OUTER RATIO ASYMPTOTICS FOR BIORTHOGONAL MATRIX POLYNOMIALS WITH UNBOUNDED RECURRENCE COEFFICIENTS.

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ABSTRACT. In this work a study of sequences of matrix biorthogonal polynomials that satisfy a nonsymmetric recurrence relation with unbounded coefficients is presented. The outer ratio asymptotics for this family of matrix biorthogonal polynomials is derived in quite general assumptions. Some illustrative examples are shown.

1. INTRODUCTION

The study of the outer ratio asymptotics, i.e. the limit of the ratio of two consecutive polynomials p_n and p_{n+1} of a sequence of polynomials, $\{p_n\}_{n\in\mathbb{N}}$, orthogonal with respect to an inner product, outside the convex hull of the support of the measure of orthogonality has attracted the interest of many researchers in the last decades. Since the first study of Nevai in 1979 (cf. [14]) for orthogonal polynomials with respect to a measure supported on a infinite subset of the real line with convergent recurrence coefficients, more general situations have been considered, such as the case of asymptotically periodic recurrence coefficients with a finite number of accumulation points (cf. [16], [19], [20]) or the case of unbounded recurrence coefficients (cf. [19]).

In 1993 Durán (cf. [9]) gave the characterization of symmetric bilinear forms for which the multiplication operator by a polynomial is symmetric. There, necessary and sufficient conditions were deduced, so that, a sequence of scalar polynomials $\{p_n\}_{n\in\mathbb{N}}$ satisfying a $(2N+1)$-term recurrence relation

$$h(x)\,p_n(x) = c_{n,0}\,p_n(x) + \sum_{k=1}^{N}\left(\overline{c}_{n,k}\,p_{n-k}(x) + c_{n+k,k}\,p_{n+k}(x)\right),$$

is orthogonal with respect to a symmetric bilinear form (generalization of the Favard's theorem). In particular, the attention was focused on the so called Sobolev type inner products. In that work, the author gave a first idea to connect scalar polynomials, orthogonal with respect to a bilinear form, and matrix orthogonal polynomials with respect to a positive definite matrix of measures. From the above result, Durán and Van Assche [13] proved that if $\{p_n\}_{n\in\mathbb{N}}$ is a sequence of scalar polynomials satisfying

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a $(2N+1)$-term recurrence relation, they are related to a matrix polynomial sequence $\{P_n\}_{n\in\mathbb{N}}$ satisfying a matrix three-term recurrence relation

$$x\,P_n(x) = A_{n+1}\,P_{n+1}(x) + B_n\,P_n(x) + A_n^*\,P_{n-1}(x)\,, \quad n\in\mathbb{N}\,, \tag{1}$$

with initial conditions $P_0(x) = I_N$ and $P_{-1}(x) = \mathbf{0}_N$, where for each $n\in\mathbb{N}$, A_n is an upper triangular, nonsingular matrix and B_n is a Hermitian matrix.

The above results increased the interest on matrix orthogonal polynomials (cf. the survey paper [7]). So, in [12] Durán and López-Rodríguez studied properties of the zeros of a sequence of matrix polynomials $\{P_n\}_{n\in\mathbb{N}}$ which are orthonormal with respect to a positive definite matrix of measures W. Next, Durán in [10] showed two important results: the first one is a quadrature formula for matrix polynomials and the second one is the Markov theorem for matrix polynomials when, again, the matrix of measures is positive definite. In [8], the author deals with the outer ratio asymptotics for matrix orthogonal polynomials. Therein, Durán obtained the asymptotic behavior of two consecutive polynomials belonging to the matrix Nevai class, i.e. these polynomials satisfy a three-term recurrence relation as in (1) where, again, A_n are nonsingular, upper triangular matrices and B_n are Hermitian matrices for all $n\in\mathbb{N}$, and such that $A_n\to A$ and $B_n\to B$. Later on, Durán and Daneri-Vias analyzed the above case but when the matrix sequences $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$ diverge in a particular way (cf. [11] for details).

Recently, Yakhlef and Marcellán [18] have studied the outer relative asymptotics of sequences of matrix orthogonal polynomials for Uvarov perturbations in the degenerate case, i.e. given a positive definite matrix of measures α, and its corresponding sequences of matrix orthonormal polynomials, $\{P_n^\alpha\}_{n\in\mathbb{N}}$, satisfying a three-term recurrence relation as in (1), they define a new matrix of measures β as

$$d\beta(u) = d\alpha(u) + M\,\delta(u-c)\,,$$

where M is a positive definite matrix, $\delta(u-c)$ is the Dirac measure supported at c that is located outside the support of $d\alpha$, and the sequence of matrix orthonormal polynomial associated with $d\beta$, $\{P_n^\beta\}_{n\in\mathbb{N}}$. Then, they study the outer relative asymptotic between the sequences $\{P_n^\beta\}_{n\in\mathbb{N}}$ and $\{P_n^\alpha\}_{n\in\mathbb{N}}$ under quite general assumptions on the coefficients of the three-term recurrence relation $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$.

When the matrix of measures, W, is no longer Hermitian, we can define a bilinear (respectively, sesquilinear form) in $\mathbb{R}^{N\times N}[x]$ (respectively, in $\mathbb{C}^{N\times N}[x]$) and deal with sequences of biorthogonal matrix polynomials, $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$, which play the role of the left and right-orthogonalities (cf. Definition 1). It can be proven that these sequences satisfy a three-term recurrence relation

$$x\,V_n(x) = \alpha_n\,V_{n+1}(x) + \beta_n\,V_n(x) + \gamma_n\,V_{n-1}(x)\,, \tag{2}$$

$$x\,G_n(x) = G_{n+1}(x)\,a_n + G_n(x)\,b_n + G_{n-1}(x)\,c_n\,, \tag{3}$$

where a_n, α_n, c_n, γ_n, are nonsingular matrices. Without loss of generality we can suppose that a_n, α_n are lower triangular matrices and c_n, γ_n are upper triangular matrices (cf. [13]).

As we have said above, matrix polynomials defined by the recurrence relations as (2) appear in a natural way in the literature and its study paid an increasing attention in the last decades. For example in [4], the authors give a matrix interpretation of the multiple orthogonality in terms of matrix orthogonal polynomials satisfying the same kind of recurrence relations. On the other hand, in [1], [2], [15] were studied perturbations of measures (Christoffel, Geronimus, and Geronimus-Uvarov) which yield to non-positive definite matrix of measures, and thus, to the concept of biorthogonality.

