COHERENT PAIRS OF BIVARIATE ORTHOGONAL POLYNOMIALS

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Abstract. Coherent pairs of measures were introduced in 1991 and constitute a very useful tool in the study of Sobolev orthogonal polynomials on the real line. In this work, coherence and partial coherence in two variables appear as the natural extension of the univariate case. Given two families of bivariate orthogonal polynomials expressed as *polynomial systems*, they are a *partial coherent pair* if a polynomial of the second family can be given as a linear combination of the first partial derivatives of (at most) three consecutive polynomials of the first family. A *full coherent pair* is a pair of families of bivariate orthogonal polynomials related by means of partial coherent relations in each variable. Consequences of this kind of relations concerning both families of bivariate orthogonal polynomials are studied. Finally, some illustrative examples are provided.

1. Introduction

Coherent pairs of measures supported on the real line appear in [9] in the framework of the theory of Sobolev orthogonal polynomials associated with a vector of nontrivial positive measures (μ_0, μ_1) supported on infinite subsets of the real line. Indeed, the Sobolev inner product is defined as

$$\langle p, q\rangle_S = \int_{\mathbb{R}} p(x)q(x)d\mu_0(x) + \lambda \int_{\mathbb{R}} p'(x)q'(x)d\mu_1(x),$$

where p and q are polynomials with real coefficients and λ is a nonnegative real number. The vector of measures (μ_0, μ_1) is said to be coherent if the corresponding sequences of monic orthogonal polynomials $\{P_n(x;\mu_0)\}_{n\geqslant 0}$ and $\{P_n(x;\mu_1)\}_{n\geqslant 0}$ satisfy

$$nP_{n-1}(x;\mu_1) = P_n'(x;\mu_0) + a_n P_{n-1}'(x;\mu_0), \quad n \geqslant 2, \tag{1.1}$$

with $a_n \neq 0$ for $n \geqslant 2$.

If (1.1) holds, then the following relation between the sequence of monic orthogonal polynomials $\{S_n(x;\lambda)\}_{n\geqslant 0}$ associated with the Sobolev inner product and the

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sequence of monic orthogonal polynomials $\{P_n(x;\mu_0)\}_{n\geqslant 0}$ with respect to the measure μ_0 holds

$$S_n(x;\lambda) + b_n(\lambda)\, S_{n-1}(x;\lambda) = P_n(x;\mu_0) + a_n P_{n-1}(x;\mu_0), \quad n \geqslant 1.$$

In [17], H. G. Meijer proved that if (μ_0, μ_1) is a coherent pair of positive measures supported on the real line, then one of the measures has to be classical (Laguerre or Jacobi). In fact, the above result is obtained in a more general framework. That is, he deals with orthogonal polynomials with respect to a pair of quasi-definite moment functionals on the set of polynomials with real coefficients and proves that one of the functionals is the Laguerre, Jacobi or Bessel functional. Observe that positive definite moment functionals are associated with nontrivial probability measures supported on the real line. Thus, Meijer [17] also determines all the possible coherent pairs of positive measures supported on the real line. The study of polynomials orthogonal with respect to a Sobolev inner product has attracted the interest of many researchers during the last thirty years. Asymptotic properties of such polynomials as well as properties concerning the distribution of their zeros have been deeply analyzed (see [16] for a short history, and [15] as an updated survey on these topics).

The aim of this contribution is to analyze the concept of coherent pairs of measures supported on some domains in the plane. Notice that in this case partial derivatives with respect to each variable must be involved in the definition of coherent pairs and, in such a sense, we will distinguish between partial and full coherence. In addition, we will need the concept of classical orthogonal polynomials in the bivariate case as well as some of their characterizations.

The structure of the paper is as follows. In Section 2, we deal with the basic definitions and facts related to bivariate orthogonal polynomials with respect to a linear functional $\mathbf{u}$ in the algebraic dual of the linear space of polynomials in two variables. In particular, we focus our attention on $\mathcal{D}$-classical and semiclassical bivariate orthogonal polynomials. In Section 3, partial x_k-coherent pairs of bivariate orthogonal polynomials, $k = 1, 2$, are introduced and the relation between the corresponding linear functionals is given (Theorem 3.5). A discussion when one of the moment functionals is classical is presented. The concept of full coherence is also analyzed and, in particular, is proved that a moment functional is self-coherent if and only if it is a $\mathcal{D}$-classical linear functional. In Section 4, some illustrative examples concerning coherent pairs of measures supported on product domains, the unit disk, the unit disk with two mass points, the parabolic biangle and the simplex, are discussed. The analysis of bivariate polynomials orthogonal with respect to a Sobolev inner product associated with a pair of either partial or full coherent measures remains an interesting open problem.

2. Basic tools

This section is devoted to present the basic background about bivariate polynomials we need throughout this paper. We follow mainly [5], and use the notations introduced in [14].

Let Π_n denote the linear space of real polynomials in two variables of total degree not greater than n, and $\Pi = \cup_{n\geqslant 0}\Pi_n$ the set of all bivariate real polynomials.

A useful tool in the theory of orthogonal polynomials in several variables is the representation of a basis of polynomials as a *polynomial system* (PS).

Definition 2.1. *A polynomial system (PS) is a sequence of column vectors of increasing size, $\{\mathbb{P}_n\}_{n\geqslant 0}$, whose entries are linearly independent polynomials of exact total degree n*

$$\mathbb{P}_n = \mathbb{P}_n(x,y) = (P_{n,0}(x,y), P_{n,1}(x,y), \ldots, P_{n,n}(x,y))^t,$$

where, as usual, the superscript t means the transpose.

Clearly, for $n \geqslant 0$, the entries of $\{\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_n\}$ constitute a basis of Π_n. A particular example of a PS is the *canonical basis*, defined by

$$\mathbb{X}_n = (x^{n-m}y^m)_{0\leqslant m\leqslant n} = (x^n, x^{n-1}y, \ldots, y^n)^t,$$

and every polynomial $p \in \Pi_n$ can be represented as

$$p(x,y) = \sum_{i=0}^{n} c_i^t\, \mathbb{X}_i,$$

where c_i are coefficient column vectors of size $i+1$, for $0 \leqslant i \leqslant n$.

Moreover, if $\{\mathbb{P}_n\}_{n\geqslant 0}$ is a PS, then

$$\mathbb{P}_n = C_n^n \mathbb{X}_n + C_n^{n-1}\mathbb{X}_{n-1} + \cdots + C_n^0 \mathbb{X}_0, \tag{2.1}$$

where C_n^i are matrices of size $(n+1)\times(i+1)$ whose entries are real numbers, such that the $n+1$ order square matrix C_n^n is non-singular and called *the leading coefficient* of the vector polynomial $\mathbb{P}_n$. The PS $\{\hat{\mathbb{P}}_n\}_{n\geqslant 0}$ with $\hat{\mathbb{P}}_n = (C_n^n)^{-1}\mathbb{P}_n$, $n \geqslant 0$, is said to be *monic PS*, since every entry of

$$\hat{\mathbb{P}}_n = (\hat{P}_{n,0}(x,y), \ldots, \hat{P}_{n,n}(x,y))^t,$$

is a monic polynomial of the form

$$\hat{P}_{n,m}(x,y) = x^{n-m}y^m + \text{lower total degree terms}, \quad 0 \leqslant m \leqslant n,$$

and $\hat{C}_n^n = I_{n+1}$, the identity matrix of size $n+1$.

As in [5], for $n \geqslant 0$, we define the matrices

$$L_{n,1} = \left(\begin{array}{cccc|c} 1 & & & \bigcirc & 0 \\ & 1 & & & 0 \\ & & \ddots & & \vdots \\ \bigcirc & & & 1 & 0 \end{array}\right), \qquad L_{n,2} = \left(\begin{array}{c|cccc} 0 & 1 & & & \bigcirc \\ 0 & & 1 & & \\ \vdots & & & \ddots & \\ 0 & \bigcirc & & & 1 \end{array}\right), \tag{2.2}$$

both of size $(n+1)\times(n+2)$. We can then write

$$x\,\mathbb{X}_n = L_{n,1}\,\mathbb{X}_{n+1}, \qquad y\,\mathbb{X}_n = L_{n,2}\,\mathbb{X}_{n+1}.$$

Observe that, for $k = 1, 2$, $\operatorname{rank} L_{n,k} = n+1$. For $n \geqslant 1$, let define the matrices

$$\Omega_{n,1} = \left(\begin{array}{cccc|c} \frac{1}{n} & & & \bigcirc & 0 \\ & \frac{1}{n-1} & & & 0 \\ & & \ddots & & \vdots \\ \bigcirc & & & 1 & 0 \end{array}\right), \qquad \Omega_{n,2} = \left(\begin{array}{c|cccc} 0 & 1 & & & \bigcirc \\ 0 & & \frac{1}{2} & & \\ \vdots & & & \ddots & \\ 0 & \bigcirc & & & \frac{1}{n} \end{array}\right), \tag{2.3}$$

both of size $n\times(n+1)$. It is clear that $\operatorname{rank}\Omega_{n,k} = n$ for $k = 1, 2$. Then, for $n \geqslant 1$, we get

$$\Omega_{n,1}\,\partial_1\mathbb{X}_n = \mathbb{X}_{n-1} \quad \text{and} \quad \Omega_{n,2}\,\partial_2\mathbb{X}_n = \mathbb{X}_{n-1}, \tag{2.4}$$

where, as usual, we denote $\partial_1 = \partial_x = \partial/\partial_x$ and $\partial_2 = \partial_y = \partial/\partial_y$.

Now, we show that it is possible to construct monic polynomial systems associated with each partial derivative of a given PS. In fact, let $\{\mathbb{P}_n\}_{n\geqslant 0}$ be a PS. For $n \geqslant 0$, we define

$$\begin{aligned}\mathbb{P}_n^{(1)} &= \Omega_{n+1,1}(C_{n+1}^{n+1})^{-1}\partial_1\,\mathbb{P}_{n+1},\\ \mathbb{P}_n^{(2)} &= \Omega_{n+1,2}(C_{n+1}^{n+1})^{-1}\partial_2\,\mathbb{P}_{n+1},\end{aligned} \tag{2.5}$$

where C_{n+1}^{n+1} is the leading coefficient of $\mathbb{P}_{n+1}$ defined in (2.1), and $\Omega_{n+1,1}$ and $\Omega_{n+1,2}$ are given in (2.3). In the particular case when $\{\mathbb{P}_n\}_{n\geqslant 0}$ is monic, the above relations become

$$\begin{aligned}\mathbb{P}_n^{(1)} &= \Omega_{n+1,1}\partial_1\,\mathbb{P}_{n+1},\\ \mathbb{P}_n^{(2)} &= \Omega_{n+1,2}\partial_2\,\mathbb{P}_{n+1}.\end{aligned} \tag{2.6}$$

Lemma 2.2. *$\{\mathbb{P}_n^{(1)}\}_{n\geqslant 0}$ and $\{\mathbb{P}_n^{(2)}\}_{n\geqslant 0}$ defined as in* (2.5) *are both monic PS.*

Proof. Suppose that $\{\mathbb{P}_n\}_{n\geqslant 0}$ is a PS not necessarily monic. In this case, we use the explicit expression (2.1) and the fact that the leading coefficient C_{n+1}^{n+1} is non-singular. Then, for $n \geqslant 0$, the following PS

$$(C_{n+1}^{n+1})^{-1}\mathbb{P}_{n+1} = \mathbb{X}_{n+1} + \sum_{i=0}^{n}(C_{n+1}^{n+1})^{-1}C_{n+1}^{i}\,\mathbb{X}_i,$$

is monic. Taking partial derivatives, multiplying times $\Omega_{n+1,k}$ and using (2.4), we get

$$\Omega_{n+1,k}(C_{n+1}^{n+1})^{-1}\partial_k\mathbb{P}_{n+1} = \mathbb{X}_n + \sum_{i=0}^{n}\Omega_{n+1,k}(C_{n+1}^{n+1})^{-1}C_{n+1}^{i}\,\partial_k\mathbb{X}_i,$$

and the result follows. □

Given a sequence of real numbers $\{\mu_{h,k}\}_{h,k\geqslant 0}$, a linear functional $\mathbf{u} : \Pi \to \mathbb{R}$ can be defined by means of its moments

$$\langle \mathbf{u}, x^h\, y^k\rangle = \mu_{h,k},$$

and extended by linearity to all bivariate polynomials. We usually call $\mathbf{u}$ a *moment functional*.

