_n R_{n-1}(x).
\end{aligned}
$$

Assuming $b_n \neq -1$ for every n, then $\{P_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ satisfy a $(4,3)-$coherence relation, namely

$$
P_{n+2}(x) + \sum_{k=1}^{4} A_{n,k} P_{n+2-k}(x) = R_{n+2}(x) + \sum_{k=1}^{3} B_{n,k} R_{n+2-k}(x),
$$

with $A_{n,4}B_{n,3} \neq 0$. According to [1], u and w are related by

$$
\widetilde{\phi}(x)u = \widetilde{\psi}(x)w,
$$

where $\widetilde{\phi}$ and $\widetilde{\psi}$ are polynomials such that $\deg \widetilde{\psi} = 4$ and $\deg \widetilde{\phi} = 3$. Here

$$
D_1 = \begin{bmatrix} 1 & d_0 & 0 & 0 \\ 0 & 1+b_0 & d_1 & 0 \\ 0 & 0 & 1+b_1 & d_2 \\ 1 & a_0^{[1]} + \|P_1\|_u^2 & a_1^{[2]} + \|P_1\|_u^2\, a_1^{[1]} & \|P_1\|_u^2\, a_2^{[2]} \end{bmatrix}.
$$

From the above theorem, if

$$
\begin{aligned}
& |D_1| \\
= \;& \left(d_0 - a_0^{[1]} - \|P_1\|_u^2\right) d_1 d_2 + \left(\left(a_1^{[2]} + \|P_1\|_u^2\, a_1^{[1]}\right) d_2 - \|P_1\|_u^2\, a_2^{[2]}\,(1+b_1)\right)(1+b_0) \\
\neq \;& 0,
\end{aligned}
$$

then the pair (u,w) satisfies

$$
\phi_3(x)w = x\varphi_1(x)u.
$$

Thus, under the conditions of such a particular framework, it is possible to improve the result obtained in [1].

4. Case $i = 1$

In this section we consider the relation (1.6) with $i = 1$. Then

$$\begin{aligned} & \frac{P'_{n+2}(x)}{n+2} + a_n^{[1]} \frac{P'_{n+1}(x)}{n+1} a_n^{[2]} \frac{P'_n(x)}{n} + b_n \left(Q_{n+1}(x) + c_n Q_n(x)\right) \\ = \quad & \left(1+b_n\right) R_{n+1}(x) + d_n R_n(x), \quad n \geq 0, \end{aligned}$$

holds. We assume that $1 + b_n \neq 0, n \geq 0$. Here $\{P_n(x)\}_{n\geq 0}$, $\{Q_n(x)\}_{n\geq 0}$ and $\{R_n(x)\}_{n\geq 0}$ are orthogonal with respect to u, v and w, respectively. Besides, let ρ be a polynomial such that $\deg \rho = m \geq 1$ and $\rho(x)u = v$. Additionally, in the sequel we assume that u is semiclassical of class at most s. Under these conditions (see Theorem 4) there exists a polynomial σ, with degree $t \leq s+2$, such that

$$\sigma(x) \frac{P'_{n+1}(x)}{n+1} = \sum_{k=n-s}^{n+t} \beta_{n,k} P_k(x), \tag{4.1}$$

where $\beta_{n,k} = \dfrac{\left\langle u, \sigma(x) \frac{P'_{n+1}(x)}{n+1} P_k(x) \right\rangle}{\|P_k\|_u^2}$ and $\beta_{n,n-s} \neq 0$. As in the previous section, let us define $T_{n+1}(x) := \left(1+b_n\right) R_{n+1}(x) + d_n R_n(x)$ and $\{\mathbf{t}_n\}_{n\geq 0}$ be the dual basis associated with the sequence $\{T_n(x)\}_{n\geq 0}$. Now, we consider the expansion of $\sigma \mathbf{q}_n$ in terms of the basis $\{\mathbf{t}_n\}_{n\geq 0}$, where

$$\begin{aligned} & \langle \sigma \mathbf{q}_n, T_k \rangle \\ = \quad & \left\langle \sigma \mathbf{q}_n, \frac{P'_{k+1}(x)}{k+1} + a_{k-1}^{[1]} \frac{P'_k(x)}{k} + a_{k-1}^{[2]} \frac{P'_{k-1}(x)}{k-1} \right\rangle \\ & + b_{k-1} \langle \sigma \mathbf{q}_n, Q_k(x) + c_{k-1} Q_{k-1}(x) \rangle . \end{aligned}$$