Example 1. Let μ be a scalar measure supported on the real line, and $\{p_n\}_{n\in\mathbb{N}}$ its corresponding sequence of monic orthogonal polynomials. If W is a matrix polynomial of degree M, then we can define a new measure $d\hat{\mu} = W\,d\mu$ (Christoffel transformation of the measure) which clearly is non-positive definite as W needs not to be identical to W^{T}. In particular, taking $W(x) = \left(\begin{smallmatrix} x & 1 \\ 0 & x \end{smallmatrix}\right)$, then following the techniques developed in [1], it is easy to see that the sequences of matrix polynomials

$$V_n(x) = \begin{pmatrix} K_n(x,0)/p_n(0) & \frac{p_n(0)\,K_n(x,0) - W(p_n,p_{n+1})(0)\,p_n(x)}{p_n^2(0)\,x} \\ 0 & K_n(x,0)/p_n(0) \end{pmatrix},$$

$$G_n(x) = -\begin{pmatrix} K_n(x,0)/p_n(0) & 0 \\ -\big(K_n(x,0)p_n'(0) + \frac{\partial K_n(x,0)}{\partial y}\big)/p_n(0) & K_n(x,0)/p_n(0) \end{pmatrix}.$$

with $W(p_n,p_{n+1})(x) = p_n(x)p_{n+1}'(x) - p_{n+1}(x)p_n'(x)$, and

$$K_n(x,0) = \big(p_n(0)\,p_{n+1}(x) - p_{n+1}(0)\,p_n(x)\big)/x$$

satisfies

$$\int V_m(x)\,W(x)\,G_n(x)\,d\mu(x) = I_N\,\delta_{n,m}\,,$$

i.e. $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ are sequences of biorthonormal polynomials with respect to $W\,d\mu$ and satisfy the three-term recurrence relations (2) and (3) with $a_n = \gamma_{n+1}$, $b_n = \beta_n$, and $c_n = \alpha_{n-1} = I_N$ (cf. Theorem 1) .

Outer ratio asymptotics for the class of matrix orthogonal polynomials satisfying a three-term recurrence formula as in (2) was studied for the first time in [6]. There, the authors analyzed the case of convergent recurrence coefficients by introducing an analog of the Nevai class of matrix polynomials for the nonsymmetric case, the generalized matrix Nevai class.

In the present contribution our aim is to generalize those results when the coefficients of the nonsymmetric recurrence formula diverge in a particular way.

The structure of the manuscript is as follows. Section 2 provides the basic background about matrix biorthogonal polynomials $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ that satisfy dual nonsymmetric recurrence relations. Here, we establish the relations between the zeros of these two families of polynomials and discuss the most appropriate way of scaling these matrix polynomials in order to obtain its asymptotic behavior. We also deduce a quadrature and a Liouville-Ostrogradski formulas for the right-orthogonal polynomials. Section 3 deals with the outer ratio asymptotics for left and right-orthogonal matrix polynomials with varying recurrence coefficients. Section 4 is focused on our main result (Theorem 8), the outer ratio asymptotics of matrix orthogonal polynomials satisfying recurrence formulas with nonsymmetric and nonsingular recurrence coefficients diverging in a particular way. Section 5 deals with the study of the case when certain matrix appearing in the recurrence formula is singular.

2. MATRIX BIORTHOGONAL POLYNOMIALS

Let us a consider a *quasidefinite* $N\times N$ *matrix of measures*, W, i.e. $W=\left(w_{i,j}\right)_{i,j=0}^{N-1}$, with measures, $w_{i,j}$, $i,j\in\{0,\dots,N-1\}$, supported on the real line but not necessarily positive definite with finite moments, $U_n=\int x^n\,dW(x)$, $n\in\mathbb{N}$, and such that the Hankel determinants satisfy

$$\det\left(\left(U_{i+j}\right)_{i=0,\dots,n}^{j=0,\dots,n}\right)\neq 0\,,\quad n\in\mathbb{N}\,.$$

We will assume, without loss of generality, that the matrix of measures is normalized by $U_0=\int dW(x)=I_N$. If P and R are matrix polynomials in $\mathbb{C}^{N\times N}[x]$, then we introduce the following *sesquilinear form*,

$$\langle P\,,R\rangle=\int P(x)\,dW(x)R^{\mathsf{T}}(x)\,,\ P\,,\ R\in\mathbb{C}^{N\times N}[x]\,.$$

Definition 1. Let W be a quasidefinite $N\times N$ matrix of measures. The matrix polynomial sequences, $\{V_n\}_{n\in\mathbb{N}}$ (respectively, $\{G_n\}_{n\in\mathbb{N}}$), such that for every $n,m\in\mathbb{N}$, $\deg V_n(x)=n$ (respectively, $\deg G_m(x)=m$) and

$$\int V_n(x)\,dW(x)\,x^m=\Omega_n^{(1)}\,\delta_{n,m}\,,\quad m=0,\dots,n$$

$$\text{(respectively,}\quad\int x^n\,dW(x)\,G_m(x)=\Omega_m^{(2)}\,\delta_{n,m}\,,\quad n=0,\dots,m\,),$$

with $\delta_{n,m}$ is the Kronecker symbol and $\Omega_n^{(1)}$ (respectively, $\Omega_n^{(2)}$) nonsingular matrices for $n\in\mathbb{N}$, are said to be *the left* (respectively, *right*) *orthogonal polynomial sequences* with respect to W.

We also refer to $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ as biorthonormal polynomial sequences with respect to the quasidefinite matrix of measures W when

$$\int V_n(x)\,dW(x)\,G_m(x)=I_N\,\delta_{n,m}\,,\quad n,m\in\mathbb{N}\,.$$

Theorem 1 (cf. [5], Theorem 4)**.** *Given a quasidefinite matrix of measures W, then its biorthogonal polynomial sequences, $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ satisfy the three-term recurrence relations*

$$x\,V_n(x)=A_n\,V_{n+1}(x)+B_n\,V_n(x)+C_n\,V_{n-1}(x)\,,\quad n\in\mathbb{N}\,,\tag{4}$$

$$x\,G_n(x)=G_{n-1}(x)A_{n-1}+G_n(x)B_n+G_{n+1}(x)\,C_{n+1}\,,\quad n\in\mathbb{N}\,,\tag{5}$$

with $V_{-1}(x)=\mathbf{0}_N$, $V_0(x)=I_N$, and $G_{-1}(x)=\mathbf{0}_N$, $G_0(x)=I_N$. Here, A_n and C_n are nonsingular matrices for every $n\in\mathbb{N}$.

Without loss of generality we can suppose that $(A_n)_{n\in\mathbb{N}}$ (respectively, $(C_n)_{n\in\mathbb{N}}$) is a sequence of lower (respectively, upper) triangular matrices (cf. [13]).

In the same way as in the scalar case, the Favard's Theorem for matrix polynomials can be find in the literature (cf. [5], Theorem 7).