We will say that a PS $\{\mathbb{P}_n\}_{n\geqslant 0}$ is an *orthogonal polynomial system (OPS)* with respect to $\mathbf{u}$ if

$$\langle \mathbf{u}, \mathbb{P}_n\,\mathbb{P}_m^t\rangle = \begin{cases} \mathtt{0}, & m \neq n,\\ H_n, & m = n,\end{cases}$$

where H_n is a $n+1$ symmetric and non-singular real matrix, and $\mathtt{0}$ denotes the zero matrix of adequate size.

A moment functional $\mathbf{u}$ is said to be *quasi-definite* if there exists an OPS associated with it. As in the univariate case, $\mathbf{u}$ is called *positive definite* if $\langle \mathbf{u}, p^2\rangle > 0$ for all $p \in \Pi$, $p \not\equiv 0$ ([5, p. 63]). If $\mathbf{u}$ is positive definite, then it is quasi-definite, and the $n+1$ non-singular matrix H_n is positive definite, for $n \geqslant 0$.

We must remark that in the multivariate case, orthogonal polynomial systems associated with a quasi-definite moment functional $\mathbf{u}$ are not unique. However, if $\mathbf{u}$ is quasi-definite, there exists a unique monic OPS.

As usual, we define the *left product of a polynomial* $q \equiv q(x, y) \in \Pi$ *times* $\mathbf{u}$ as a new functional $q\,\mathbf{u}$ satisfying

$$\langle q\,\mathbf{u}, p\rangle = \langle \mathbf{u}, q\,p\rangle, \quad \forall p \in \Pi,$$

and the *distributional partial derivatives* are defined by

$$\langle \partial_1 \mathbf{u}, p\rangle = -\langle \mathbf{u}, \partial_1 p\rangle, \quad \langle \partial_2 \mathbf{u}, p\rangle = -\langle \mathbf{u}, \partial_2 p\rangle, \quad \forall p \in \Pi.$$

Following [14], if $\mathbf{u}$ is quasi-definite, the sequence of row functionals

$$\mathbf{U}_n = \mathbb{P}_n^t \, H_n^{-1} \mathbf{u} \tag{2.7}$$

satisfy the duality conditions

$$\langle \mathbf{U}_n, \mathbb{P}_m^t\rangle = \begin{cases} 0, & n \neq m, \\ I_{n+1}, & n = m. \end{cases}$$

In [14], a relation between the differentiation on Π and the derivative of a moment functional was established.

Lemma 2.3 ([14])**.** *Let $\{\mathbb{P}_n\}_{n\geqslant 0}$ be a PS and let $\{\mathbf{U}_n\}_{n\geqslant 0}$ be its corresponding dual basis. For $k = 1, 2$, let $\{\mathbf{U}_n^{(k)}\}_{n\geqslant 0}$ be the dual basis for the PS $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ defined in* (2.5)*. Then*

$$\partial_k \, \mathbf{U}_n^{(k)} = -\mathbf{U}_{n+1} \, \Gamma_{n+1,k},$$

where $\Gamma_{n+1,k}$ are $(n+2) \times (n+1)$ real matrices.

2.1. **Classical and semiclassical orthogonal polynomials in two variables.**

Definition 2.4 ([6, 7])**.** *A quasi-definite moment functional* $\mathbf{u}$ *is said to be* classical *if there exist polynomials $a_k(x,y), b(x,y), d_k(x,y)$, $k = 1, 2$, such that* $\mathbf{u}$ *satisfies the matrix Pearson-type equation*

$$\begin{aligned} \partial_1(a_1 \, \mathbf{u}) + \partial_2(b \, \mathbf{u}) &= d_1 \, \mathbf{u}, \\ \partial_1(b \, \mathbf{u}) + \partial_2(a_2 \, \mathbf{u}) &= d_2 \, \mathbf{u}, \end{aligned} \tag{2.8}$$

where $\deg a_k(x,y) \leqslant 2, \deg b(x,y) \leqslant 2$, $\deg d_k(x,y) = 1$, $k = 1, 2$, *and*

$$\det \begin{pmatrix} \langle \mathbf{u}, a_1\rangle & \langle \mathbf{u}, b\rangle \\ \langle \mathbf{u}, b\rangle & \langle \mathbf{u}, a_2\rangle \end{pmatrix} \neq 0.$$

An OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ associated with $\mathbf{u}$ *is called a classical OPS.*

In the above definition, the moment functional $\mathbf{u}$ satisfies the matrix Pearson-type equation in the distributional sense. This means that for every polynomial $p \in \Pi$, the following relations hold

$$\begin{aligned} \langle \mathbf{u}, a_1 \, \partial_1 \, p + b \, \partial_2 \, p + d_1 \, p\rangle &= 0, \\ \langle \mathbf{u}, b \, \partial_1 \, p + a_2 \, \partial_2 \, p + d_2 \, p\rangle &= 0. \end{aligned}$$

OPS associated with classical moment functionals satisfy several properties that characterize them ([6, 7]). In particular, an OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ is classical if and only if for each $n \geqslant 0$ there exists a $n+1$ square real matrix Λ_n, and Λ_1 non-singular, such that $\mathbb{P}_n$ satisfies the second order partial linear differential equation

$$a_1 \, \partial_1^2 \mathbb{P}_n + 2 \, b \, \partial_1 \partial_2 \mathbb{P}_n + a_2 \, \partial_2^2 \mathbb{P}_n + d_1 \, \partial_1 \mathbb{P}_n + d_2 \, \partial_2 \mathbb{P}_n = \Lambda_n \, \mathbb{P}_n.$$

This characterization is the non trivial extension of the definition for classical orthogonal polynomials given by Krall and Sheffer in [12]. Therein, the authors considered as classical the particular case when the matrix in the partial differential equation is a scalar matrix, that is, $\Lambda_n = \lambda_n \, I_{n+1}$.

Semiclassical orthogonal polynomials in two variables were introduced in [2] as the natural extension of classical orthogonal polynomials, when there are no restrictions about the degrees of the polynomials in (2.8). They are characterized by means of an analogue of the structure relation in one variable ([2]), as well as by means of a differential-difference equation ([3]).

2.2. $\mathcal{D}$-classical orthogonal polynomials.

Definition 2.5 ([14]). *A quasi-definite moment functional* $\mathbf{u}$ *is* $\mathcal{D}$*-classical if it is classical and satisfies a diagonal matrix Pearson-type equation in the form*

$$\partial_k(a_k(x,y)\,\mathbf{u}) = d_k(x,y)\,\mathbf{u},$$

for $k=1,2$*, where* $a_k(x,y)$ *and* $d_k(x,y)$ *are polynomials of degree less than or equal to 2 and 1, respectively.*

In [14], $\mathcal{D}$-classical moment functionals where characterized. However, the next theorem can be proved separately for each variable.

Theorem 2.6. *Let* $\mathbf{u}$ *be a quasi-definite moment functional with associated monic OPS* $\{\mathbb{P}_n\}_{n\geqslant 0}$*, and let* $k=1$ *or* $k=2$ *be fixed. The following statements are equivalent:*

(i) $\partial_k(a_k(x,y)\,\mathbf{u}) = d_k(x,y)\,\mathbf{u}$.

(ii) For each $n\geqslant 0$*, there exist square real matrices* $\Lambda_n^{(k)}$ *of size* $n+1$ *such that*

$$a_k\,\partial_k^2\,\mathbb{P}_n + d_k\,\partial_k\,\mathbb{P}_n = \Lambda_n^{(k)}\,\mathbb{P}_n,$$

where $\operatorname{rank}\Lambda_n^{(k)} = n$*, for* $n\geqslant 0$.

(iii) $\mathbf{u}^{(k)} = a_k(x,y)\,\mathbf{u}$ *is quasi-definite and* $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ *is its corresponding monic OPS.*

(iv) There exist real matrices $M_{n,k}$*,* $n\geqslant 1$*, and* $N_{n,k}$*,* $n\geqslant 2$*, of dimensions* $(n+1)\times n$ *and* $(n+1)\times(n-1)$*, respectively, such that*

$$\mathbb{P}_n = \mathbb{P}_n^{(k)} + M_{n,k}\,\mathbb{P}_{n-1}^{(k)} + N_{n,k}\,\mathbb{P}_{n-2}^{(k)}, \quad n\geqslant 1. \tag{2.9}$$

Here $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ *is the monic PS defined in* (2.6).

When above result holds for $k=1$ and $k=2$, we have the equivalence with the $\mathcal{D}$-classical character for $\mathbf{u}$.

3. Partial x_k-coherent pairs and Coherent pairs in two variables

In this section, we define and study the *partial* x_k*-coherence* in two variables that relates two PS just for one fixed variable x or y, that is, for the first variable ($k=1$) or the second variable ($k=2$). When a PS satisfies two partial x_k-coherence relations, we talk about *full coherence*.

3.1. Partial x_k-coherence.

Definition 3.1. *Let* $k=1$ *or* $k=2$ *be fixed. Given two monic PS* $\{\mathbb{P}_n\}_{n\geqslant 0}$ *and* $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$*, we say that they constitute a* partial x_k-coherent pair *if they are connected by means of the relation*

$$\begin{aligned} \mathbb{Q}_{n,k} &= \mathbb{P}_n^{(k)} + M_{n,k}\mathbb{P}_{n-1}^{(k)} + N_{n,k}\mathbb{P}_{n-2}^{(k)}, \quad n\geqslant 1,\\ \mathbb{Q}_{0,k} &= \mathbb{P}_0^{(k)}, \end{aligned} \tag{3.1}$$

where $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0} = \{\Omega_{n+1,k}\partial_k\mathbb{P}_{n+1}\}_{n\geqslant 0}$ *was defined in* (2.6)*, and* $M_{n,k}$*, for* $n\geqslant 1$*,* $N_{n,k}$*,* $n\geqslant 2$*, are real matrices of sizes* $(n+1)\times n$ *and* $(n+1)\times(n-1)$*, respectively.*

We can write relation (3.1) in an alternative way

$$\mathbb{Q}_{n,k} = \Omega_{n+1,k}\partial_k\mathbb{P}_{n+1} + M_{n,k}\Omega_{n,k}\partial_k\mathbb{P}_n + N_{n,k}\Omega_{n-1,k}\partial_k\mathbb{P}_{n-1}.$$

Our definition of partial x_k-coherence in two variables encompasses the analogue of the usual definition of *standard partial x_k-coherence* in which $N_{n,k} \equiv \mathtt{0}$, for $n \geqslant 2$,

$$\mathbb{Q}_{n,k} = \Omega_{n+1,k}\,\partial_k\mathbb{P}_{n+1} + M_{n,k}\Omega_{n,k}\,\partial_k\mathbb{P}_n, \quad n \geqslant 1,$$

that is,

$$\mathbb{Q}_{n,k} = \mathbb{P}_n^{(k)} + M_{n,k}\mathbb{P}_{n-1}^{(k)}, \quad n \geqslant 1, \tag{3.2}$$

as well as the so-called *symmetric partial x_k-coherence*, when $M_{n,k} \equiv \mathtt{0}$, for $n \geqslant 1$,

$$\mathbb{Q}_{n,k} = \Omega_{n+1,k}\,\partial_k\mathbb{P}_{n+1} + N_{n,k}\Omega_{n-1,k}\,\partial_k\mathbb{P}_{n-1}, \quad n \geqslant 2,$$

equivalently,

$$\mathbb{Q}_{n,k} = \mathbb{P}_n^{(k)} + N_{n,k}\mathbb{P}_{n-2}^{(k)}, \quad n \geqslant 2, \tag{3.3}$$

with $\mathbb{Q}_{0,k} = \mathbb{P}_0^{(k)}$, and $\mathbb{Q}_{1,k} = \mathbb{P}_1^{(k)}$.