As a consequence of (4.1) we get

$$\begin{aligned} & \sigma(x) \frac{P'_{k+1}(x)}{k+1} + a_{k-1}^{[1]} \sigma(x) \frac{P'_k(x)}{k} + a_{k-1}^{[2]} \sigma(x) \frac{P'_{k-1}(x)}{k-1} \\ = \quad & \left(\sum_{j=k-s}^{k+t} \beta_{k,j} P_j(x) \right) + a_{k-1}^{[1]} \left(\sum_{j=k-s-1}^{k+t-1} \beta_{k-1,j} P_j(x) \right) \\ & + a_{k-1}^{[2]} \left(\sum_{j=k-s-2}^{k+t-2} \beta_{k-2,j} P_j(x) \right) \\ = \quad & \sum_{j=k-s-2}^{k+t} \beta^*_{k,j} P_j(x), \end{aligned}$$

where

$$\beta^*_{k,j} := \begin{cases} \beta_{k,j}, & \text{if } j = k+t, \\ \beta_{k,j} + a_{k-1}^{[1]} \beta_{k-1,j}, & \text{if } j = k+t-1, \\ \beta_{k,j} + a_{k-1}^{[1]} \beta_{k-1,j} + a_{k-1}^{[2]} \beta_{k-2,j}, & \text{if } k+t-2 \leq j \leq k-s, \\ a_{k-1}^{[1]} \beta_{k-1,j} + a_{k-1}^{[2]} \beta_{k-2,j}, & \text{if } j = k-s-1, \\ a_{k-1}^{[2]} \beta_{k-2,j}, & \text{if } j = k-s-2. \end{cases}$$

Then

$$
\begin{aligned}
&\left\langle \sigma \mathbf{q}_n, \frac{P'_{k+1}(x)}{k+1} + a^{[1]}_{k-1}\frac{P'_k(x)}{k} + a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1} \right\rangle \\
= &\ \frac{1}{\|Q_n\|_v^2}\left\langle v, \sum_{j=k-2-s}^{k+t} \beta^*_{k,j} Q_n(x) P_j(x) \right\rangle \\
= &\ \frac{1}{\|Q_n\|_v^2} \sum_{j=k-2-s}^{k+t} \beta^*_{k,j} \left\langle v, Q_n(x) P_j(x)\right\rangle,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&\left\langle \sigma \mathbf{q}_n, \frac{P'_{k+1}(x)}{k+1} + a^{[1]}_{k-1}\frac{P'_k(x)}{k} + a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1} \right\rangle \\
= &\ \left\langle \frac{Q_n(x)}{\|Q_n\|_v^2}\rho u, \sum_{j=k-2-s}^{k+t} \beta^*_{k,j} P_j(x) \right\rangle \\
= &\ \frac{1}{\|Q_n\|_v^2} \sum_{j=k-2-s}^{k+t} \beta^*_{k,j} \left\langle u, P_j(x)\rho(x) Q_n(x)\right\rangle.
\end{aligned}
$$

Thus, we have proved the following

Lemma 4.1. *For $k < n-t$ and $k > n+m+s+2$*

$$
\left\langle \sigma \mathbf{q}_n, \frac{P'_{k+1}(x)}{k+1} + a^{[1]}_{k-1}\frac{P'_k(x)}{k} + a^{[2]}_{k-1}\frac{P'_{k-1}(x)}{k-1} \right\rangle = 0.
$$

On the other hand,

$$
\begin{aligned}
&\left\langle \sigma \mathbf{q}_n, Q_k(x) + c_{k-1}Q_{k-1}(x)\right\rangle \\
= &\ \frac{1}{\|Q_n\|_v^2}\left\langle v, Q_n(x)\sigma(x)\left(Q_k(x) + c_{k-1}Q_{k-1}(x)\right)\right\rangle
\end{aligned}
$$

and, as a consequence, we get

Lemma 4.2. *For $k < n-t$ and $k > n+t+1$*

$$
\left\langle \sigma \mathbf{q}_n, Q_k(x) + c_{k-1}Q_{k-1}(x)\right\rangle = 0.
$$

Since $t \leq m+s+1$, from the above lemmas we deduce

Proposition 4.3.

$$
\sigma \mathbf{q}_n = \sum_{k=n-t}^{n+m+s+2} \mu_{n,k} \mathbf{t}_k, \qquad n \geq t, \tag{4.2}
$$

where $\mu_{n,k} = \left\langle \mathbf{q}_n, \sigma T_k\right\rangle$.