Theorem 2. *Given a quasidefinite matrix of measures W, then its biorthogonal polynomial sequences, $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$, are such that for each n, V_n and G_n have the same zeros.*

Proof. Notice that the N-block Jacobi matrix associated with the recurrence relation (5) for the polynomials G_n is the transpose of the N-block Jacobi matrix

$$J=\begin{pmatrix} B_0 & A_0 & \mathbf{0}_N & \\ C_1 & B_1 & A_1 & \ddots \\ \mathbf{0}_N & C_2 & B_2 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

associated with the recurrence relation (4) for the polynomials V_n. The result follows by taking into account that the zeros of V_n (G_n, respectively) are the eigenvalues of J_n (J_n^{T}, respectively), where J_n is the truncated matrix of J, with dimension $nN\times nN$. □

Definition 2. Let W be a quasidefinite matrix of measures and $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ the corresponding sequences of biorthogonal matrix polynomials. We define the sequence of first kind associated polynomials $\{V_n^{(1)}\}_{n\in\mathbb{N}}$ and $\{G_n^{(1)}\}_{n\in\mathbb{N}}$, as follows,

$$V_{n-1}^{(1)}(x)=\int \frac{V_n(x)-V_n(y)}{x-y}\,dW(y),\ G_{n-1}^{(1)}(x)=\int dW(y)\,\frac{G_n(x)-G_n(y)}{x-y}.$$

The first kind associated polynomial sequences $\{V_n^{(1)}\}_{n\in\mathbb{N}}$, $\{G_n^{(1)}\}_{n\in\mathbb{N}}$ also satisfy the three-term recurrence relations (4) and (5) with initial conditions, $V_{-1}^{(1)}(x)=\mathbf{0}_N$, $V_0^{(1)}(x)=A_0^{-1}$, and $G_{-1}^{(1)}(x)=\mathbf{0}_N$, $G_0^{(1)}(x)=C_1^{-1}$.

In order to obtain the outer ratio asymptotics for polynomials with varying recurrence coefficients we will need some auxiliary results such as quadrature and Liouville-Ostrogradsky type formulas for biorthogonal polynomials. For the left-orthogonal polynomials these results can be found in the literature.

Lemma 1 (cf. [12], Lemma 2.2)**.** *Let A be an $N\times N$ matrix polynomials and let $\mathbf{a}$ be a zero of A of multiplicity p, i.e. $\mathbf{a}$ is a zero of* $\det A$ *of multiplicity p. We put $L(\mathbf{a},A)=\{v\in\mathbb{C}^N : \overline{v}^{\mathsf{T}}A(\mathbf{a})=\mathbf{0}_N\}$ and $R(\mathbf{a},A)=\{v\in\mathbb{C}^N : A(\mathbf{a})v=\mathbf{0}_N\}$. If* $\dim L(\mathbf{a},A)=\dim R(\mathbf{a},A)=$ *p, then $\left(\operatorname{Adj}(A(x))\right)^{(j)}(\mathbf{a})=\mathbf{0}_N$, for $j=0,\dots,p-2$, and $\left(\operatorname{Adj}(A(x))\right)^{(p-1)}(\mathbf{a})\neq\mathbf{0}_N$. Moreover,* $\operatorname{rank}\left(\operatorname{Adj}(A(x))\right)^{(p-1)}(\mathbf{a})=p$.

Lemma 2 (cf. [3], Proposition 5.14)**.** *Let P_n be a matrix polynomial of degree n with m different zeros $\{x_{n,1},\dots,x_{n,m}\}$ and with $\{\ell_1,\dots,\ell_m\}$ as corresponding multiplicities. For any matrix polynomial R of degree less than or equal to $n-1$ and $x\in\mathbb{C}\setminus\{x_{n,1},\dots,x_{n,m}\}$ we have*

$$R(x)\,(P_n(x))^{-1}=\sum_{k=1}^{m}\frac{C_{n,k}}{x-x_{n,k}},\quad (P_n(x))^{-1}R(x)=\sum_{k=1}^{m}\frac{D_{n,k}}{x-x_{n,k}},$$

where

$$C_{n,k}=\frac{\ell_k}{(\det P_n)^{(\ell_k)}(x_{n,k})}R(x_{n,k})\left(\operatorname{Adj}P_n(x)\right)^{(\ell_k-1)}(x_{n,k}),$$
$$D_{n,k}=\frac{\ell_k}{(\det P_n)^{(\ell_k)}(x_{n,k})}\left(\operatorname{Adj}P_n(x)\right)^{(\ell_k-1)}(x_{n,k})R(x_{n,k}).$$

Theorem 3 (Quadrature formula)**.** *Let $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures W, and let*

$\{V_n^{(1)}\}_{n\in\mathbb{N}}$, $\{G_n^{(1)}\}_{n\in\mathbb{N}}$ be its first kind associated polynomial sequences. Given the different zeros of V_n, $\{x_{n,1},\dots,x_{n,s}\}$, with multiplicities $\{\ell_1,\dots,\ell_s\}$, we define the matrices $\Gamma_{m,k}$, $\widetilde{\Gamma}_{m,k}$, as

$$\Gamma_{n,k}=\frac{\ell_k}{(\det V_n)^{(\ell_k)}(x_{n,k})}\big(\operatorname{Adj} V_n(x)\big)^{(\ell_k-1)}(x_{n,k})V_{n-1}^{(1)}(x_{n,k}),$$

$$\widetilde{\Gamma}_{n,k}=G_{n-1}^{(1)}(x_{n,k})\frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})}\big(\operatorname{Adj} G_n(x)\big)^{(\ell_k-1)}(x_{n,k}).$$

Then, for any polynomial P of degree less than or equal to $2n-1$ the following quadrature formula holds

$$\int P(x)\,dW(x)=\sum_{k=1}^{s}P(x_{n,k})\,\Gamma_{n,k}\,,\quad \int dW(x)\,P(x)=\sum_{k=1}^{s}\widetilde{\Gamma}_{n,k}\,P(x_{n,k}).$$

Proof. We will prove the quadrature formula for the right orthogonal polynomials, because the left one is already proved in [6]. Let P be a matrix polynomial of degree less than or equal to $n-1$. Since G_n is a polynomial with nonsingular leading coefficient, then $P(x)=G_n(x)\,C(x)+R(x)$. Here $C(x)$ and $R(x)$ are matrix polynomials with degree of $R(x)$ less than or equal to $2n-1$. Using Lemma 2 we get

$$G_n^{-1}(x)R(x)=\sum_{k=1}^{s}\frac{D_{n,k}}{x-x_{n,k}}\,,$$

where the matrices $D_{n,k}$ are

$$D_{n,k}=\frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})}\big(\operatorname{Adj} G_n(x)\big)^{(\ell_k-1)}(x_{n,k})R(x_{n,k}).$$

Taking into account that $R(x_{n,k})=P(x_{n,k})-G_n(x_{n,k})\,C(x_{n,k})$ and

$$\big(\operatorname{Adj} G_n\big)^{(\ell_k-1)}(x_{n,k})\,G_n(x_{n,k})=G_n(x_{n,k})\big(\operatorname{Adj} G_n\big)^{(\ell_k-1)}(x_{n,k})=\mathbf{0}_N\,, \tag{6}$$

the previous expression becomes

$$D_{n,k}=\frac{\ell_k}{(\det G_n)^{(\ell_k)}(x_{n,k})}\big(\operatorname{Adj} G_n(x)\big)^{(\ell_k-1)}(x_{n,k})\,P(x_{n,k}).$$