The monic character of the PS involved in the partial x_k-coherence relation is superfluous. In fact, for $n \geqslant 0$, let E_n and $F_{n,k}$ be non-singular matrices of size $n+1$, and define the new polynomial systems $\{\widetilde{\mathbb{P}}_n\}_{n\geqslant 0}$ and $\{\widetilde{\mathbb{Q}}_{n,k}\}_{n\geqslant 0}$ by means of

$$\widetilde{\mathbb{P}}_n = E_n\,\mathbb{P}_n, \qquad \widetilde{\mathbb{Q}}_{n,k} = F_{n,k}\,\mathbb{Q}_{n,k}, \quad n \geqslant 0.$$

Since $\mathbb{P}_n$ and $\mathbb{Q}_{n,k}$ are monic, then E_n and $F_{n,k}$ are the leading coefficients of $\widetilde{\mathbb{P}}_n$ and $\widetilde{\mathbb{Q}}_{n,k}$, respectively. Multiplying (3.1) by $F_{n,k}$ and using (2.5), we get

$$\begin{aligned}\widetilde{\mathbb{Q}}_{n,k} &= F_{n,k}\mathbb{Q}_{n,k} = F_{n,k}\Omega_{n+1,k}E_{n+1}^{-1}\partial_k\widetilde{\mathbb{P}}_{n+1} + F_{n,k}M_{n,k}\Omega_{n,k}E_n^{-1}\partial_k\widetilde{\mathbb{P}}_n \\ &\qquad + F_{n,k}N_{n,k}\Omega_{n-1,k}E_{n-1}^{-1}\partial_k\widetilde{\mathbb{P}}_{n-1} \\ &= F_{n,k}\widetilde{\mathbb{P}}_n^{(k)} + F_{n,k}M_{n,k}\widetilde{\mathbb{P}}_{n-1}^{(k)} + F_{n,k}N_{n,k}\widetilde{\mathbb{P}}_{n-2}^{(k)}.\end{aligned}$$

Then,

$$\widetilde{\mathbb{Q}}_{n,k} = F_{n,k}\widetilde{\mathbb{P}}_n^{(k)} + \widetilde{M}_{n,k}\widetilde{\mathbb{P}}_{n-1}^{(k)} + \widetilde{N}_{n,k}\widetilde{\mathbb{P}}_{n-2}^{(k)},$$

that is, $\{\widetilde{\mathbb{P}}_n\}_{n\geqslant 0}$ and $\{\widetilde{\mathbb{Q}}_{n,k}\}_{n\geqslant 0}$ satisfy a similar relation as (3.1), where $\widetilde{M}_{n,k} = F_{n,k}M_{n,k}$, and $\widetilde{N}_{n,k} = F_{n,k}N_{n,k}$. In this way, the coherence property does not depend on a particular basis. Moreover, $\operatorname{rank}\widetilde{M}_{n,k} = \operatorname{rank} M_{n,k}$, and $\operatorname{rank}\widetilde{N}_{n,k} = \operatorname{rank} N_{n,k}$, $n \geqslant 0$, since the rank is unchanged upon left or right multiplication by a non-singular matrix.

For $k = 1$ or $k = 2$ fixed, the partial x_k-coherence relation can be translated to the dual basis as follows.

Proposition 3.2. *Let $\{\mathbb{P}_n\}_{n\geqslant 0}$, $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$ be two monic PS satisfying a relation like* (3.1)*, and let $\{\mathbf{U}_n\}_{n\geqslant 0}$, $\{\mathbf{V}_{n,k}\}_{n\geqslant 0}$ be the corresponding dual bases. Consider the monic PS $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ defined in* (2.6) *and its corresponding dual basis $\{\mathbf{U}_n^{(k)}\}_{n\geqslant 0}$. Then, for $n \geqslant 0$, it follows that*

(*i*) $\quad \mathbf{U}_n^{(k)} = \mathbf{V}_{n+2,k}\,N_{n+2,k} + \mathbf{V}_{n+1,k}\,M_{n+1,k} + \mathbf{V}_{n,k},$

(*ii*) $\quad -\mathbf{U}_{n+1}\,\Gamma_{n+1,k} = \partial_k[\mathbf{V}_{n+2,k}\,N_{n+2,k} + \mathbf{V}_{n+1,k}\,M_{n+1,k} + \mathbf{V}_{n,k}].$

Proof. Since $\{\mathbf{V}_{n,k}\}_{n\geq 0}$ is a basis of the dual space, then

$$\mathbf{U}_n^{(k)} = \sum_{m=0}^{+\infty} \mathbf{V}_{m,k}\, D_m^{(k)},$$

where, using (3.1),

$$\begin{aligned} D_m^{(k)} &= \langle \mathbf{U}_n^{(k)}, \mathbb{Q}_{m,k}^t \rangle^t = \langle \mathbf{U}_n^{(k)}, (\mathbb{P}_m^{(k)} + M_{m,k}\mathbb{P}_{m-1}^{(k)} + N_{m,k}\mathbb{P}_{m-2}^{(k)})^t \rangle^t \\ &= \begin{cases} I_{n+1} & \text{for} \quad m = n, \\ M_{n+1,k} & \text{for} \quad m = n+1, \\ N_{n+2,k} & \text{for} \quad m = n+2, \\ \mathtt{0} & \text{otherwise}, \end{cases} \end{aligned}$$

and (i) follows. Differentiating (i), and using Lemma 2.3, we get (ii). □

When both monic PS are orthogonal, using (2.7) and Lemma 2.3, above proposition can be written as

Corollary 3.3. *Suppose that $\{\mathbb{P}_n\}_{n\geqslant 0}$, $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$ are two monic OPS associated with the quasi-definite moment functionals $\mathbf{u}$ and $\mathbf{v_k}$, respectively, and suppose that they satisfy a relation like* (3.1). *Then, for $n \geqslant 0$,*

$$\mathbf{U}_n^{(k)} = \mathbb{C}_{n+2,k}(x,y)\mathbf{v_k}, \tag{3.4}$$

$$\mathbb{B}_{n+1,k}(x,y)\mathbf{u} = \partial_k \left[\mathbb{C}_{n+2,k}(x,y)\mathbf{v_k}\right], \tag{3.5}$$

where

$$\begin{aligned} \mathbb{B}_{n+1,k}(x,y) &= -\mathbb{P}_{n+1}^t H_{n+1}^{-1}\Gamma_{n+1,k}, \\ \mathbb{C}_{n+2,k}(x,y) &= \mathbb{Q}_{n+2,k}^t \widetilde{H}_{n+2,k}^{-1} N_{n+2,k} + \mathbb{Q}_{n+1,k}^t \widetilde{H}_{n+1,k}^{-1} M_{n+1,k} + \mathbb{Q}_{n,k}^t \widetilde{H}_{n,k}^{-1}, \end{aligned}$$

are $1\times(n+1)$ vector of polynomials of degrees $n+1$ and $n+2$, respectively.

Here, for $n \geqslant 0$, we denote

$$H_n = \langle \mathbf{u}, \mathbb{P}_n\, \mathbb{P}_n^t \rangle, \quad \widetilde{H}_{n,k} = \langle \mathbf{v_k}, \mathbb{Q}_{n,k}\, \mathbb{Q}_{n,k}^t \rangle.$$

Obviously, these $(n+1)$-size matrices are symmetric and non-singular.

In the partial x_k-coherence relation (3.1), when both monic PS are orthogonal, we extend the definition to quasi-definite moment functionals.

Definition 3.4. *Let $\{\mathbb{P}_n\}_{n\geqslant 0}$ and $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$ be two monic OPS associated with the quasi-definite moment functionals $\mathbf{u}$ and $\mathbf{v_k}$, respectively. We say that $\{\mathbf{u}, \mathbf{v_k}\}$ is a* partial x_k-coherent pair, *for $k=1$ or $k=2$ fixed, if* (3.1) *holds.*

When we fix $k=1$ or $k=2$, we will show that if $\{\mathbf{u},\mathbf{v_k}\}$ is a partial x_k-coherent pair of moment functionals, then both functionals $\mathbf{u}$ and $\mathbf{v_k}$ satisfy a x_k-partial differential equation, and they are related by means of a rational relation.

Theorem 3.5. *Let $\{\mathbf{u}, \mathbf{v_k}\}$ be a partial x_k-coherent pair and let $\{\mathbb{P}_n\}_{n\geqslant 0}$ and $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$ be the corresponding monic OPS.*

(i) There exists a polynomial $\lambda_k(x,y)$ of degree not greater than 2 such that
$$\mathbf{U}_0^{(k)} = \lambda_k(x,y)\,\mathbf{v_k}, \tag{3.6}$$
where
$$\lambda_k(x,y) = \mathbb{Q}_{2,k}^t \widetilde{H}_{2,k}^{-1} N_{2,k} + \mathbb{Q}_{1,k}^t \widetilde{H}_{1,k}^{-1} M_{1,k} + \mathbb{Q}_{0,k}^t \widetilde{H}_{0,k}^{-1}. \tag{3.7}$$
(ii) There exist polynomials $\tilde{a}_k(x,y)$ and $\tilde{d}_k(x,y)$ of respective degrees 4 and 3, such that
$$\partial_k(\tilde{a}_k(x,y)\,\mathbf{v_k}) = \tilde{d}_k(x,y)\,\mathbf{v_k}. \tag{3.8}$$
(iii) The moment functionals $\mathbf{u}$ and $\mathbf{v_k}$ are related by means of a rational relation as
$$\tilde{a}_k(x,y)\,\mathbf{u} = \rho_k(x,y)\,\mathbf{v_k}, \tag{3.9}$$
where $\deg \tilde{a}_k(x,y) \leqslant 4$ *and* $\deg \rho_k(x,y) \leqslant 4$.
(iv) There exist polynomials $\alpha_k(x,y)$ and $\delta_k(x,y)$ of respective degrees 8 and 7 such that
$$\partial_k(\alpha_k(x,y)\,\mathbf{u}) = \delta_k(x,y)\,\mathbf{u}. \tag{3.10}$$

Proof. (i) Taking $n=0$ in (3.4), and denoting
$$\lambda_k(x,y) = \mathbb{C}_{2,k}(x,y) = \mathbb{Q}_{2,k}^t \widetilde{H}_{2,k}^{-1} N_{2,k} + \mathbb{Q}_{1,k}^t \widetilde{H}_{1,k}^{-1} M_{1,k} + \mathbb{Q}_{0,k}^t \widetilde{H}_{0,k}^{-1},$$
we get the result.

(ii) Next, consider (3.5) for $n=0$ and $n=1$,
$$\begin{aligned} \mathbb{B}_{1,k}(x,y)\,\mathbf{u} &= \partial_k\left(\mathbb{C}_{2,k}(x,y)\,\mathbf{v_k}\right), \\ \mathbb{B}_{2,k}(x,y)\,\mathbf{u} &= \partial_k\left(\mathbb{C}_{3,k}(x,y)\,\mathbf{v_k}\right), \end{aligned}$$
where $\mathbb{B}_{1,k}(x,y)$ and $\mathbb{C}_{2,k}(x,y)$ are polynomials of degrees 1 and 2, respectively, and
$$\mathbb{B}_{2,k}(x,y) = (b_{2,1}^{(k)}(x,y), b_{2,2}^{(k)}(x,y)), \quad \mathbb{C}_{3,k}(x,y) = (c_{3,1}^{(k)}(x,y), c_{3,2}^{(k)}(x,y)),$$
are two vector of polynomials of degree not greater than 2 and 3, respectively. Then, denoting $b_{1,1}^{(k)}(x,y) = \mathbb{B}_{1,k}(x,y)$ and $c_{2,1}^{(k)}(x,y) = \mathbb{C}_{2,k}(x,y)$, we have the following linear system
$$\begin{aligned} b_{1,1}^{(k)}(x,y)\,\mathbf{u} &= \partial_k(c_{2,1}^{(k)}(x,y))\,\mathbf{v_k} + c_{2,1}^{(k)}(x,y)\,\partial_k\,\mathbf{v_k}, \\ b_{2,1}^{(k)}(x,y)\,\mathbf{u} &= \partial_k(c_{3,1}^{(k)}(x,y))\,\mathbf{v_k} + c_{3,1}^{(k)}(x,y)\,\partial_k\,\mathbf{v_k}, \\ b_{2,2}^{(k)}(x,y)\,\mathbf{u} &= \partial_k(c_{3,2}^{(k)}(x,y))\,\mathbf{v_k} + c_{3,2}^{(k)}(x,y)\,\partial_k\,\mathbf{v_k}. \end{aligned} \tag{3.11}$$
Eliminating $\mathbf{u}$ from the two first equations, we get
$$\partial_k(\tilde{a}_k(x,y)\,\mathbf{v_k}) = \tilde{d}_k(x,y)\,\mathbf{v_k},$$
where
$$\begin{aligned} \tilde{a}_k(x,y) &= c_{2,1}^{(k)}(x,y) b_{2,1}^{(k)}(x,y) - c_{3,1}^{(k)}(x,y) b_{1,1}^{(k)}(x,y), \\ \tilde{d}_k(x,y) &= c_{2,1}^{(k)}(x,y)\partial_k(b_{2,1}^{(k)}(x,y)) - c_{3,1}^{(k)}(x,y)\partial_k(b_{1,1}^{(k)}(x,y)), \end{aligned}$$
are polynomials of degree not greater than 4 and 3, respectively.