Since $\beta^*_{t+m+s+2,m+t} = a^{[2]}_{t+m+s+1}\beta_{t+m+s,m+t} \neq 0$, we point out that in the particular case $n = t$ we get

$$\begin{aligned}
\mu_{t,t+m+s+2} &= \left\langle \sigma\mathbf{q}_t, \frac{P'_{t+m+s+3}(x)}{t+m+s+3} + a^{[1]}_{t+m+s+1}\frac{P'_{t+m+s+2}(x)}{t+m+s+2} + a^{[2]}_{t+m+s+1}\frac{P'_{t+m+s+1}(x)}{t+m+s+1} \right\rangle \\
&\quad + b_{t+m+s+1}\left\langle \sigma\mathbf{q}_t, Q_{t+m+s+2}(x) + c_{t+m+s+1}Q_{t+m+s+1}(x)\right\rangle \\
&= \frac{1}{\|Q_t\|_v^2}\sum_{j=m+t}^{2t+m+s+2} \beta^*_{t+m+s+2,j}\left\langle u, P_j(x)\rho(x)Q_t(x)\right\rangle \\
&= \frac{1}{\|Q_t\|_v^2}\beta^*_{t+m+s+2,m+t}\left\langle u, P_{m+t}(x)\rho(x)Q_t(x)\right\rangle \\
&\neq 0.
\end{aligned}$$

We consider (3.5) with $n = 0, 1, \cdots, t+s+m+1$, and (4.2) with $n = t$. Thus we get a system of linear equations with $t+m+s+3$ unknowns $\{\mathbf{t}_k\}_{k=0}^{t+m+s+2}$. Indeed,

$$D_{m,s,t}\begin{pmatrix} \mathbf{t}_0 \\ \mathbf{t}_1 \\ \vdots \\ \mathbf{t}_{t+m+s+2} \end{pmatrix} = \begin{pmatrix} \mathbf{r}_0 \\ \mathbf{r}_1 \\ \vdots \\ \mathbf{r}_{t+m+s+1} \\ \sigma\mathbf{q}_t, \end{pmatrix}$$

where the entries of $D_{m,s,t} = \left(h^{[m,s,t]}_{i,j}\right)_{i,j=1}^{m+s+t+3}$ are

$$h^{[m,s,t]}_{i,j} = \begin{cases} 0, & if \ \ m+s+t+3 > i > j \text{ and } i+1 < j, \\ 1 + b_{j-2}, & if \ i = j \neq m+s+t+3, \\ d_{i-1}, & if \ i+1 = j, \\ \mu_{t,j-1}, & if \ i = m+s+t+3, \end{cases}$$

with $b_{-1} := 0$. Assuming that $D_{m,s,t}$ is nonsingular the above system of linear equations has solution and, as a consequence, there exists a matrix $(\zeta_{k,i})_{k,i=0}^{t+m+s+2}$ such that each $\mathbf{t}_k$, $k = 0,..., m+s+t+2$, can be written in terms of the linear functionals $\{\mathbf{r}_k\}_{k=0}^{m+s+t+1}$ and $\sigma\mathbf{q}_t$ as follows

$$\mathbf{t}_k = \zeta_{k,0}\sigma\mathbf{q}_t + \zeta_{k,1}\mathbf{r}_0 + \zeta_{k,2}\mathbf{r}_1 + \cdots + \zeta_{k,t+m+s+2}\mathbf{r}_{t+s+m+1} = \zeta_{k,0}\sigma\mathbf{q}_t + \sum_{j=1}^{t+m+s+2}\zeta_{k,j}\mathbf{r}_{j-1}.$$