Thus,

$$P(x)=G_n(x)\,C(x)+\sum_{k=1}^{s}G_n(x)\,\frac{D_{n,k}}{x-x_{n,k}}\,.$$

Using again (6), we have

$$P(x)=G_n(x)\,C(x)+\sum_{k=1}^{s}\frac{G_n(x)-G_n(x_{n,k})}{x-x_{n,k}}\,D_{n,k}\,.$$

and by the definition of the first kind associated polynomial, it follows that

$$\int dW(x)\,P(x)=\int dW(x)\,G_n(x)\,C(x)+\sum_{k=1}^{s}G_{n-1}^{(1)}(x_{n,k})\,D_{n,k}\,.$$

So, from orthogonality we have

$$\int dW(x)\,P(x) = \sum_{k=1}^{s} G_{m-1}^{(1)}(x_{m,k})\,D_{m,k}\,,$$

and the result follows. □

Theorem 4 (Liouville-Ostrogradski formula). *Let $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ be the sequence of matrix biorthogonal polynomials with respect to a quasidefinite matrix of measures W and $\{V_n^{(1)}\}_{n\in\mathbb{N}}$, $\{G_n^{(1)}\}_{n\in\mathbb{N}}$ be, respectively, the first kind associated polynomial sequences. Then,*

$$V_n(x)\,G_n^{(1)}(x) - V_{n+1}^{(1)}(x)\,G_{n+1}(x) = A_n^{-1}\,, \tag{7}$$

$$V_n(x)\,G_n^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_{n+1}(x) = C_{n+1}^{-1}\,, \tag{8}$$

where A_n, C_n are the nonsingular matrices in (5).

Proof. Equation (7) was already proved in [6]. To prove (8) we proceed by induction on n. For $n = 0$ the result follows from the initial conditions. We assume that the formula

$$V_p(x)\,G_p^{(1)}(x) - V_{p-1}^{(1)}(x)\,G_{p+1}(x) = C_{p+1}^{-1}\,,$$

holds for $p = 1,\dots,n-1$. First, we use the recurrence relation in $G_n^{(1)}$ and G_{n+1} to obtain

$$\begin{aligned}V_n(x)\,G_n^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_{n+1}(x) &= (V_n(x)\,G_{n-1}^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_n(x))\\ &\quad\times(xI_N - B_n)\,C_{n+1}^{-1} - (V_n(x)\,G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_{n-1}(x))A_{n-1}\,C_{n+1}^{-1}\,.\end{aligned}$$

Second, we prove that

$$V_n(x)\,G_{n-1}^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_n(x) = \mathbf{0}_N\,.$$

Using the definition of the first kind associated polynomial, we get

$$\begin{aligned}V_n(x)\,G_{n-1}^{(1)}(x) &- V_{n-1}^{(1)}(x)\,G_n(x)\\ &= \int \frac{V_n(y)\,dW(y)}{x-y}\,G_n(x) - V_n(x)\int\frac{dW(y)\,G_n(y)}{x-y}\,.\end{aligned}$$

Adding and subtracting $\displaystyle\int \frac{V_n(y)\,dW(y)\,G_n(y)}{x-y}$ in the last relation and taking into account the left and right-orthogonalities, the result follows. With this in mind

$$\begin{aligned}V_n(x)\,G_n^{(1)}(x) &- V_{n-1}^{(1)}(x)\,G_{n+1}(x)\\ &= -(V_n(x)\,G_{n-2}^{(1)}(x) - V_{n-1}^{(1)}(x)\,G_{n-1}(x))A_{n-1}\,C_{n+1}^{-1}\,.\end{aligned} \tag{9}$$

Now, using the recurrence relations for V_n and $V_{n-1}^{(1)}$,

$$\begin{aligned}V_n(x)\,G_{n-2}^{(1)}(x) &- V_{n-1}^{(1)}(x)\,G_{n-1}(x)\\ &= A_{n-1}^{-1}(x - B_{n-1})\,(V_{n-1}(x)\,G_{n-2}^{(1)}(x) - V_{n-2}^{(1)}(x)\,G_{n-1}(x))\\ &\quad - A_{n-1}^{-1}C_{n-1}\,(V_{n-2}(x)\,G_{n-2}^{(1)}(x) - V_{n-3}^{(1)}(x)\,G_{n-1}(x))\,.\end{aligned}$$

Since $V_{n-1}(x)\,G^{(1)}_{n-2}(x)-V^{(1)}_{n-2}(x)\,G_{n-1}(x)=\mathbf{0}_p$, we deduce

$$V_n(x)\,G^{(1)}_{n-2}(x)-V^{(1)}_{n-1}(x)\,G_{n-1}(x) \\ =-A^{-1}_{n-1}\,C_{n-1}(V_{n-2}(x)\,G^{(1)}_{n-2}(x)-V^{(1)}_{n-3}(x)\,G_{n-1}(x)).$$

Using this relation in (9) we obtain

$$V_n(x)\,G^{(1)}_{n}(x)-V^{(1)}_{n-1}(x)\,G_{n+1}(x) \\ =A^{-1}_{n-1}\,C_{n-1}\,(V_{n-2}(x)\,G^{(1)}_{n-2}(x)-V^{(1)}_{n-3}(x)\,G_{n-1}(x))A_{n-1}\,C^{-1}_{n+1}.$$

According to the induction hypothesis the result follows. □

In the sequel, we will assume that the matrix recurrence coefficients diverge in a particular way. We will suppose that there exists a sequence of positive definite matrices $(D_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases}\lim_{n\to\infty} D_n^{-1/2}A_n\,D_n^{-1/2}=A\,,\ \lim_{n\to\infty} D_n^{-1/2}B_n\,D_n^{-1/2}=B\,,\\ \lim_{n\to\infty} D_n^{-1/2}C_n\,D_n^{-1/2}=C\,,\ \lim_{n\to\infty} D_n^{-1/2}D_{n-1}^{1/2}=I_N\,.\end{cases} \tag{10}$$

When unbounded coefficients are considered in the scalar case (assuming the same hypothesis given by (10)), the outer ratio asymptotics is then obtained for the scaled polynomials $p_n(c_n z)$. However, in the matrix case there is a large range of possibilities to define the scaled matrix polynomial $P(Hx)$ (cf. [11]). From now on, we are going to work with two notions of scaled matrix polynomials depending on the kind of orthogonality (left or right) that we will deal with. In the case of left-orthogonality, the suitable definition of scaled matrix polynomials was introduced by Durán in [11].