(iii) Now, we eliminate $\partial_k \mathbf{v}$ in the two first equations of (3.11), and here
$$\rho_k(x,y) = c_{2,1}^{(k)}(x,y)\partial_k(c_{3,1}^{(k)}(x,y)) - c_{3,1}^{(k)}(x,y)\partial_k(c_{2,1}^{(k)}(x,y)).$$

(iv) Using (ii) and (iii), we compute

$$\begin{aligned}\partial_k(\tilde{a}_k^2\,\mathbf{u}) &= \partial_k(\tilde{a}_k\,\rho_k\mathbf{v_k}) = \partial_k(\rho_k)\tilde{a}_k\,\mathbf{v_k} + \rho_k\partial_k(\tilde{a}_k\,\mathbf{v_k})\\ &= \partial_k(\rho_k)\tilde{a}_k\,\mathbf{v_k} + \rho_k\,\tilde{d}_k\,\mathbf{v_k} = [\partial_k(\rho_k)\tilde{a}_k + \rho_k\tilde{d}_k]\mathbf{v_k}.\end{aligned}$$

And then,

$$\alpha_k(x,y) = \tilde{a}_k(x,y)^2, \quad \delta_k(x,y) = \partial_k(\rho_k(x,y))\tilde{a}_k(x,y) + \rho_k(x,y)\tilde{d}_k(x,y).$$

□

Remark 3.6. *When we work with* standard partial x_k-coherence (3.2)*, that is, $N_{n,k} \equiv \mathtt{0}$ for $n \geqslant 2$, the degree of the polynomials involved in all the statements of the above theorem is lowered. In fact, for $n \geqslant 0$, $\mathbb{C}_{n+2}^{(k)}(x,y)$ is a $1\times(n+1)$ row vector of polynomials of total degree $n+1$ and, then,*

$$\begin{aligned}&\deg\lambda_k(x,y) \leqslant 1, && \deg\tilde{a}_k(x,y) \leqslant 3, && \deg\tilde{d}_k(x,y) \leqslant 2,\\ &\deg\rho_k(x,y) \leqslant 2, && \deg\alpha_k(x,y) \leqslant 6, && \deg\delta_k(x,y) \leqslant 4.\end{aligned}$$

We now deduce that if the rank of the first matrices $M_{1,k}$ and $N_{2,k}$ in (3.1) is zero, then the family of polynomials $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$ coincides with $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$.

Proposition 3.7. *Assume that the hypotheses of Theorem 3.5 hold. If* rank $M_{1,k} =$ rank $N_{2,k} = 0$*, then $M_{n,k} \equiv \mathtt{0}$, $n \geqslant 1$, and $N_{n,k} \equiv \mathtt{0}$, $n \geqslant 2$. As a consequence,*

$$\mathbb{Q}_{n,k} = \mathbb{P}_n^{(k)}, \quad n \geqslant 0.$$

Proof. Observe that $M_{1,k}$ and $N_{2,k}$ are column vectors of sizes 2×1, and 3×1, respectively. If

$$\operatorname{rank} M_{1,k} = \operatorname{rank} N_{2,k} = 0,$$

then both matrices must be zero and, from (3.6), and (3.7), we get

$$\mathbf{U}_0^{(k)} = \mathbb{Q}_{0,k}^t\,\widetilde{H}_{0,k}^{-1}\,\mathbf{v_k}.$$

In this way, $\mathbf{U}_0^{(k)}$ is a real non-zero multiple of $\mathbf{v_k}$. Therefore,

$$\mathbb{Q}_{n,k} = \mathbb{P}_n^{(k)}, \quad n \geqslant 0.$$

Finally, from (3.1), we get $M_{n,k} \equiv \mathtt{0}$, $n \geqslant 1$, and $N_{n,k} \equiv \mathtt{0}$, $n \geqslant 2$, and the PS $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ is orthogonal with respect to $\widetilde{H}_{0,k}^{-1}\,\mathbf{v_k}$. □

An interesting case arises when $\mathbf{u}$ is a $\mathcal{D}$-classical moment functional. Then, for $k=1$ or $k=2$, Theorem 2.6 means that $\{\mathbb{P}_n^{(k)}\}_{n\geqslant 0}$ is an OPS. In this case, we can deduce additional properties for the rank of the matrices involved in the coherence relation (3.1). We fix $k=1$ or $k=2$.

Proposition 3.8. *Let $\{\mathbf{u}, \mathbf{v_k}\}$ be a coherent pair of moment functionals with associated monic OPS $\{\mathbb{P}_n\}_{n\geqslant 0}$ and $\{\mathbb{Q}_{n,k}\}_{n\geqslant 0}$, respectively. If $\mathbf{u}$ is $\mathcal{D}$-classical, then*

$$\operatorname{rank} N_{n,k} = n-1,\ n \geqslant 2, \text{ if and only if } \deg\lambda_k(x,y) = 2.$$

Otherwise,

$$\operatorname{rank} N_{n,k} = 0,\ n \geqslant 2,\ \operatorname{rank} M_{n,k} = n,\ n \geqslant 1, \text{ if and only if } \deg\lambda_k(x,y) = 1.$$

Proof. On the one hand, if $\operatorname{rank} N_{n,k} = n-1$, for $n \geqslant 2$, then $\operatorname{rank} N_{2,k} = 1$ and from (3.7), $\lambda_k(x,y)$ is a polynomial of exact total degree 2.

On the other hand, if $\operatorname{rank} N_{n,k} = 0$, $n \geqslant 2$, and $\operatorname{rank} M_{n,k} = n$, $n \geqslant 1$, then $N_{2,k} \equiv \mathtt{0}$ and $\operatorname{rank} M_{1,k} = 1$. Again, from (3.7), $\lambda_k(x,y)$ has total degree 1.

Conversely, let

$$\lambda_k(x,y) = a_1^{(k)} x^2 + a_2^{(k)} xy + a_3^{(k)} y^2 + b_1^{(k)} x + b_2^{(k)} y + c_0^{(k)},$$

be the explicit expression of the polynomial $\lambda_k(x,y)$ given in (3.7) and suppose that $\deg \lambda_k(x,y) = 2$. Then,

$$|a_1^{(k)}| + |a_2^{(k)}| + |a_3^{(k)}| > 0.$$

Since $\mathbf{u}$ is $\mathcal{D}$-classical, then $\mathbf{u}^{(k)}$ is a quasi-definite moment functional and, using (3.1), we get

$$\langle \mathbf{u}^{(k)}, \mathbb{Q}_{n,k} (\mathbb{P}_{n-2}^{(k)})^t \rangle = N_{n,k} \langle \mathbf{u}^{(k)}, \mathbb{P}_{n-2}^{(k)} (\mathbb{P}_{n-2}^{(k)})^t \rangle = N_{n,k}\, H_{n-2}^{(k)},$$

where $H_{n-2}^{(k)}$ is non-singular. From (i) in Theorem 3.5, we get $\mathbf{u}^{(k)} = \lambda_k(x,y)\mathbf{v_k}$ and, then,

$$\begin{aligned}
\langle \mathbf{u}^{(k)}, \mathbb{Q}_{n,k} (\mathbb{P}_{n-2}^{(k)})^t \rangle &= \langle \lambda_k(x,y)\, \mathbf{v_k}, \mathbb{Q}_{n,k}\, (\mathbb{P}_{n-2}^{(k)})^t \rangle = \langle \mathbf{v_k}, \mathbb{Q}_{n,k} \lambda_k(x,y) (\mathbb{P}_{n-2}^{(k)})^t \rangle \\
&= \langle \mathbf{v_k}, \mathbb{Q}_{n,k}\, (a_1^{(k)} x^2 + a_2^{(k)} xy + a_3^{(k)} y^2) \mathbb{X}_{n-2}^t \rangle \\
&= \langle \mathbf{v_k}, \mathbb{Q}_{n,k}\, \mathbb{X}_n^t \rangle A_n^{(k)} = \widetilde{H}_{n,k}\, A_n^{(k)},
\end{aligned}$$

where

$$A_n^{(k)} = a_1^{(k)} L_{n-1,1}^t L_{n-2,1}^t + a_2^{(k)} L_{n-1,2}^t L_{n-2,1}^t + a_3^{(k)} L_{n-1,2}^t L_{n-2,2}^t.$$

The special shape of the matrices $L_{n,i}$ given in (2.2), allows us to deduce the explicit expression of the $(n+1) \times (n-1)$ matrix $A_n^{(k)}$ as

$$A_n^{(k)} = \begin{pmatrix} a_1^{(k)} & & & \\ a_2^{(k)} & a_1^{(k)} & & \\ a_3^{(k)} & a_2^{(k)} & \ddots & \\ & a_3^{(k)} & \ddots & a_1^{(k)} \\ & & \ddots & a_2^{(k)} \\ & & & a_3^{(k)} \end{pmatrix},$$

and, thus, $\operatorname{rank} A_n^{(k)} = n-1$. It follows that $N_{n,k}\, H_{n-2}^{(k)} = \widetilde{H}_{n,k}\, A_n^{(k)}$, and then, $\operatorname{rank} N_{n,k} = \operatorname{rank} A_n^{(k)} = n-1$, $n \geqslant 2$.

Moreover, if $\deg \lambda_k(x,y) = 1$, that is, $A_n^{(k)} \equiv \mathtt{0}$, but $|b_1^{(k)}| + |b_2^{(k)}| > 0$, we obtain that $N_{n,k} \equiv \mathtt{0}$. Working as above, we deduce that

$$\langle \mathbf{u}^{(k)}, \mathbb{Q}_{n,k} (\mathbb{P}_{n-1}^{(k)})^t \rangle = M_{n,k} \langle \mathbf{u}^{(k)}, \mathbb{P}_{n-1}^{(k)} (\mathbb{P}_{n-1}^{(k)})^t \rangle = M_{n,k}\, H_{n-1}^{(k)},$$

and

$$\langle \lambda_k(x,y) \mathbf{v_k}, \mathbb{Q}_{n,k} (\mathbb{P}_{n-1}^{(k)})^t \rangle = \langle \mathbf{v_k}, \mathbb{Q}_{n,k} (b_1^{(k)} x + b_2^{(k)} y) \mathbb{X}_{n-1}^t \rangle = \widetilde{H}_{n,k} B_n^{(k)},$$

where

$$B_n^{(k)} = b_1^{(k)} L_{n-1,1}^t + b_2^{(k)} L_{n-1,2}^t = \begin{pmatrix} b_1^{(k)} & & \\ b_2^{(k)} & \ddots & \\ & \ddots & b_1^{(k)} \\ & & b_2^{(k)} \end{pmatrix}.$$

Now, $B_n^{(k)}$ is a $(n+1)\times n$ matrix with $\operatorname{rank} B_n^{(k)} = n$. In this case, $M_{n,k}\, H_{n-1}^{(k)} = \widetilde{H}_{n,k} B_n^{(k)}$, for $n \geqslant 1$, and then $\operatorname{rank} M_{n,k} = \operatorname{rank} B_n^{(k)} = n$. □

3.2. **Full coherence.** Full coherence is defined when we have two coherence relations, for $k = 1$ and for $k = 2$. That is, given three PS $\{\mathbb{P}_n\}_{n\geqslant 0}$, $\{\mathbb{Q}_{n,1}\}_{n\geqslant 0}$ and $\{\mathbb{Q}_{n,2}\}_{n\geqslant 0}$, and matrices $M_{n,k}$, $n \geqslant 1$, and $N_{n,k}$, $n \geqslant 2$, of adequate size, we get that (3.1) for $k = 1$, and $k = 2$ hold, that is,

$$\mathbb{Q}_{n,1} = \mathbb{P}_n^{(1)} + M_{n,1}\mathbb{P}_{n-1}^{(1)} + N_{n,1}\mathbb{P}_{n-2}^{(1)}, \quad n \geqslant 1,$$
$$\mathbb{Q}_{n,2} = \mathbb{P}_n^{(2)} + M_{n,2}\mathbb{P}_{n-1}^{(2)} + N_{n,2}\mathbb{P}_{n-2}^{(2)}, \quad n \geqslant 1,$$

and $\mathbb{Q}_{0,k} = \mathbb{P}_0^{(k)}$, for $k = 1, 2$.