If we consider (4.2) with $n = t+1$, (3.5) with $n = t+m+s+2$, multiplying them by $d_{t+m+s+2}$ and $\mu_{t+1,t+m+s+3}$, respectively, and subtracting the corresponding expressions we get

$$d_{t+m+s+2}\sigma\mathbf{q}_{t+1} - \mu_{t+1,t+m+s+3}\mathbf{r}_{t+m+s+2} = d_{t+m+s+2}\sum_{k=1}^{t+m+s+2}\widetilde{\mu}_{t+1,k}\mathbf{t}_k,$$

where $\widetilde{\mu}_{t+1,k} := \mu_{t+1,k}$ for $1 \leq k \leq t+m+s+1$ and $\widetilde{\mu}_{t+1,t+m+s+2} :=$

$d_{t+m+s+2}\mu_{t+1,t+m+s+2} - (1 + b_{t+m+s+2})\,\mu_{t+1,t+m+s+3}$. Finally, replacing $\mathbf{t}_k$, with $k = 0, ..., t + m + s + 2$, in the above relation, we get

$$\begin{aligned} & d_{t+m+s+2}\sigma(x)\mathbf{q}_{t+1} - \left(\sum_{k=1}^{t+m+s+2} d_{t+m+s+2}\widetilde{\mu}_{t+1,k}\zeta_{k,0}\sigma(x)\right)\mathbf{q}_t \\ = \; & \mu_{t+1,t+m+s+3}\mathbf{r}_{t+m+s+2} + \sum_{j=1}^{t+m+s+2}\left(\sum_{k=1}^{t+m+s+2} d_{t+m+s+2}\widetilde{\mu}_{t+1,k}\zeta_{k,j}\right)\mathbf{r}_{j-1}. \end{aligned}$$

Thus,

Theorem 4.4. *Let u be a semiclassical linear functional of class s. Assume that the relation (1.6) holds and there exists a monic polynomial ρ, with $\deg\rho = m \geq 1$, such that $\rho(x)u = v$ as well as the matrix $\left(h_{i,j}^{[m,s,t]}\right)_{i,j=1}^{m+s+t+3}$ with entries*

$$h_{i,j}^{[m,s,t]} = \begin{cases} 0, & if \;\; m+s+t+3 > i > j \;\; and \; i+1 < j, \\ 1 + b_{j-2} & if \;\; i = j \neq m+s+t+3, \\ d_{i-1}, & if \;\; i+1 = j, \\ \mu_{t,j-1} & if \;\; i = m+s+t+3, \end{cases} \tag{4.3}$$

is nonsingular, then there exist polynomials ϕ_{t+1}, $\psi_{t+m+s+2}$, with $\deg(\phi_{t+1}) = t+1$ and $\deg(\psi_{t+m+s+2}) = t+m+s+2$, such that

$$\phi_{t+1}(x)\sigma(x)v = \phi_{t+1}(x)\sigma(x)\rho(x)u = \psi_{t+m+s+2}(x)w, \tag{4.4}$$

where

$$\phi_{t+1}(x) = \frac{d_{t+m+s+2}}{\|Q_{t+1}\|_v^2}\left(Q_{t+1}(x) - \frac{\|Q_{t+1}\|_v^2}{\|Q_t\|_v^2}\left(\sum_{k=1}^{t+m+s+2}\widetilde{\mu}_{t+1,k}\zeta_{k,0}\right)Q_t(x)\right),$$

and

$$\begin{aligned} & \psi_{t+m+s+2}(x) \\ = \; & \frac{\mu_{t+1,t+m+s+3}}{\|R_{t+m+s+2}\|_w^2}R_{t+m+s+2}(x) + d_{t+m+s+2}\sum_{j,k=1}^{t+m+s+2}\frac{\widetilde{\mu}_{t+1,k}\zeta_{k,j}}{\|R_{j-1}\|_w^2}R_{j-1}(x), \end{aligned}$$

where σ is the polynomial satisfying (4.1).