Definition 3 (cf. [11])**.** Given the sequences of recurrence coefficients $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, and $(C_n)_{n\in\mathbb{N}}$, we can define a sequence of matrix polynomials in a matrix variable, $\{\boldsymbol{V}_n\}_{n\in\mathbb{N}}$, as

$$T\,\boldsymbol{V}_n(T)=A_n\,\boldsymbol{V}_{n+1}(T)+B_n\,\boldsymbol{V}_n(T)+C_n\,\boldsymbol{V}_{n-1}(T)\,,$$

with initial conditions $\boldsymbol{V}_{-1}(T)=\mathbf{0}_N, \boldsymbol{V}_0(T)=I_N$. We define the *left-scaled matrix polynomials* as $V_n^H(x)=\boldsymbol{V}_n(H\,x)$. On the other hand, the natural definition in order to scale the right-orthogonal polynomials is the following one. Using the recurrence coefficients we can define another matrix polynomial sequence of a matrix variable, $\{\boldsymbol{G}_n\}_{n\in\mathbb{N}}$, as

$$\boldsymbol{G}_n(T)\,T=\boldsymbol{G}_{n-1}(T)A_{n-1}+\boldsymbol{G}_n(T)B_n+\boldsymbol{G}_{n+1}(T)\,C_{n+1}\,,$$

with initial conditions $\boldsymbol{G}_{-1}(x)=\mathbf{0}_N$ and $\boldsymbol{G}_0(x)=I_N$. Now, we define the *right-scaled matrix polynomials* as $G_n^H(x)=\boldsymbol{G}_n(H\,x)$.

Notice that, in particular, for each non-negative integer k the scaled polynomial sequences $\{V_n^{D_k}(z)\}_{n\in\mathbb{N}}$ and $\{G_n^{D_k}(z)\}_{n\in\mathbb{N}}$, are biorthogonal with respect to a certain varying matrix of measures W_k. We will say that $\{V_n^{D_k}\}_{n\in\mathbb{N}}$ and $\{G_n^{D_k}\}_{n\in\mathbb{N}}$ are matrix biorthogonal polynomials with varying recurrence coefficients. In this way, our main result Theorem 8 will be a consequence of a more general theorem on outer ratio asymptotics cf. Theorem 6.

In the sequel, we associate with three given matrices A, C, B, where A, C, are nonsingular, the left-orthogonal Chebyshev matrix polynomials of second kind $\{U_n^{C,B,A}\}_{n\in\mathbb{N}}$

which are defined by the recurrence relation

$$x\,U_n^{C,B,A}(x) = C\,U_{n+1}^{C,B,A}(x) + B\,U_n^{C,B,A}(x) + A\,U_{n-1}^{C,B,A}(x)\,,\quad n\in\mathbb{N}\,, \tag{11}$$

with initial conditions $U_0^{C,B,A}(x) = I_N$ and $U_{-1}^{C,B,A}(x) = \mathbf{0}_N$, as well as the right-orthogonal Chebyshev matrix polynomial of second kind $\{T_m^{A,B,C}\}_{m\in\mathbb{N}}$ given by

$$x\,T_n^{A,B,C}(x) = T_{n+1}^{A,B,C}(x)A + T_n^{A,B,C}(x)B + T_{n-1}^{A,B,C}(x)\,C\,,\quad n\in\mathbb{N}\,, \tag{12}$$

where $T_0^{A,B,C}(x) = I_N$ and $T_{-1}^{A,B,C}(x) = \mathbf{0}_N$. We denote by $W_{C,B,A}$ the matrix weight for which the polynomial sequences $\{U_n^{C,B,A}\}$, $\{T_n^{A,B,C}\}$ are biorthogonal.

3. Outer ratio asymptotics for orthogonal polynomials with varying recurrence coefficients

For each $k = 1,2,\ldots$, we consider orthogonal matrix polynomials $\{R_{n,k}\}_{n\in\mathbb{N}}$ and $\{S_{n,k}\}_{n\in\mathbb{N}}$, given by the recurrence relations

$$x\,R_{n,k}(x) = A_{n,k}\,R_{n+1,k}(x) + B_{n,k}\,R_{n,k}(x) + C_{n,k}\,R_{n-1,k}(x)\,,\quad n\in\mathbb{N}\,, \tag{13}$$

$$x\,S_{n,k}(x) = S_{n-1,k}(x)A_{n-1,k} + S_{n,k}(x)B_{n,k} + S_{n+1,k}(x)\,C_{n+1,k}\,,\quad n\in\mathbb{N}\,, \tag{14}$$

with $R_{0,k}(x) = I_N$, $R_{-1,k}(x) = \mathbf{0}_N$, and $S_{0,k}(x) = I_N$, $S_{-1,k}(x) = \mathbf{0}_N$.

For a fixed k, these matrix polynomial sequences are biorthogonal with respect to a certain quasidefinite matrix of measures which we denote by W_k.

As far as we know, the only result on outer ratio asymptotics for matrix polynomials satisfying nonsymmetric recurrence relations is the following one.

Theorem 5 (cf. [6], Theorem 3)**.** *Let W be a quasidefinite matrix of measures, and $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ biorthogonal polynomial sequences with respect to W, satisfying,respectively, the three-term recurrence relation* (4)*,* (5)*. Let us assume*

$$\lim_{n\to\infty} A_n = A\,,\ \lim_{n\to\infty} B_n = B\,,\ \lim_{n\to\infty} C_n = C\,,$$

with A, C nonsingular matrices. We denote by Δ_n the set of zeros of V_n, and $\Gamma = \bigcap_{N\geq 0} M_N$, where $M_N = \overline{\bigcup_{n\geq N}\Delta_n}$. Then,

$$\lim_{n\to\infty} V_{n-1}(x)\,V_n^{-1}(x)A_{n-1}^{-1} = \int \frac{dW_{C,B,A}(y)}{x-y}\,,\quad x\in\mathbb{C}\setminus\Gamma\,,$$

$$\lim_{n\to\infty} C_n^{-1}\,G_n^{-1}(x)\,G_{n-1}(x) = \int \frac{dW_{C,B,A}(y)}{x-y}\,,\quad x\in\mathbb{C}\setminus\Gamma\,,$$

where $W_{C,B,A}$ is the matrix of measures associated with the second kind Chebyshev matrix polynomials. Moreover, the convergence is locally uniformly on compact subsets of $\mathbb{C}\setminus\Gamma$.

In the previous case, the coefficients in the recurrence relation are assumed to be convergent.

The following result generalizes the previous one in two directions: we consider a case of varying recurrence coefficients and, for a fixed k, the recurrence coefficients will diverge in a particular way.