From Theorem 3.5, if $\{\mathbf{u}, \mathbf{v_k}\}$ is a partial x_k-coherent pair of functionals, for $k = 1$ and $k = 2$, then $\mathbf{u}$, $\mathbf{v_1}$, and $\mathbf{v_2}$ are semiclassical, and from (3.9) $\mathbf{u}$ with $\mathbf{v_k}$ are related by means of a rational expression. If both partial x_k-coherent relations hold for $k = 1$ and $k = 2$, then the moment functionals $\mathbf{v_1}$ and $\mathbf{v_2}$ are also related by a rational expression, namely

$$\tilde{a}_2(x, y)\rho_1(x, y)\mathbf{v_1} = \tilde{a}_1(x, y)\rho_2(x, y)\mathbf{v_2}$$

Then, we define the *full coherence* in terms of two functionals.

Definition 3.9. *Let $\{\mathbf{u}, \mathbf{v}\}$ be a pair of quasi-definite moment functionals. Suppose that there exist two non-zero polynomials $\theta_1(x, y)$ and $\theta_2(x, y)$ such that the moment functionals $\mathbf{v_1} = \theta_1(x, y)\mathbf{v}$, and $\mathbf{v_2} = \theta_2(x, y)\mathbf{v}$ are quasi-definite.*

Then, we say that $\{\mathbf{u}, \mathbf{v}\}$ is a full coherent pair if $\{\mathbf{u}, \mathbf{v_1}\}$ and $\{\mathbf{u}, \mathbf{v_2}\}$ are both partial x_k-coherent pairs, for $k = 1, 2$.

We must remark that if $\theta_1(x, y) = \theta_2(x, y) = 1$, then

$$\mathbf{v} = \mathbf{v_1} = \mathbf{v_2},$$

and $\{\mathbb{Q}_{n,1}\}_{n\geqslant 0} \equiv \{\mathbb{Q}_{n,2}\}_{n\geqslant 0} \equiv \{\mathbb{Q}_n\}_{n\geqslant 0}$.

We will now focus our attention on the case when the quasi-definite moment functionals satisfy

$$\mathbf{u} = \mathbf{v} = \mathbf{v_1} = \mathbf{v_2},$$

that is, when $\mathbf{u}$ is *self-coherent*, that is, for $k = 1, 2$, the coherence relation (3.1) takes the following form,

$$\mathbb{P}_n = \mathbb{P}_n^{(k)} + M_{n,k}\, \mathbb{P}_{n-1}^{(k)} + N_{n,k}\, \mathbb{P}_{n-2}^{(k)}, \quad n \geqslant 0.$$

Then, using (3.5) for $n = 0$ and $k = 1, 2$, and Theorem 2.6, we can deduce the next result.

Proposition 3.10. *Let $\mathbf{u}$ be a quasi-definite moment functional. Then $\mathbf{u}$ is self-coherent if and only if it is $\mathcal{D}$-classical.*

4. Examples

In this final section, we apply our results to several examples of bivariate orthogonal polynomials that can be related by means of a coherence relation. Moreover, we study the matrix coefficients in the (partial) coherence relation.

First, we study the case of tensor product of univariate classical orthogonal polynomials, and then, we study bivariate classical orthogonal polynomials such as polynomials that are orthogonal on the disk, the biangle and the simplex, and we analyze the case when we deal with a measure generate by means of a univariate Uvarov perturbation of Gegenbauer polynomials.

We will use the standard representation for classical orthogonal polynomials considered in the usual literature on the subject (see for instance [1, 4, 18]).

As usual, univariate Jacobi polynomials will be denoted by $\{P_n^{(\alpha,\beta)}\}_{n\geqslant 0}$, and they are orthogonal in $[-1,1]$ with respect to the weight function

$$w^{(\alpha,\beta)}(t) = (1-t)^{\alpha}(1+t)^{\beta}, \quad \alpha,\beta > -1,$$

normalized by the condition [18, (4.1.1)]

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}.$$

We denote by $\{C_n^{(\mu)}(t)\}_{n\geqslant 0}$ the classical Gegenbauer polynomials, orthogonal with respect to the weight function $w(t) = (1-t^2)^{\mu-1/2}$, for $\mu > -1/2$, $t \in (-1,1)$, such that [18, (4.7.1)]

$$C_n^{(\mu)}(t) = \frac{\Gamma(\mu+1/2)}{\Gamma(2\mu)} \frac{\Gamma(n+2\mu)}{\Gamma(n+\mu+1/2)} P_n^{(\mu-1/2,\mu-1/2)}(t). \tag{4.1}$$

4.1. Tensor product of univariate classical orthogonal polynomials.

Let $\mathbf{u}^{(x)}$ and $\mathbf{v}^{(y)}$ be univariate classical moment functionals acting on the variables x and y, respectively. Then, there exist polynomials $a_1(x)$, $a_2(y)$ of degree not greater than 2 and $d_1(x)$, $d_2(y)$ of exact degree 1, such that

$$\frac{d}{dx}(a_1\,\mathbf{u}^{(x)}) = d_1\,\mathbf{u}^{(x)}, \qquad \frac{d}{dy}(a_2\,\mathbf{v}^{(y)}) = d_2\,\mathbf{v}^{(y)}.$$

Let $\{p_n(x)\}_{n\geqslant 0}$, and $\{q_n(y)\}_{n\geqslant 0}$ be the sequences of monic orthogonal polynomials associated with $\mathbf{u}^{(x)}$ and $\mathbf{v}^{(y)}$, respectively.

It is well known [13] that there exist real numbers b_n, c_n, $\tilde{b}_n, \tilde{c}_n$ such that

$$p_n(x) = \frac{p'_{n+1}(x)}{n+1} + b_n \frac{p'_n(x)}{n} + c_n \frac{p'_{n-1}(x)}{n-1}, \tag{4.2}$$

$$q_n(y) = \frac{q'_{n+1}(y)}{n+1} + \tilde{b}_n \frac{q'_n(y)}{n} + \tilde{c}_n \frac{q'_{n-1}(y)}{n-1}. \tag{4.3}$$

For $n \geqslant 0$, the bivariate polynomials

$$P_{n,m}(x,y) = p_{n-m}(x)\,q_m(y), \quad 0 \leqslant m \leqslant n,$$

are orthogonal with respect to the moment functional

$$\langle \mathbf{w}, p(x,y)\rangle = \left\langle \mathbf{u}^{(x)}, \langle \mathbf{v}^{(y)}, p(x,y)\rangle \right\rangle = \left\langle \mathbf{v}^{(y)}, \langle \mathbf{u}^{(x)}, p(x,y)\rangle \right\rangle, \quad \forall p \in \Pi.$$

Obviously, $\mathbf{w}$ is a $\mathcal{D}$-classical moment functional since

$$\partial_1(a_1\,\mathbf{w}) = d_1\,\mathbf{w}, \qquad \partial_2(a_2\,\mathbf{w}) = d_2\,\mathbf{w}.$$

From (4.2), we can write

$$P_{n,m}(x,y) = \frac{\partial_1 P_{n+1,m}(x,y)}{n+1-m} + b_{n-m}\frac{\partial_1 P_{n,m}(x,y)}{n-m} + c_{n-m}\frac{\partial_1 P_{n-1,m}(x,y)}{n-1-m},$$

and, in a similar way, from (4.3) we get

$$P_{n,m}(x,y) = \frac{\partial_2 P_{n+1,m+1}(x,y)}{m+1} + \tilde{b}_m\frac{\partial_2 P_{n,m}(x,y)}{m} + \tilde{c}_m\frac{\partial_2 P_{n-1,m-1}(x,y)}{m-1}.$$

It follows that $\mathbf{w}$ is a self-coherent in the form

$$\mathbb{P}_n = \mathbb{P}_n^{(k)} + M_{n,k}\,\mathbb{P}_{n-1}^{(k)} + N_{n,k}\,\mathbb{P}_{n-2}^{(k)}, \quad k=1,2,$$

where

$$M_{n,1} = \left(\begin{array}{cccc} b_n & & & \\ & b_{n-1} & & \\ & & \ddots & \\ & & & b_1 \\ \hline 0 & 0 & \cdots & 0 \end{array}\right), \quad M_{n,2} = \left(\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \hline \tilde{b}_1 & & & \\ & \tilde{b}_2 & & \\ & & \ddots & \\ & & & \tilde{b}_n \end{array}\right),$$

and

$$N_{n,1} = \left(\begin{array}{cccc} c_n & & & \\ & c_{n-1} & & \\ & & \ddots & \\ & & & c_2 \\ \hline 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{array}\right), \quad N_{n,2} = \left(\begin{array}{cccc} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \hline \tilde{c}_2 & & & \\ & \tilde{c}_3 & & \\ & & \ddots & \\ & & & \tilde{c}_n \end{array}\right).$$

4.2. **Classical orthogonal polynomials on the unit disk.** Let $\mathbf{B}^2 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \leqslant 1\}$ be the unit disk and define the moment functional

$$\langle \mathbf{u}_\mu, p\rangle = \iint_{\mathbf{B}^2} p(x,y)\,(1-x^2-y^2)^{\mu-1/2}\,dxdy, \quad \forall p \in \Pi,$$

with $\mu > -1/2$. Clearly, $\mathbf{u}_\mu$ is positive definite and classical since it satisfies (2.8) for $a_1(x,y) = 1-x^2; b(x,y) = -x\,y; a_2(x,y) = 1-y^2; d_1(x,y) = -2(\mu+1)x$, and $d_2(x,y) = -2(\mu+1)y$.

Moreover, $\mathbf{u}_\mu$ is also $\mathcal{D}$-classical satisfying

$$\partial_1((1-x^2-y^2)\,\mathbf{u}_\mu) = -(2\mu+1)\,x\,\mathbf{u}_\mu,$$
$$\partial_2((1-x^2-y^2)\,\mathbf{u}_\mu) = -(2\mu+1)\,y\,\mathbf{u}_\mu.$$

Then, by Proposition 3.10, $\mathbf{u}_\mu$ is self-coherent, and using Theorem 2.6 the derivatives of disk polynomials are again disk polynomials, and they are orthogonal with respect to the moment functionals $\mathbf{u}_\mu^{(1)} = \mathbf{u}_\mu^{(2)} = (1-x^2-y^2)\mathbf{u}_\mu = \mathbf{u}_{\mu+1}$. In this way, if $\{\mathbb{P}_{n,\mu}\}_{n\geqslant 0}$ is an OPS on the unit disk, then $\mathbb{P}_{n,\mu}^{(1)} = \mathbb{P}_{n,\mu}^{(2)}$, both defined in (2.5), and

$$\mathbb{P}_{n,\mu+1} = K_{n,\mu+1}\,\mathbb{P}_{n,\mu}^{(k)} = K_{n,\mu+1}\,\Omega_{n+1,k}\,(K_{n+1,\mu})^{-1}\,\partial_k\mathbb{P}_{n+1,\mu}, \tag{4.4}$$

where, following (2.1), $K_{n,\mu}$ denotes the (matrix) leading coefficient of $\mathbb{P}_{n,\mu}$, for $n \geqslant 0$.