Remark 1. The above result is based on the choice $n = t$ in (4.2) and $n = 0, 1, \cdots, t+s+m+1$, in relation (3.5). In a general way, we also could consider $n = t + i$ in (4.2) and $n = i, 1, ..., t + i + s + m + 1$, in (3.5) for $i \in \mathbb{Z}^+ \cup \{0\}$. Thus (4.4) becomes

$$\phi_{t+i+1}(x)\sigma(x)\rho(x)u = \psi_{t+i+m+s+2}(x)w. \tag{4.5}$$

Remark 2. When $m=0$, (4.2) can be written

$$\sigma \mathbf{q}_n = \sum_{k=n-t}^{n+\Phi_{t,s}} \mu_{n,k} \mathbf{t}_k, \qquad n \geq t, \tag{4.6}$$

where $\Phi_{t,s} = \max\{t+1, s+2\}$. As above, we can consider (3.5) with $n = 0, 1, ..., \Phi_{t,s}$ and (4.6) with $n = t$. In this way, we get $t + \Phi_{t,s} + 1$ unknowns $\{\mathbf{t}_k\}_{k=0}^{t+\Phi_{t,s}}$, namely, we have the system of linear equations

$$D_{s,t} \left(\mathbf{t}_0, \mathbf{t}_1, ..., \mathbf{t}_{t+\Phi_{t,s}}\right)^T = \left(\mathbf{r}_0, \mathbf{r}_1, ..., \mathbf{r}_{t+\Phi_{t,s}-1}, \sigma \mathbf{q}_t\right)^T,$$

where $D_{s,t}$ has the same structure as in (4.3) with $\Phi_{t,s}$ instead of $m+s+2$. In this way, we get

$$\phi_{t+1}(x)\sigma(x)u = \psi_{t+\Phi_{t,s}}(x)w. \tag{4.7}$$

Now, we want to study the relation between the formal Stieltjes series associated with the functionals u and w. Let $\mathbf{P}_k^{[1]}$ be the $k-th$ term of the dual basis associated with the sequence $\left\{\frac{P'_{n+1}(x)}{n+1}\right\}_{n\geq 0}$. Then we expand the linear functional $\sigma \mathbf{P}_n$ in terms of such a basis. Here $\{\mathbf{P}_n\}_{n\geq 0}$ is the dual basis associated with $\{P_n(x)\}_{n\geq 0}$. From (2.6) we get

$$\left\langle \sigma \mathbf{P}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = \left\langle \mathbf{P}_n, \sum_{j=k-s}^{k+t} \beta_{k,j} P_j(x) \right\rangle, \ \ \beta_{k,k-s} \neq 0.$$

If either $k+t < n$ or $k-s > n$, then $\left\langle \sigma \mathbf{P}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = 0$. Therefore, from $\lambda_{n,k} := \left\langle \sigma \mathbf{P}_n, \frac{P'_{k+1}(x)}{k+1} \right\rangle = \left\langle \mathbf{P}_n, \sum_{j=k-s}^{k+t} \beta_{k,j} P_j(x) \right\rangle$, we get

$$\sigma \mathbf{P}_n = \sum_{k=n-t}^{n+s} \lambda_{n,k} \mathbf{P}_k^{[1]}. \tag{4.8}$$

In particular, $\lambda_{t,t+s} = \left\langle \mathbf{P}_t, \sum_{j=t}^{2t+s} \beta_{t+s,j} P_j(x) \right\rangle = \beta_{t+s,t} \neq 0$. For $n = t$ and taking derivatives, in the distributional sense, from (2.3) the above relation becomes

$$D\left(\sigma \mathbf{P}_t\right) = \xi_{t+s+1}(x)u, \tag{4.9}$$

where $\xi_{t+s+1}(x) := \sum_{k=0}^{t+s} \widetilde{\lambda}_{t,k} P_{k+1}(x)$ and $\widetilde{\lambda}_{t,k} = -\frac{(k+1)}{\|P_{k+1}\|_u^2} \lambda_{t,k}$. In particular, notice that $\widetilde{\lambda}_{t,t+s} = -\frac{(t+s+1)}{\|P_{t+s+1}\|_u^2} \lambda_{t,t+s} \neq 0$ and, therefore, $\deg(\xi_{t+s+1}) = t+s+1$. Multiplying both sides of (4.9) by $\phi_{t+1}\rho$, from (4.4) we get

$$\phi_{t+1}(x)\rho(x)D\left(\frac{1}{\|P_t\|_u^2}P_t(x)\sigma(x)u\right) = \phi_{t+1}(x)\rho(x)\xi_{t+s+1}(x)u.$$

As a consequence, from (4.4) we obtain

Theorem 4.5. *The linear functionals w and u are related through the differential relation*

$$D(P_t(x)\psi_{t+m+s+2}(x)w) = \omega_{2t+m+s+2}(x)u, \tag{4.10}$$

where

$$\omega_{2t+m+s+2}(x) = \|P_t\|_u^2\,\phi_{t+1}(x)\rho(x)\xi_{t+s+1}(x) + (\phi_{t+1}(x)\rho(x))'\,P_t(x)\sigma(x)$$

and $\deg(\omega_{2t+m+s+2}) = 2t+m+s+2$.