Theorem 6. *Let W_k be for each $k\in\mathbb{N}$ a quasidefinite matrix of measures, and $\{R_{n,k}\}_{n\in\mathbb{N}}$, $\{S_{n,k}\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal matrix polynomials depending on a parameter k, $k=1,2,\dots$, satisfying (13), (14). Let $(n_m)_{m\in\mathbb{N}}$, $(k_m)_{m\in\mathbb{N}}$, be two increasing sequences of positive integers and we will assume that there exist three matrices A, B, C, with A and C nonsingular, such that for all $l\in\mathbb{N}$,*

$$\lim_{m\to\infty} A_{n_m-l,k_m}=A\,,\ \lim_{m\to\infty} B_{n_m-l,k_m}=B\,,\ \lim_{m\to\infty} C_{n_m-l,k_m}=C\,. \tag{15}$$

We denote by $\widetilde{\Delta}_{n,k}$ the set of zeros of S_{n_m,k_m} and $\widetilde{\Gamma}=\bigcap_{N\geq 0}\widetilde{M}_{N,k}$, where $\widetilde{M}_{N,k}=\overline{\bigcup_{n\geq N}\widetilde{\Delta}_{n,k}}$. Then,

$$\lim_{m\to\infty} R_{n_m-1,k_m}(x)R^{-1}_{n_m,k_m}(x)A^{-1}_{n_m-1,k_m}=\int\frac{dW_{C,B,A}(t)}{x-t}\,,\quad x\in\mathbb{C}\setminus\Gamma\,, \tag{16}$$

$$\lim_{m\to\infty} C^{-1}_{n_m,k_m}S^{-1}_{n_m,k_m}(x)S_{n_m-1,k_m}(x)=\int\frac{dW_{C,B,A}(y)}{x-y}\,,\quad x\in\mathbb{C}\setminus\widetilde{\Gamma}\,, \tag{17}$$

where $W_{C,B,A}$ is the matrix weight for the generalized Chebyshev matrix polynomials defined in (12). Moreover, the convergence is locally uniformly on compact subsets of $\mathbb{C}\setminus\Gamma$.

Proof. We will prove (17). Notice that (16) follows by using analogous arguments. First, we consider the sequence of discrete measures $\{\mu_{n,k}\}_{n\in\mathbb{N}}$ defined by

$$\mu_{n,k}=\sum_{j=1}^{s}\delta_{x_{m,k,j}}R_{n-1,k}(x_{n,k,j})\widetilde{\Gamma}_{n,k,j}S_{n-1,k}(x_{n,k,j})\,,\quad n\in\mathbb{N}\,,$$

where $x_{n,k,j}$, $j=1,\dots,s$, are the different zeros of the matrix polynomial, $R_{n,k}$, or, equivalently, the zeros of $S_{n,k}$ (cf. Theorem 2) with multiplicities $\{\ell_1,\dots,\ell_s\}$, and

$$\widetilde{\Gamma}_{n,k,j}=S^{(1)}_{n-1,k}(x_{n,k,j})\frac{\ell_j\big(\operatorname{Adj}(S_{n,k}(x))\big)^{(\ell_j-1)}(x_{n,k,j})}{(\det(S_{n,k}(x)))^{(\ell_j)}(x_{n,k,j})}\,.$$

Notice that from the quadrature formula

$$\begin{aligned}\int d\mu_{n,k}(x)&=\sum_{j=1}^{s}R_{n-1,k}(x_{n,k,j})\widetilde{\Gamma}_{n,k,j}S_{n-1,k}(x_{n,k,j})\\&=\int R_{n-1,k}(x)\,dW_k(x)\,S_{n-1,k}(x)=I_N\,,\quad n=1,2,\dots\,.\end{aligned}$$

According to Lemma 2, we get

$$\big(S_{n,k}(x)\big)^{-1}S_{n-1,k}(x)=\sum_{k=1}^{s}\frac{D_{n,k,j}}{x-x_{n,k,j}}\,,$$

where

$$D_{n,k,j}=\frac{\ell_j}{(\det S_{n,k})^{(\ell_j)}(x_{n,k,j})}\big(\operatorname{Adj}\,S_{n,k}(x)\big)^{(\ell_j-1)}(x_{n,k,j})S_{n-1,k}(x_{n,k,j})\,. \tag{18}$$

Multiplying in the left hand side of (18) by $C^{-1}_{n,k}$

$$C^{-1}_{n,k}D_{n,k,j}=C^{-1}_{n,k}\frac{\ell_j}{(\det S_{n,k})^{(\ell_j)}(x_{n,k,j})}\big(\operatorname{Adj}\,S_{n,k}(x)\big)^{(\ell_k-1)}(x_{n,k,j})S_{n-1,k}(x_{n,k,j})\,,$$

and applying the Liouville-Ostrogradski formula (8)

$$S_{n,k}(x_{n,k,j})\big(\operatorname{Adj} S_{n,k}\big)^{(\ell_k-1)}(x_{n,k,j}) = \mathbf{0}_N\,,$$

we get

$$C_{n,k}^{-1}D_{n,k,j} = R_{n-1,k}(x_{n,k,j})\,\widetilde{\Gamma}_{n,k}\,S_{n-1,k}(x_{n,k,j})\,.$$

From the definition of the matrices $\widetilde{\Gamma}_{n,k}$, we have

$$C_{n,k}^{-1}S_{n,k}^{-1}(x)\,S_{n-1,k}(x) = \int \frac{d\mu_{n,k}(y)}{x-y}\,,\quad x\in\mathbb{C}\setminus\widetilde{\Gamma}\,.$$

For two given nonnegative integers n, k let us consider the generalized Chebyshev matrix polynomials of the second kind, $\{T_n^{A,B,C}(x)\}_{n\in\mathbb{N}}$, defined in (12). We can prove by induction that

$$\lim_{m\to\infty}\int d\mu_{n_m,k_m}(x)T_l^{A,B,C}(x) = I_N\,\delta_{l,0}\,. \tag{19}$$

To this end, we can write

$$S_{n-1,k}(x)\,T_l^{A,B,C}(x) = S_{n,k}(x)\,K_{l,n-1,k}(x) + \sum_{i=1}^{n} S_{n-i,k}(x)\,\Delta_{i,l,n-1,k}\,, \tag{20}$$

where $K_{l,n-1,k}(x)$ is a matrix polynomial with degree less than or equal to $n-1$. Thus,

$$\begin{aligned}\int d\mu_{n,k}(x)T_l^{A,B,C}(x) &= \sum_{j=1}^{s}R_{n-1,k}(x_{n,k,j})\widetilde{\Gamma}_{n,k,j}S_{n-1,k}(x_{n,k,j})T_l^{A,B,C}(x_{n,k,j})\\ &= \sum_{j=1}^{s}R_{n-1,k}(x_{n,k,j})\widetilde{\Gamma}_{n,k,j}\Big(S_{n,k}(x_{n,k,j})K_{l,n-1,k}(x_{n,k,j}) + \sum_{i=1}^{n}S_{n-i,k}(x_{n,k,j})\Delta_{i,l,n-1,k}\Big)\,.\end{aligned}$$