For $\mu > 0$, an orthogonal basis is given in terms of Gegenbauer polynomials in the following form ([5, p. 31])

$$P_{n,m}^{(\mu)}(x,y) = C_{n-m}^{(\mu+1/2+m)}(x)\,(1-x^2)^{m/2}\,C_m^{(\mu)}\left(\frac{y}{\sqrt{1-x^2}}\right), \tag{4.5}$$

for $0 \leqslant m \leqslant n$, $n \geqslant 0$. This basis holds in the limit $\lim_{\mu\to 0}\mu^{-1}C_k^{(\mu)}(t) = (2/k)T_k(t)$ if $\mu = 0$, where $T_k(t)$ corresponds to the k-th Chebyshev polynomial of the first kind.

We define the OPS $\{\mathbb{P}_{n,\mu}\}_{n\geqslant 0}$ by means of above polynomials, that is,

$$\mathbb{P}_{n,\mu} = (P_{n,0}^{(\mu)}(x,y), P_{n,1}^{(\mu)}(x,y), \ldots, P_{n,n}^{(\mu)}(x,y))^t.$$

In this case, substituting the explicit expression of Gegenbauer polynomials, given in [1, (22.3.4)], in (4.5), we deduce that the leading coefficient matrix $K_{n,\mu} = (k_{i,j}^n(\mu))_{i,j=0}^n$ is a lower triangular matrix where

$$k_{i,j}^n(\mu) = \frac{\Gamma(n+\mu+1/2)}{\Gamma(\mu+i+1/2)(n-i)!\Gamma(\mu)}\frac{2^{n-(i-j)}\Gamma(\mu+j+(i-j)/2)}{j!((i-j)/2)!},$$

for $j = i, i-2, \ldots, i-2\lfloor i/2\rfloor$, and $c_{i,j}^n(\mu) = 0$ otherwise.

Now, substituting [1, (22.7.23)] in the second Gegenbauer polynomial in (4.5), and then the three term recurrence relation [1, (22.7.3)] in [1, (22.7.22)], a straightforward computation yields

$$\begin{aligned} P_{n,m}^{(\mu)}(x,y) =& a_{0,n}^{(m)}\,P_{n,m}^{(\mu+1)}(x,y) + a_{2,n}^{(m)}\,P_{n,m-2}^{(\mu+1)}(x,y) \\ &+ c_{0,n}^{(m)}\,P_{n-2,m}^{(\mu+1)}(x,y) + c_{2,n}^{(m)}\,P_{n-2,m-2}^{(\mu+1)}(x,y), \end{aligned}$$

where

$$\begin{aligned} a_{0,n}^{(m)} =& \frac{\mu}{m+\mu}\frac{m+\mu+1/2}{n+\mu+1/2}, && 0 \leqslant m \leqslant n, \\ a_{2,n}^{(m)} =& \frac{1}{4}\frac{\mu}{m+\mu}\frac{(n-m+1)_2}{(n+\mu+1/2)(m+\mu-1/2)}, && 2 \leqslant m \leqslant n, \\ c_{0,n}^{(m)} =& -\,a_{0,n}^{(m)}, && 0 \leqslant m \leqslant n-2, \\ c_{2,n}^{(m)} =& -\frac{(n+m+2\,\mu-1)_2}{(n-m+1)_2}\,a_{2,n}^{(m)}, && 2 \leqslant m \leqslant n-2, \end{aligned}$$

and $(\nu)_n = \nu(\nu+1)\cdots(\nu+n-1)$, $\nu \in \mathbb{R}$, $n \geqslant 1$, $(\nu)_0 = 1$, denotes the standard Pochhammer symbol. When $\mu = 0$, above constants can be considered as the corresponding limits.

Hence, ball polynomials satisfy the symmetric full coherence relations

$$\mathbb{P}_{n,\mu} = F_n\,K_{n,\mu+1}\,\mathbb{P}_{n,\mu}^{(k)} + N_n\,K_{n-2,\mu+1}\,\mathbb{P}_{n-2,\mu}^{(k)}, \quad k = 1, 2,$$

where F_n is a non-singular lower triangular matrix in the form

$$F_n = \begin{pmatrix} a_{0,n}^{(0)} & & & & \\ 0 & a_{0,n}^{(1)} & & & \\ a_{2,n}^{(2)} & 0 & a_{0,n}^{(2)} & & \\ & \ddots & \ddots & \ddots & \\ & & a_{2,n}^{(n)} & 0 & a_{0,n}^{(n)} \end{pmatrix},$$

and

$$N_n = \left(\begin{array}{ccccc} c_{0,n}^{(0)} & & & & \\ 0 & c_{0,n}^{(1)} & & & \\ c_{2,n}^{(2)} & 0 & c_{0,n}^{(2)} & & \\ & \ddots & \ddots & \ddots & \\ & & c_{2,n}^{(n-2)} & 0 & c_{0,n}^{(n-2)} \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

4.3. **Disk orthogonal polynomials with masses.** Let us consider $d\nu_\mu$, the classical Gegenbauer univariate measure

$$d\nu_\mu(t) = (1-t^2)^{\mu-1/2}dt, \quad \mu > -1/2,$$

and the measure $d\eta$ obtained from $d\nu_\mu$ by adding two symmetrical mass points at the ends of the supporting interval $[-1,1] \subset \mathbb{R}$,

$$d\eta(t) = (1-t^2)^{\mu-1/2}dt + \lambda\,\delta(t-1) + \lambda\,\delta(t+1), \quad \lambda > 0.$$

We denote by $\{C_n^{(\mu,\lambda)}(t)\}_{n\geqslant 0}$ the univariate orthogonal polynomials with respect to $d\eta$, such that

$$C_n^{(\mu,\lambda)}(t) = \frac{\Gamma(\mu+1/2)}{\Gamma(2\mu)}\frac{\Gamma(n+2\mu)}{\Gamma(n+\mu+1/2)}P_n^{(\mu-1/2,\mu-1/2,\lambda,\lambda)}(t),$$

where $\{P_n^{(\alpha,\beta,M,N)}\}_{n\geqslant 0}$, for $\alpha,\beta > -1$, $M,N \geqslant 0$, are the Jacobi-type polynomials studied in [11]. Observe that $C_n^{(\mu,0)}(t) = C_n^{(\mu)}(t)$, for $n \geqslant 0$. Then the bivariate polynomials

$$Q_{n,m}^{(\mu,\lambda)}(x,y) = C_{n-m}^{(\mu+m+1/2)}(x)\,\left(\sqrt{1-x^2}\right)^m\,C_m^{(\mu,\lambda)}\left(\frac{y}{\sqrt{1-x^2}}\right),$$

$0 \leqslant m \leqslant n$, $n \geqslant 0$, are orthogonal with respect to the positive definite moment functional $\mathbf{v}$ defined by

$$\begin{aligned}\langle \mathbf{v}, p(x,y)\rangle = &\iint_{\mathbf{B}^2} p(x,y)\,(1-x^2-y^2)^{\mu-1/2}\,dxdy \\ &+ \lambda\int_{-1}^{1} p(x,\sqrt{1-x^2})(1-x^2)^{\mu-1/2}dx \\ &+ \lambda\int_{-1}^{1} p(x,-\sqrt{1-x^2})(1-x^2)^{\mu-1/2}dx, \quad \forall p \in \Pi.\end{aligned}$$

For $n \geqslant 0$, let $\mathbb{Q}_n^{(\mu,\lambda)} = (Q_{n,0}^{(\mu,\lambda)}(x,y), Q_{n,1}^{(\mu,\lambda)}(x,y), \ldots, Q_{n,n}^{(\mu,\lambda)}(x,y))^t$ be the corresponding OPS.

Observe that $d\nu_{\mu+1}(t) = (1-t^2)\,d\eta(t)$, and both measures are symmetric. Therefore, $d\nu_{\mu+1}$ is a Christoffel transformation of the measure $d\eta$ by mens of the polynomial $1-t^2$ (see [8] as well as [20] and the references therein). Then, $\{C_n^{(\mu+1)}(t)\}_{n\geqslant 0}$ and $\{C_n^{(\mu,\lambda)}(t)\}_{n\geqslant 0}$ are related by

$$C_m^{(\mu,\lambda)}(t) = d_m^{(\mu)}\,C_m^{(\mu+1)}(t) + e_m^{(\mu)}\,C_{m-2}^{(\mu+1)}(t), \quad m \geqslant 0.$$

Using formula (4.4) in [11], expression [1, (22.3.4)], and the relation between Jacobi and Gegenbauer polynomials given in (4.1), we get

$$\begin{aligned} d_m^{(\mu)} =& \frac{\mu}{m+\mu}\left[1+\lambda\frac{(2\mu+1)_n}{(n-1)!(\mu+1/2)}\right]\left[1+\lambda\frac{(2\mu+1)_{n-1}}{(n-2)!(\mu+1/2)}\right], \\ e_m^{(\mu)} =& -\frac{\mu}{m+\mu}\left[1+\lambda\frac{(2\mu+1)_n}{(n-1)!(\mu+1/2)}\right] \\ &\times\left[1+\lambda\frac{(2\mu+1)_{n-1}}{(n-2)!(\mu+1/2)}+4\lambda\frac{(2\mu+1)_{n-1}(\mu+n)}{n!}\right]. \end{aligned}$$

Using the above relation and the properties of the Gegenbauer polynomials as in the previous example, we deduce

$$\begin{aligned} Q_{n,m}^{(\mu,\lambda)}(x,y) =& \tilde{a}_{0,n}^{(m)}\, P_{n,m}^{(\mu+1)}(x,y) + \tilde{a}_{2,n}^{(m)}\, P_{n,m-2}^{(\mu+1)}(x,y) \\ &+ \tilde{c}_{0,n}^{(m)}\, P_{n-2,m}^{(\mu+1)}(x,y) + \tilde{c}_{2,n}^{(m)}\, P_{n-2,m-2}^{(\mu+1)}(x,y), \end{aligned}$$

where

$$\begin{aligned} \tilde{a}_{0,n}^{(m)} &= d_m^{(\mu)}\,\frac{m+\mu+1/2}{n+\mu+1/2}, && 0\leqslant m\leqslant n, \\ \tilde{a}_{2,n}^{(m)} &= \frac{e_m^{(\mu)}}{4}\frac{(n-m+1)_2}{(n+\mu+1/2)\,(m+\mu-1/2)}, && 2\leqslant m\leqslant n, \\ \tilde{c}_{0,n}^{(m)} &= -\tilde{a}_{1,n}^{(m)}, && 0\leqslant m\leqslant n-2, \\ \tilde{c}_{2,n}^{(m)} &= -\frac{(n+m+2\,\mu-1)_2}{(n-m+1)_2}\,\tilde{a}_{2,n}^{(m)}, && 2\leqslant m\leqslant n-2. \end{aligned}$$

Hence $\{\mathbb{Q}_n^{(\mu,\lambda)}\}_{n\geqslant 0}$ and disk polynomials $\{\mathbb{P}_{n,\mu}\}_{n\geqslant 0}$ satisfy the full symmetric coherence relation

$$\mathbb{Q}_n^{(\mu,\lambda)} = \widetilde{F}_n\,\widetilde{K}_{n,\mu+1}\,\mathbb{P}_{n,\mu}^{(k)} + \widetilde{N}_n\,\widetilde{K}_{n-2,\mu+1}\,\mathbb{P}_{n-2,\mu}^{(k)}, \quad k=1,2,$$

where $\widetilde{K}_{n,\mu+1}$, $n\geqslant 0$, are the corresponding leading coefficients, computed as in the above example, and $\widetilde{F}_n$, $\widetilde{N}_n$ are constructed as in Example 4.2. Then, $\{\mathbf{u}_\mu,\mathbf{v}\}$ is a full coherent pair of moment functionals.