From (4.4) and (4.10) we can deduce an upper bound for the class of w. Indeed, after straightforward computations we get

Corollary 4.6. *The linear functional w satisfies the differential relation*

$$D(\widetilde{\phi}w) = \widetilde{\psi}w,$$

where $\widetilde{\phi}(x) = \phi_{t+1}(x)\sigma(x)\rho(x)P_t(x)\psi_{t+m+s+2}(x)$ *and*

$$\widetilde{\psi}(x) = \psi_{t+m+s+2}(x)\left((\phi_{t+1}(x)\sigma(x)\rho(x))'\,P_t(x) + \omega_{2t+m+s+2}(x)\right).$$

Thus, w is semiclassical of class almost $2m+2s+3t+3$.

Remark 3. As above, when $m=0$ we get

$$D(P_t(x)\psi_{t+\Phi_{t,s}}(x)w) = \omega_{2t+s+2}(x)u,$$

with

$$\omega_{2t+s+2}(x) = \|P_t\|_u^2\,\phi_{t+1}(x)\xi_{t+s+1}(x) + \phi'_{t+1}(x)P_t(x)\sigma(x).$$

Remark 4. Notice that we can write the differential relation obtained in the above theorem as follows

$$D(P_{t+i}(x)\psi_{t+i+m+s+2}(x)w) = \omega_{2t+2i+m+s+2}(x)u, \qquad i \in \mathbb{Z}^+ \cup \{0\}. \tag{4.11}$$

Let $S_u(z) = -\sum_{n\geq 0}\frac{u_n}{z^{n+1}}$ and $S_w(z) = -\sum_{n\geq 0}\frac{w_n}{z^{n+1}}$ be the formal Stieltjes series associated with u and w, respectively. Here $\{u_n\}_{n\geq 0}$ and $\{w_n\}_{n\geq 0}$ denote the corresponding moment sequences, respectively. Taking into account the particular cases $i=0$ and $i=1$ in (4.11) and, as a consequence of Theorem 7, we get

Corollary 4.7. *Under the same conditions of the above theorem, the Stieltjes series $S_u(z)$ and $S_w(z)$ satisfy the relation*

$$\omega_{2t+2n+m+s+2}(z)S_u(z) - (\psi_{t+n+m+s+2}(z)P_{t+n}(z)S_w(z))' = A_n^{[m,t,s]}(z),$$

where

$$A_n^{[m,t,s]}(z) = \left(w\theta_0\left(P_{t+n}\psi_{t+n+m+s+2}\right)\right)'(z) - \left(u\theta_0\omega_{2t+2n+m+s+2}\right)(z).$$

On the other hand, $S_w(z)$ satisfies the following first order non-homogeneous linear differential equation

$$B_1(z)S_w'(z) + B_0(z)S_w(z) = C(z),$$

where, for $j = 0, 1$,

$$\begin{aligned}&B_j(z)\\ =\ &\binom{1}{j}\left(\omega_{2t+m+s+2}(z)P_{t+1}(x)\psi^{(1-j)}_{t+m+s+3}(z) - \omega_{2t+m+s+4}(z)P_t(x)\psi^{(1-j)}_{t+m+s+2}(z)\right)\end{aligned}$$

and

$$C(z) = \omega_{2t+m+s+4}(z)A_0^{[m,t,s]}(z) - \omega_{2t+m+s+2}(z)A_1^{[m,t,s]}(z).$$

In order to illustrate the results of this section, we consider $\rho(x) = x$, $s = 0$, and $u = \mathcal{L}^{(\alpha)}$, the classical Laguerre functional with $\alpha > -1$. Notice that $\sigma(x) = x$. Thus,

$$L^{\alpha+1}_{n+1}(x) + \frac{a_n^{[1]} + b_nc_n}{1+b_n}L^{\alpha+1}_n(x) + \frac{a_n^{[2]}}{1+b_n}L^{\alpha+1}_{n-1}(x) = R_{n+1}(x) + \frac{d_n}{1+b_n}R_n(x), n \geq 0. \tag{4.12}$$