According to the definition of the matrices $\widetilde{\Gamma}_{n,k}$ and taking into account that

$$\begin{aligned}&\big(\operatorname{Adj}(S_{n,k})(x)\big)^{(\ell_k-1)}(x_{n,k,j})\,S_{n,k}(x_{n,k,j})\\ &\qquad = S_{n,k}(x_{n,k,j})\,\big(\operatorname{Adj}(S_{n,k})(x)\big)^{(\ell_k-1)}(x_{n,k,j}) = \mathbf{0}_N\,,\end{aligned}$$

we get

$$\int d\mu_{n,k}(x)\,T_l^{A,B,C}(x) = \sum_{j=1}^{s}R_{n-1,k}(x_{n,k,j})\,\widetilde{\Gamma}_{n,k,j}\,\Big(\sum_{i=1}^{n}S_{n-i,k}(x_{n,k,j})\,\Delta_{i,l,n-1,k}\Big)\,.$$

Using the quadrature formula given in Theorem 3, we conclude

$$\begin{aligned}&\int d\mu_{n,k}(x)\,T_l^{A,B,C}(x)\\ &\qquad = \sum_{i=1}^{n}\int R_{n-1,k}(x)\,dW_k(x)\,S_{n-i,k}(x)\,\Delta_{i,l,n-1,k} = \Delta_{1,l,n-1,k}\,.\end{aligned}$$

So, (19) follows when $\lim_{m\to\infty}\Delta_{j,l,n_m-1,k_m} = I_N\,\delta_{j,l+1}$ holds. We use induction on l. When $l=0$ the result is immediate. Now, assuming that the result is valid up to l, the three-term recurrence relation for the matrix polynomials $\{T_n^{A,B,C}\}_{n\in\mathbb{N}}$ yields

$$S_{n-1,k}(x)\,T_{l+1}^{A,B,C}(x) = S_{n-1,k}(x)\big(x\,T_l^{A,B,C}(x) - T_l^{A,B,C}(x)\,B - T_{l-1}^{A,B,C}(x)\,C\big)A^{-1}\,.$$

Using (20) and the three-term recurrence relation for $\{S_{n,k}\}_{n\in\mathbb{N}}$

$$\Delta_{j,l+1,n-1,k} = A_{n-j,k}\,\Delta_{j-1,l,n-1,k}A^{-1} + B_{n-j,k}\,\Delta_{j,l,n-1,k}A^{-1} + C_{n-j}\,\Delta_{j+1,l,n-1,k}A^{-1} - \Delta_{j,l,n-1,k}\,BA^{-1} - \Delta_{j,l-1,n-1,k}\,CA^{-1}\,.$$

For $j \geq l+3$ or $j \leq l-1$ the induction hypothesis yields

$$\lim_{m\to\infty} \Delta_{j,l+1,n_m-1,k_m} = \mathbf{0}_N\,.$$

We study the cases $j = l$, $j = l+1$, and $j = l+2$ separately:
Case 1. $j = l$.

$$\lim_{m\to\infty} \Delta_{j,l+1,n_m-1,k_m} = \lim_{m\to\infty}\big(A_{n_m-l,k_m}\,\Delta_{l-1,l,n_m-1,k_m}A^{-1} + B_{n_m-l,k_m}\,\Delta_{l,l,n_m-1,k_m}A^{-1} + C_{n_m-l}\,\Delta_{l+1,l,n_m-1,k_m}A^{-1} - \Delta_{l,l,n_m-1,k_m}\,BA^{-1} - \Delta_{l,l-1,n_m-1,k_m}\,CA^{-1}\big) = (C-C)A^{-1} = \mathbf{0}_N\,.$$

Case 2. $j = l+1$.

$$\lim_{m\to\infty} \Delta_{l+1,l+1,n_m-1,k_m} = \lim_{m\to\infty}\big(A_{n_m-l-1,k_m}\Delta_{l,l,n_m-1,k_m}A^{-1} + B_{n_m-l-1,k_m}\Delta_{l+1,l,n_m-1,k_m}A^{-1} + C_{n_m-l-1}\Delta_{l+2,l,n_m-1,k_m}A^{-1} - \Delta_{l+1,l,n_m-1,k_m}\,BA^{-1} - \Delta_{l+1,l-1,n_m-1,k_m}\,CA^{-1}\big) = (B-B)A^{-1} = \mathbf{0}_N\,.$$

Case 3. $j = l+2$.

$$\lim_{m\to\infty} \Delta_{l+2,l+1,n_m-1,k_m} = \lim_{m\to\infty}\big(A_{n_m-l-2,k_m}\Delta_{l+1,l,n_m-1,k_m}A^{-1} + B_{n_m-l-2,k_m}\Delta_{l+2,l,n_m-1,k_m}A^{-1} + C_{n_m-l-2}\Delta_{l+3,l,n_m-1,k_m}A^{-1} - \Delta_{l+2,l,n_m-1,k_m}\,BA^{-1} - \Delta_{l+2,l-1,n_m-1,k_m}\,CA^{-1}\big) = AA^{-1} = I_N\,.$$

Now, in the same way as in [11], one can prove

$$\lim_{m\to\infty}\int \frac{d\mu_{n_m,k_m}(y)}{x-y} = \int \frac{dW_{C,B,A}(y)}{x-y}\,, \quad x\in\mathbb{C}\setminus\Gamma\,,$$

by using the so called *method of moments*. □

4. Main result

Definition 4 (cf. [17] Section 7.7)**.** Let $A, B \in \mathbb{C}^{N\times N}$ be Hermitian matrices. We write $A \geq B$ if the matrix $A-B$ is positive semi-definite. Similarly, $A > B$ means that $A-B$ is positive definite.

It is easy to see that the relation $\geq$ (respectively, $>$) is transitive and reflexive.

Definition 5. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of Hermitian matrices. We say that $(A_n)_{n\in\mathbb{N}}$ is an increasing sequence if $A_{n+1} \geq A_n$ for every $n \in \mathbb{N}$.