4.4. **Orthogonal polynomials on the parabolic biangle.** For $\alpha,\beta>-1$, define the positive definite moment functional

$$\langle \mathbf{u}_{(\alpha,\beta)}, p\rangle = \iint_\Omega p(x,y)\,(1-x)^\alpha\,(x-y^2)^\beta\,dydx, \quad \forall p\in\Pi,$$

where $\Omega=\{(x,y)\in\mathbb{R}^2 : y^2\leqslant x\leqslant 1\}$. It is well known [5, 10] that the polynomials

$$P_{n,m}^{(\alpha,\beta)}(x,y) = P_{n-m}^{(\alpha,\beta+m+1/2)}(2x-1)\,\left(\sqrt{x}\right)^m\,C_m^{(\beta+1/2)}\left(\frac{y}{\sqrt{x}}\right),$$

for $0\leqslant m\leqslant n$, $n\geqslant 0$, are orthogonal with respect to $\mathbf{u}_{(\alpha,\beta)}$. Let $\{\mathbb{P}_{n,(\alpha,\beta)}\}_{n\geqslant 0}$ be the OPS defined by

$$\mathbb{P}_{n,(\alpha,\beta)} = (P_{n,0}^{(\alpha,\beta)}(x,y), P_{n,1}^{(\alpha,\beta)}(x,y),\ldots,P_{n,n}^{(\alpha,\beta)}(x,y))^t, \quad n\geqslant 0.$$

Although the moment functional $\mathbf{u}_{(\alpha,\beta)}$ is classical, it is not $\mathcal{D}$-classical. Nevertheless, it satisfies

$$\partial_2((x-y^2)\mathbf{u}_{(\alpha,\beta)}) = -2\,(\beta+1)\,y\,\mathbf{u}_{(\alpha,\beta)}.$$

Then, by Theorem 2.6, $\{\mathbb{P}^{(2)}_{n,(\alpha,\beta)}\}_{n\geqslant 0}$ constitutes an orthogonal polynomial system with respect to the moment functional $\mathbf{u}^{(2)} = (x-y^2)\,\mathbf{u}_{(\alpha,\beta)} = \mathbf{u}_{(\alpha,\beta+1)}$. Thus,

$$\mathbb{P}_{n,(\alpha,\beta+1)} = \widehat{C}_{n,\alpha,\beta+1}\mathbb{P}^{(2)}_{n,(\alpha,\beta)},$$

where $\widehat{C}_{n,\alpha,\beta+1}$ denotes the leading coefficient of $\mathbb{P}_{n,(\alpha,\beta+1)}$. From the explicit expressions for Jacobi polynomials [1, (22.3.2)] and Gegenbauer polynomials [1, (22.3.4)] we deduce that $\widehat{C}_{n,\alpha,\beta}$ is a diagonal matrix with entries

$$\hat{c}^{\alpha,\beta}_{i,i} = \frac{2^i\Gamma(2n+\alpha+\beta-i+3/2)\Gamma(\beta+i+1/2)}{(n-i)!\,i!\Gamma(n+\alpha+\beta+3/2)\Gamma(\beta+1/2)}, \quad 0\leqslant i\leqslant n.$$

Therefore, the moment functional $\mathbf{u}_{(\alpha,\beta)}$ is self-partial 2-coherent. Indeed, using relations [1, (22.7.16)] and [1, (22.7.19)] for Jacobi polynomials, we get

$$\begin{aligned}P^{(\alpha,\beta)}_{n,m}(x,y) = \hat{a}^{(m)}_n\, &P^{(\alpha,\beta+1)}_{n,m}(x,y)\\ &+ \hat{b}^{(m)}_{0,n}\,P^{(\alpha,\beta+1)}_{n-1,m}(x,y) + \hat{b}^{(m)}_{2,n}\,P^{(\alpha,\beta+1)}_{n-1,m-2}(x,y)\\ &\qquad + \hat{c}^{(m)}_n\,P^{(\alpha,\beta+1)}_{n-2,m-2}(x,y),\end{aligned}$$

where

$$\begin{aligned}
\hat{a}^{(m)}_n &= \frac{(\beta+1/2)\,(n+\alpha+\beta+3/2)}{(2\,n-m+\alpha+\beta+3/2)\,(m+\beta+1/2)}, && 0\leqslant m\leqslant n,\\
\hat{b}^{(m)}_{0,n} &= \frac{(\beta+1/2)\,(n-m+\alpha)}{(2\,n-m+\alpha+\beta+3/2)\,(m+\beta+1/2)}, && 0\leqslant m\leqslant n-1,\\
\hat{b}^{(m)}_{2,n} &= -\frac{(\beta+1/2)\,(n-m+1)}{(2\,n-m+\alpha+\beta+3/2)\,(m+\beta+1/2)}, && 2\leqslant m\leqslant n-1,\\
\hat{c}^{(m)}_n &= -\frac{(\beta+1/2)\,(n+\beta+1/2)}{(2\,n-m+\alpha+\beta+3/2)\,(m+\beta+1/2)}, && 2\leqslant m\leqslant n-2.
\end{aligned}$$

The case $\beta=-1/2$ is treated as the corresponding limit in the Gegenbauer polynomial described in Example 4.2.

Then, biangle polynomials satisfy the partial 2-coherence relation

$$\begin{aligned}\mathbb{P}_{n,(\alpha,\beta)} =& \widehat{F}_n\widehat{C}_{n,\alpha,\beta+1}\mathbb{P}^{(2)}_{n,(\alpha,\beta)} + \widehat{M}_n\widehat{C}_{n-1,\alpha,\beta+1}\mathbb{P}^{(2)}_{n-1,(\alpha,\beta)}\\ &+ \widehat{N}_n\widehat{C}_{n-2,\alpha,\beta+1}\mathbb{P}^{(2)}_{n-2,(\alpha,\beta)},\end{aligned}$$

where

$$\widehat{F}_n = \begin{pmatrix} \hat{a}^{(0)}_n & & & \\ & \hat{a}^{(1)}_n & & \\ & & \ddots & \\ & & & \hat{a}^{(n)}_n \end{pmatrix}$$

is a non-singular and diagonal matrix,

$$\widehat{M}_n = \left(\begin{array}{ccccc} \hat{b}_{0,n}^{(0)} & & & & \\ 0 & \hat{b}_{0,n}^{(1)} & & & \\ \hat{b}_{2,n}^{(2)} & 0 & \hat{b}_{0,n}^{(2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{b}_{2,n}^{(n-1)} & 0 & \hat{b}_{0,n}^{(n-1)} \\ \hline 0 & \cdots & 0 & 0 & 0 \end{array}\right),$$

and

$$\widehat{N}_n = \left(\begin{array}{ccccc} 0 & & & & \\ 0 & 0 & & & \\ \hat{c}_n^{(2)} & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & \hat{c}_n^{(n-2)} & 0 & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

4.5. **Orthogonal polynomials on the simplex.** For $\alpha, \beta, \gamma > -1$, define the positive definite moment functional

$$\langle \mathbf{u}_{(\alpha,\beta,\gamma)}, p\rangle = \iint_{\mathbf{T}^2} p(x,y)\, x^\alpha\, y^\beta\, (1-x-y)^\gamma\, dxdy, \quad \forall p \in \Pi,$$

where $\mathbf{T}^2 = \{(x,y) \in \mathbb{R}^2 \,:\, x \geqslant 0, y \geqslant 0, 1-x-y \geqslant 0\}$.

It is well known [5, 10] that the bivariate polynomials

$$P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) = \mathcal{P}_{n-m}^{(\beta+\gamma+2m+1,\alpha)}(2x-1)\,(1-x)^m\,\mathcal{P}_m^{(\gamma,\beta)}\left(\frac{2y}{1-x}-1\right), \qquad (4.6)$$

$0 \leqslant m \leqslant n$, $n \geqslant 0$, are orthogonal with respect to the moment functional $\mathbf{u}_{(\alpha,\beta,\gamma)}$. For $n \geqslant 0$, let

$$\mathbb{P}_{n,(\alpha,\beta,\gamma)} = (P_{n,0}^{(\alpha,\beta,\gamma)}(x,y), P_{n,1}^{(\alpha,\beta,\gamma)}(x,y), \ldots, P_{n,n}^{(\alpha,\beta,\gamma)}(x,y))^t.$$

be the corresponding OPS.

The moment functional $\mathbf{u}_{(\alpha,\beta,\gamma)}$ is $\mathcal{D}$-classical since it satisfies

$$\begin{aligned} \partial_1\left(x(1-x-y)\mathbf{u}_{(\alpha,\beta,\gamma)}\right) &= [(\alpha+1)(1-x-y)-(\gamma+1)x]\,\mathbf{u}_{(\alpha,\beta,\gamma)}, \\ \partial_2\left(y\,(1-x-y)\mathbf{u}_{(\alpha,\beta,\gamma)}\right) &= [(\beta+1)(1-x-y)-(\gamma+1)y]\mathbf{u}_{(\alpha,\beta,\gamma)}, \end{aligned}$$

and thus, it is self-coherent.

It was shown ([14], [19]) that $\{\mathbb{P}_{n,(\alpha,\beta,\gamma)}^{(1)}\}_{n\geqslant 0}$ and $\{\mathbb{P}_{n,(\alpha,\beta,\gamma)}^{(2)}\}_{n\geqslant 0}$ are OPS associated with the moment functionals

$$\begin{aligned} \mathbf{u}^{(1)} &= x(1-x-y)\mathbf{u}_{(\alpha,\beta,\gamma)} = \mathbf{u}_{(\alpha+1,\beta,\gamma+1)}, \\ \mathbf{u}^{(2)} &= y(1-x-y)\mathbf{u}_{(\alpha,\beta,\gamma)} = \mathbf{u}_{(\alpha,\beta+1,\gamma+1)}, \end{aligned}$$

respectively. Therefore,

$$\begin{aligned} \mathbb{P}_{n,(\alpha+1,\beta,\gamma+1)} &= K_{n,(\alpha+1,\beta,\gamma+1)}\mathbb{P}_{n,(\alpha,\beta,\gamma)}^{(1)}, \\ \mathbb{P}_{n,(\alpha,\beta+1,\gamma+1)} &= K_{n,(\alpha,\beta+1,\gamma+1)}\mathbb{P}_{n,(\alpha,\beta,\gamma)}^{(2)}, \end{aligned}$$

where $K_{n,(\alpha,\beta,\gamma)}$ denotes the non-singular matrix leading coefficient of $\mathbb{P}_{n,(\alpha,\beta,\gamma)}$.

In order to compute that matrix, we use the explicit expression of Jacobi polynomials given in [1, (22.3.2)] in (4.6), and it can be proved that the leading coefficient $K_{n,(\alpha,\beta,\gamma)} = (k_{i,j}^n)_{i,j=0}^n$ is a lower triangular matrix whose entries are given by

$$k_{i,j}^n = f_{n,i} \sum_{m=0}^{i} (-1)^{i-m} \binom{i}{m} \binom{m}{m-j} \frac{\Gamma(\beta+\gamma+i+m+1)}{2^{m-j}\Gamma(\gamma+m+1)}, \quad 0 \leqslant j \leqslant i,$$

and

$$f_{n,i} = \frac{\Gamma(2n+\alpha+\beta+\gamma+2)\Gamma(\gamma+i+1)}{(n-i)!i!\Gamma(n+i+\alpha+\beta+\gamma+2)\Gamma(\beta+\gamma+i+1)}.$$

Now, we want to know the shape of the matrices involved in the coherent relations

$$\mathbb{P}_{n,(\alpha,\beta,\gamma)} = \bar{F}_{n,k} K_n^{(k)} \mathbb{P}_{n,(\alpha,\beta,\gamma)}^{(k)} + \bar{M}_{n,k} K_{n-1}^{(k)} \mathbb{P}_{n-1,(\alpha,\beta,\gamma)}^{(k)} + \bar{N}_{n,k} K_{n-2}^{(k)} \mathbb{P}_{n-2,(\alpha,\beta,\gamma)}^{(k)}, \tag{4.7}$$

for $k = 1, 2$, where $K_n^{(1)} = K_{n,(\alpha+1,\beta,\gamma+1)}$, and $K_n^{(2)} = K_{n,(\alpha,\beta+1,\gamma+1)}$ denote the respective leading coefficient.