Also, the SMOP $\left\{L^{\alpha+1}_{n+1}(x)\right\}_{n\geq 0}$ satisfies the TTRR

$$xL^{\alpha+1}_n(x) = L^{\alpha+1}_{n+1}(x) + (2n+\alpha+2)\,L^{\alpha+1}_n(x) + n(n+\alpha+1)L^{\alpha+1}_{n-1}(x), n \geq 0, \tag{4.13}$$

and $\|L^\alpha_n\|^2 = n!\Gamma(n+\alpha+1)$. We also assume that

$$xR_n(x) = R_{n+1}(x) + \lambda_nR_n(x) + \gamma_nR_{n-1}(x), \quad n \geq 0, \tag{4.14}$$

with $R_{-1}(x) = 0$, is the TTRR for $\{R_n(x)\}_{n\geq 0}$. From (4.12), (4.13) and (4.14) we get

$$\begin{aligned}&-\frac{a_{n-1}^{[1]} + b_{n-1}c_{n-1}}{1+b_{n-1}}xL^{\alpha+1}_{n-1}(x) - \frac{a_{n-1}^{[2]}}{1+b_{n-1}}xL^{\alpha+1}_{n-2}(x)\\ &+\left(\frac{a_n^{[1]} + b_nc_n}{1+b_n} - (2n+\alpha+2)\right)L^{\alpha+1}_n(x) + \left(\frac{a_n^{[2]}}{1+b_n} - n(n+\alpha+1)\right)L^{\alpha+1}_{n-1}(x)\\ =\ &\left(\frac{d_n}{1+b_n} - \lambda_n\right)R_n(x) - \frac{d_{n-1}}{1+b_{n-1}}xR_{n-1}(x) - \gamma_nR_{n-1}(x), n \geq 0.\end{aligned}$$

The comparison of the coefficients of x^n and x^{n-1} in both hand sides yields explicit expressions for the recurrence coefficients of $\{R_n(x)\}_{n\geq 0}$. Indeed, for $n \geq 0$,

$$\lambda_n = \theta_{n-1} - \theta_n + (2n + \alpha + 2)\,, \tag{4.15}$$

and

$$\begin{aligned} & \gamma_n \\ = \; & \Lambda_{n-1} - \Lambda_n - n(n+\alpha+1)\,(2n+\alpha+1) \\ & + \left(\lambda_n - \frac{d_n}{1+b_n}\right)\Omega_n + \frac{d_{n-1}}{1+b_{n-1}}\Omega_{n-1}, \end{aligned} \tag{4.16}$$

where

$$\theta_n = \frac{a_n^{[1]} + b_n c_n - d_n}{1+b_n}, \qquad \Lambda_n = \frac{a_n^{[2]} - \left(a_n^{[1]} + b_n c_n\right) n\,(n+\alpha+1)}{1+b_n}, \tag{4.17}$$

and

$$\Omega_n = n\,(n+\alpha+1) - \theta_{n-1}. \tag{4.18}$$

If

$$a_n^{[1]} = \frac{1}{n+1},\; a_n^{[2]} = \frac{n+2}{n+1},\; b_n = \frac{3}{2n+1},\; c_n = d_n = \frac{n+1}{n+2}, \quad n \geq 0,$$

with $a_{-1}^{[1]} = a_{-1}^{[2]} = b_{-1} = c_{-1} = d_{-1} := 0$, then the recurrence coefficients $\{\lambda_n, \gamma_n\}_{n\geq 0}$ are completely determined by (4.15), (4.16), (4.17) and (4.18). For instance, if $\alpha = 0$ we get

$$\lambda_0 = \frac{3}{2},\; \lambda_1 = \frac{17}{4},\; \lambda_2 = \frac{299}{48},\; \lambda_3 = \frac{9799}{1200},\; \lambda_4 = \frac{6073}{600},\; \lambda_5 = \frac{35551}{2940},$$

and

$$\gamma_1 = \frac{19}{24},\; \gamma_2 = \frac{13559}{384},\; \gamma_3 = \frac{5518523}{57600},\; \gamma_4 = \frac{4003607}{20000},\; \gamma_5 = \frac{254335129}{705600}.$$

In this way, from (4.14) the elements of the sequence $\{R_n(x)\}_{n\geq 0}$ can be deduced. As an example, from the above data we get

$$R_0(x) = 1, \quad R_1 = x - \frac{3}{2},$$

$$R_2(x) = x^2 - \frac{23}{4}x + \frac{67}{12}, \quad R_3(x) = x^3 - \frac{575}{48}x^2 + \frac{2339}{384}x + \frac{41\,899}{2304},$$

$$R_4(x) = x^4 - \frac{4029}{200}x^3 + \frac{2431}{300}x^2 + \frac{239\,311\,997}{460\,800}x - \frac{125\,968\,831}{184\,320},$$

and

$$\begin{aligned} & R_5(x) \\ = \; & x^5 - \frac{454}{15}x^4 + \frac{11\,351}{960}x^3 + \frac{32\,662\,826\,593}{11\,520\,000}x^2 - \frac{1979\,414\,728\,109}{276\,480\,000}x + \frac{1812\,077\,996\,999}{552\,960\,000}. \end{aligned}$$

On the other hand, according to Theorem 12 u and w are related through

$$\phi_2(x)x^2 u = \psi_4(x)w,$$

and from Theorem 15 they satisfy

$$D(L_1^\alpha(x)\psi_4(x)w) = \omega_5(x)u.$$

As a consequence, for a fixed value $\alpha > -1$, we can explicitly obtain the polynomials ϕ_2, ψ_4 and ω_5. On one hand, according to definitions of ϕ_2, ψ_4 we get

$$\begin{aligned} & \left\| L_n^{\alpha+1} \right\|^2 \mu_{n,k} \\ = \; & \left\langle \mathcal{L}^{(\alpha+1)}, x L_n^{\alpha+1}(x) \left((1+b_{k-1})\, L_k^{\alpha+1}(x) + \left(a_{k-1}^{[1]} + b_{k-1}c_{k-1} \right) L_{k-1}^{\alpha+1}(x) + a_{k-1}^{[2]} L_{k-2}^{\alpha+1}(x) \right) \right\rangle. \end{aligned}$$

On the other hand, let consider

$$\left(h_{i,j}^{[1,0,1]} \right)_{i,j=1}^{5} = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 4 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 2 & \frac{3}{4} & 0 \\ 0 & 0 & 0 & \frac{8}{5} & \frac{4}{5} \\ 1 & \frac{37}{2} & \frac{109}{6} & \frac{301}{30} & \frac{29}{4} \end{bmatrix}.$$

This matrix is nonsingular and, as a consequence, the matrix $(\zeta_{k,i})_{k,i=0}^4$ is

$$\begin{bmatrix} \frac{4859}{4874} & -\frac{2707}{19\,496} & \frac{134}{7311} & -\frac{2175}{77\,984} & \frac{15}{4874} \\ \frac{15}{2437} & \frac{2707}{9748} & -\frac{268}{7311} & \frac{2175}{38\,992} & -\frac{15}{2437} \\ -\frac{90}{2437} & -\frac{405}{2437} & \frac{536}{2437} & -\frac{6525}{19\,496} & \frac{90}{2437} \\ \frac{240}{2437} & \frac{1080}{2437} & \frac{1820}{2437} & \frac{2175}{2437} & -\frac{240}{2437} \\ -\frac{480}{2437} & -\frac{2160}{2437} & -\frac{3640}{2437} & -\frac{5215}{9748} & \frac{480}{2437} \end{bmatrix}.$$

Finally, (see [12]), $\|R_n\|_w^2 := \prod_{k=0}^{n} \gamma_k$, with $\gamma_0 := 1$. Besides, the polynomials $R_i(x)$, $i = 0, 1, 2, 3, 4$, will be needed. On the other hand, according to the definition of ω_5, through straightforward calculations you can deduce the values of $\beta_{n,k} = \dfrac{\left\langle \mathcal{L}^{(\alpha+1)}, L_n^{\alpha+1}(x) L_k^\alpha(x) \right\rangle}{\left\| L_k^\alpha \right\|^2}$ and $\left\| L_n^\alpha \right\|^2 \lambda_{n,k} = \left\langle \mathcal{L}^{(\alpha+1)}, L_n^\alpha(x) L_k^{\alpha+1}(x) \right\rangle.$

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The authors declare that they have no conflict of interest

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