Indeed, using [1, (22.7.15)] and [1, (22.7.18)] in (4.6), we get

$$\begin{aligned} P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) =& \bar{a}_{0,n}^{(m)} P_{n,m}^{(\alpha+1,\beta,\gamma+1)}(x,y) + \bar{a}_{1,n}^{(m)} P_{n,m-1}^{(\alpha+1,\beta,\gamma+1)}(x,y) \\ &+ \bar{b}_{0,n}^{(m)} P_{n-1,m}^{(\alpha+1,\beta,\gamma+1)}(x,y) + \bar{b}_{1,n}^{(m)} P_{n-1,m-1}^{(\alpha+1,\beta,\gamma+1)}(x,y) \\ &+ \bar{c}_{0,n}^{(m)} P_{n-2,m}^{(\alpha+1,\beta,\gamma+1)}(x,y) + \bar{c}_{1,n}^{(m)} P_{n-2,m-1}^{(\alpha+1,\beta,\gamma+1)}(x,y), \end{aligned}$$

where

$$\bar{a}_{0,n}^{(m)} = \frac{n+m+\alpha+\beta+\gamma+3}{2n+\alpha+\beta+\gamma+3} r_{n,m}^{\gamma+1,\alpha+1}, \qquad 0 \leqslant m \leqslant n,$$

$$\bar{a}_{1,n}^{(m)} = \frac{n-m+1}{2n+\alpha+\beta+\gamma+3} r_{n,m}^{0,\alpha+1}, \qquad 1 \leqslant m \leqslant n,$$

$$\bar{b}_{0,n}^{(m)} = \left[\frac{n+m+\beta+\gamma+1}{2n+\alpha+\beta+\gamma+1} - \frac{n-m+\alpha+1}{2n+\alpha+\beta+\gamma+3}\right] r_{n,m}^{\gamma+1,\alpha+1}, \quad 0 \leqslant m \leqslant n-1,$$

$$\bar{b}_{1,n}^{(m)} = \left[\frac{n-m}{2n+\alpha+\beta+\gamma+1} - \frac{n+m+\alpha+\beta+\gamma+2}{2n+\alpha+\beta+\gamma+3}\right] r_{n,m}^{0,0}, \quad 1 \leqslant m \leqslant n-1,$$

$$\bar{c}_{0,n}^{(m)} = -\frac{n-m+\alpha}{2n+\alpha+\beta+\gamma+1} r_{n,m}^{\gamma+1,0}, \qquad 0 \leqslant m \leqslant n-2,$$

$$\bar{c}_{1,n}^{(m)} = -\frac{n+m+\beta+\gamma}{2n+\alpha+\beta+\gamma+1} r_{n,m}^{0,0}, \qquad 1 \leqslant m \leqslant n-2,$$

and

$$r_{n,m}^{a,b} = \frac{(m+\beta+a)\,(n+m+\beta+\gamma+1+b)}{(2n+\alpha+\beta+\gamma+2)\,(2m+\beta+\gamma+1)}.$$

Then, the matrices involved in the first coherent relation (4.7) for $k = 1$, $\bar{F}_{n,1}$, $\bar{M}_{n,1}$, and $\bar{N}_{n,1}$ are lower triangular and bidiagonal matrices in the form

$$\bar{F}_{n,1} = \begin{pmatrix} \bar{a}_{0,n}^{(0)} & & & & \\ \bar{a}_{1,n}^{(1)} & \bar{a}_{0,n}^{(1)} & & & \\ & \bar{a}_{1,n}^{(2)} & \bar{a}_{0,n}^{(2)} & & \\ & & \ddots & \ddots & \\ & & & \bar{a}_{1,n}^{(n)} & \bar{a}_{0,n}^{(n)} \end{pmatrix},$$

$$\bar{M}_{n,1} = \left(\begin{array}{ccccc} \bar{b}_{0,n}^{(0)} & & & & \\ \bar{b}_{1,n}^{(1)} & \bar{b}_{0,n}^{(1)} & & & \\ & \bar{b}_{1,n}^{(2)} & \bar{b}_{0,n}^{(2)} & & \\ & & \ddots & \ddots & \\ & & & \bar{b}_{1,n}^{(n-1)} & \bar{b}_{0,n}^{(n-1)} \\ \hline 0 & 0 & \cdots & 0 & 0 \end{array}\right),$$

$$\bar{N}_{n,1} = \left(\begin{array}{ccccc} \bar{c}_{0,n}^{(0)} & & & & \\ \bar{c}_{1,n}^{(1)} & \bar{c}_{0,n}^{(1)} & & & \\ & \bar{c}_{1,n}^{(2)} & \bar{c}_{0,n}^{(2)} & & \\ & & \ddots & \ddots & \\ & & & \bar{c}_{1,n}^{(n-2)} & \bar{c}_{0,n}^{(n-2)} \\ \hline 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{array}\right),$$

Next, using expressions [1, (22.7.16)] and [1, (22.7.19)] in (4.6), we get

$$\begin{aligned} P_{n,m}^{(\alpha,\beta,\gamma)}(x,y) =& \bar{d}_{0,n}^{(m)} P_{n,m}^{(\alpha,\beta+1,\gamma+1)} + \bar{d}_{1,n}^{(m)} P_{n,m-1}^{(\alpha,\beta+1,\gamma+1)} + \bar{d}_{2,n}^{(m)} P_{n,m-2}^{(\alpha,\beta+1,\gamma+1)} \\ &+ \bar{e}_{0,n}^{(m)} P_{n-1,m}^{(\alpha,\beta+1,\gamma+1)} + \bar{e}_{1,n}^{(m)} P_{n-1,m-1}^{(\alpha,\beta+1,\gamma+1)} + \bar{e}_{2,n}^{(m)} P_{n-1,m-2}^{(\alpha,\beta+1,\gamma+1)} \\ &+ \bar{f}_{0,n}^{(m)} P_{n-2,m}^{(\alpha,\beta+1,\gamma+1)} + \bar{f}_{1,n}^{(m)} P_{n-2,m-1}^{(\alpha,\beta+1,\gamma+1)} + \bar{f}_{2,n}^{(m)} P_{n-2,m-2}^{(\alpha,\beta+1,\gamma+1)}. \end{aligned}$$

where

$$\bar{d}_{0,n}^{(m)} = r_{n,m}^{\gamma+1,\alpha+1}\, u_{n,m}^{\gamma+2,\alpha+2}, \qquad 0 \leqslant m \leqslant n,$$

$$\bar{d}_{1,n}^{(m)} = \frac{(n-m+1)\,(s_{m,m+1}^{\beta,1} - s_{m,m}^{(\gamma,0)})}{2n+\alpha+\beta+\gamma+3}\, r_{n,m}^{\gamma+1,\alpha+1}, \qquad 1 \leqslant m \leqslant n,$$

$$\bar{d}_{2,n}^{(m)} = -\frac{(n-m+1)_2\,(m+\beta)\,(m+\gamma)}{(2\,n+\alpha+\beta+\gamma+2)_2\,(2\,m+\beta+\gamma)_2}, \qquad 2 \leqslant m \leqslant n,$$

$$\bar{e}_{0,n}^{(m)} = -\frac{2\,(n-m+\alpha)}{(2\,n+\alpha+\beta+\gamma)}\, r_{n,m}^{\gamma+1,\alpha+1}\, s_{m,m}^{\beta,2}, \qquad 0 \leqslant m \leqslant n-1,$$

$$\bar{e}_{1,n}^{(m)} = -\frac{r_{n,m}^{\gamma+1,\alpha+1}\,(s_{m,m+1}^{\beta,1}\,t_{n,m}^{1} - s_{m,m}^{\gamma,0}\,t_{n,m}^{0})}{n+m+\alpha+\beta+\gamma+2}, \qquad 1 \leqslant m \leqslant n-1,$$

$$\bar{e}_{2,n}^{(m)} = \frac{2\,(n-m+1)\,(2\,m+\beta)\,u_{n,m}^{0,0}\,s_{m,m}^{\gamma,0}}{(2\,n+\alpha+\beta+\gamma+3)\,(2\,m+\beta+\gamma+1)}, \qquad 2 \leqslant m \leqslant n-1,$$

$$\bar{f}_{0,n}^{(m)} = \frac{(n-m+\alpha-1)_2\,(m+\beta+\gamma+1)_2}{(2\,n+\alpha+\beta+\gamma+1)_2\,(2\,m+\beta+\gamma+1)_2}, \qquad 0 \leqslant m \leqslant n-2,$$

$$\bar{f}_{1,n}^{(m)} = \frac{(n-m+\alpha)\,(s_{m,m+1}^{\beta,1} - s_{m,m}^{\gamma,0})}{2n+\alpha+\beta+\gamma+1}\,r_{n,m}^{\gamma+1,0}, \qquad 1 \leqslant m \leqslant n-2,$$

$$\bar{f}_{2,n}^{(m)} = -\frac{(n+m+\beta+\gamma)_2\,(m+\beta)\,(m+\gamma)}{(2\,n+\alpha+\beta+\gamma+1)_2\,(2\,m+\beta+\gamma)_2}, \qquad 2 \leqslant m \leqslant n-2,$$

and

$$u_{n,m}^{a,b} = \frac{(m+\beta+a)\,(n+m+\beta+\gamma+1+b)}{(2\,n+\beta+\gamma+1+b)\,(2\,m+\beta+a)}$$

$$s_{k,l}^{a,b} = \frac{k+a+b}{2\,l+\beta+\gamma}$$

$$\begin{aligned} t_{n,m}^{a} = {} & \frac{(n+m+\beta+\gamma+1+a)\,(n+m+\alpha+\beta+\gamma+1+a)}{2\,n+\alpha+\beta+\gamma+1+2\,a} \\ & + \frac{(n-m+1-a)\,(n-m+\alpha+1-a)}{2\,n+\alpha+\beta+\gamma+3-2\,a}. \end{aligned}$$

Therefore, $\bar{F}_{n,2}$, $\bar{M}_{n,2}$ and $\bar{N}_{n,2}$ are lower three-diagonal matrices, in the form

$$\bar{F}_{n,2} = \begin{pmatrix} \bar{d}_{0,n}^{(0)} & & & & \\ \bar{d}_{1,n}^{(1)} & \bar{d}_{0,n}^{(1)} & & & \\ \bar{d}_{2,n}^{(2)} & \bar{d}_{1,n}^{(2)} & \bar{d}_{0,n}^{(2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \bar{d}_{2,n}^{(n)} & \bar{d}_{1,n}^{(n)} & \bar{d}_{0,n}^{(n)} \end{pmatrix},$$

$$\bar{M}_{n,2} = \left(\begin{array}{cccc} \bar{e}_{0,n}^{(0)} & & & \\ \bar{e}_{1,n}^{(1)} & \bar{e}_{0,n}^{(1)} & & \\ \bar{e}_{2,n}^{(2)} & \bar{e}_{1,n}^{(2)} & \bar{e}_{0,n}^{(2)} & \\ & \ddots & \ddots & \ddots \\ & & \bar{e}_{2,n}^{(n-1)} & \bar{e}_{1,n}^{(n-1)} \quad \bar{e}_{0,n}^{(n-1)} \\ \hline 0 & \cdots & 0 & 0 \quad 0 \end{array}\right),$$

$$\bar{N}_{n,2} = \left(\begin{array}{ccccc} \bar{f}^{(0)}_{0,n} & & & & \\ \bar{f}^{(1)}_{1,n} & \bar{f}^{(1)}_{0,n} & & & \\ \bar{f}^{(2)}_{2,n} & \bar{f}^{(2)}_{1,n} & \bar{f}^{(2)}_{0,n} & & \\ & \ddots & \ddots & \ddots & \\ & & \bar{f}^{(n-2)}_{2,n} & \bar{f}^{(n-2)}_{1,n} & \bar{f}^{(n-2)}_{0,n} \\ \hline 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{array}\right).$$

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