Theorem 7. *Let $(D_n)_{n\in\mathbb{N}}$ be an increasing matrix sequence. Then, for each $n \in \mathbb{N}$, the polynomials $V_n^{D_n}(x)$ and $G_n^{D_n}(x)$ have the same zeros. Moreover, denoting by $\widetilde{x}_{n,j}$ the zeros of $V_n^{D_n}(x)$ or $G_n^{D_n}(x)$, then there exists a positive constant M (independent of n) such that the the zeros $\widetilde{x}_{n,j}$ are contained in a disk $\mathrm{D} = \{z \in \mathbb{C} : |z| < M\}$.*

Proof. The N-block Jacobi matrix associated with $G_n^{D_k}(x)D_k^{1/2}$ is the transpose of the matrix

$$\widetilde{J}^{(k)} = \begin{pmatrix} D_k^{-1/2}B_0D_k^{-1/2} & D_k^{-1/2}A_0D_k^{-1/2} & \mathbf{0}_N & \\ D_k^{-1/2}C_1D_k^{-1/2} & D_k^{-1/2}B_1D_k^{-1/2} & D_k^{-1/2}A_1D_k^{-1/2} & \ddots \\ \mathbf{0}_N & D_k^{-1/2}C_2D_k^{-1/2} & D_k^{-1/2}B_2D_k^{-1/2} & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

associated with $D_k^{1/2}V_n^{D_k}(x)$. The first statement is a straightforward consequence of the fact that the zeros of $V_n^{D_n}(x)$ are the eigenvalues of $\widetilde{J}_{nN}^{(k)}$ (truncated N-block Jacobi matrix of dimension nN) and the zeros of $G_n^{D_n}(x)D_k^{1/2}$ are the eigenvalues of $\widetilde{J}_{nN}^{(k)T}$ (the transpose of the previous one). Using the Gershgorin disk theorem for the location of eigenvalues, it is enough to show that the entries of the matrix $\widetilde{J}_{nN}$ are bounded (independently of n). But the entries of this matrix are

$$\begin{aligned} D_k^{-1/2}A_n\,D_k^{-1/2} &= D_k^{-1/2}D_n^{1/2}D_n^{-1/2}A_n\,D_n^{-1/2}D_n^{1/2}D_k^{-1/2},\\ D_k^{-1/2}B_n\,D_k^{-1/2} &= D_k^{-1/2}D_n^{1/2}D_n^{-1/2}B_n\,D_n^{-1/2}D_n^{1/2}D_k^{-1/2},\\ D_k^{-1/2}C_n\,D_k^{-1/2} &= D_k^{-1/2}D_n^{1/2}D_n^{-1/2}C_n\,D_n^{-1/2}D_n^{1/2}D_k^{-1/2}. \end{aligned}$$

Since the matrices $D_n^{-1/2}A_n\,D_n^{-1/2}$, $D_n^{-1/2}B_n\,D_n^{-1/2}$, and $D_n^{-1/2}C_n\,D_n^{-1/2}$ converge and $I_N \geq D_k^{-1/2}D_n^{1/2}$ if $n \leq k$ since $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence, then the result follows. $\square$

The theory of matrix orthogonal polynomials with varying recurrence coefficients studied in the previous section allows us to prove our main result, i.e. the outer ratio asymptotics for matrix biorthogonal polynomials satisfying three-term recurrence relations as in (4) and (5) with unbounded coefficients reads as follows.

Theorem 8 (Outer ratio asymptotics). *Let $\{V_n\}_{n\in\mathbb{N}}$ and $\{G_n\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures, W, satisfying the recurrence relations (4), (5), respectively. Let assume that there exists a sequence of positive definite matrices $(D_n)_{n\in\mathbb{N}}$ such that (10) holds with A and C nonsingular matrices, and consider the scaled matrix polynomials sequences $\{V_n^{D_n}\}$, $\{G_n^{D_n}\}$ given in Definition 3. We denote by Δ_n the set of zeros of $V_n^{D_n}$ and by $\Gamma = \bigcap_{N\geq 0} M_N$, where $M_N = \overline{\bigcup_{n\geq N}\Delta_n}$. Then,*

(a) If we assume that the matrix sequence $(D_n)_{n\in\mathbb{N}}$ is increasing, then Γ is a compact set.
(b) The following outer ratio asymptotics hold

$$\lim_{n\to\infty} D_n^{1/2}V_{n-1}^{D_n}(z)\left(V_n^{D_n}(z)\right)^{-1}A_{n-1}^{-1}D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t},\quad z\in\mathbb{C}\setminus\Gamma,$$
$$\lim_{n\to\infty} D_n^{1/2}C_n^{-1}\left(G_n^{D_n}(z)\right)^{-1}G_{n-1}^{D_n}(z)\,D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t},\quad z\in\mathbb{C}\setminus\Gamma,$$

where $W_{C,B,A}$ is the matrix weight for the generalized Chebyshev matrix polynomials of the second kind defined by (11). Moreover, the convergence is locally uniformly on compact subsets of $\mathbb{C}\setminus\Gamma$.

Proof. Let $(D_n)_{n\in\mathbb{N}}$ be a sequence of $N\times N$ positive definite matrices. In order to apply Theorem 6, we consider the scaled matrix polynomials $V_n^{D_k}(x)$, $G_n^{D_k}(x)$ associated with the parameters $(A_n)_{n\in\mathbb{N}}$, $(B_n)_{n\in\mathbb{N}}$, $(C_n)_{n\in\mathbb{N}}$. Taking into account their definitions, we have

$$x\,D_k\,V_n^{D_k}(x)=A_n\,V_{n+1}^{D_k}(x)+B_n\,V_n^{D_k}(x)+C_n\,V_{n-1}^{D_k}(x)\,, \tag{21}$$

$$G_m^{D_k}(x)\,D_k\,x=G_{m-1}^{D_k}(x)A_{m-1}+G_m^{D_k}(x)\,B_m+G_{m+1}^{D_k}(x)\,C_{m+1}\,, \tag{22}$$

and so

$$\begin{aligned} xD_k^{1/2}\,V_n^{D_k}(x)&=D_k^{-1/2}A_n\,D_k^{-1/2}D_k^{1/2}\,V_{n+1}^{D_k}(x)\\ &\quad+D_k^{-1/2}B_n\,D_k^{-1/2}D_k^{1/2}\,V_n^{D_k}(x)+D_k^{-1/2}C_n\,D_k^{-1/2}D_k^{1/2}\,V_{n-1}^{D_k}(x)\,,\end{aligned}$$

$$\begin{aligned} xG_n^{D_k}(x)\,D_k^{1/2}&=G_{n-1}^{D_k}(x)\,D_k^{1/2}D_k^{-1/2}A_{n-1}\,D_k^{-1/2}\\ &\quad+G_n^{D_k}(x)\,D_k^{1/2}D_k^{-1/2}B_n\,D_k^{-1/2}+G_{n+1}^{D_k}(x)\,D_k^{1/2}D_k^{-1/2}C_{n+1}\,D_k^{-1/2}\,,\end{aligned}$$

respectively. For each k, taking

$$\begin{cases} R_{n,k}(x)=D_k^{1/2}V_n^{D_k}(x)\,,\ S_{n,k}(x)=G_n^{D_k}(x)\,D_k^{1/2}\,,\\ A_{n,k}=D_k^{-1/2}A_n\,D_k^{-1/2}\,,\ B_{n,k}=D_k^{-1/2}B_n\,D_k^{-1/2}\,,\ C_{n,k}=D_k^{-1/2}C_n\,D_k^{-1/2}\,,\end{cases}$$

the matrix polynomial sequences $\{R_{n,k}\}_{n\in\mathbb{N}}$, $\{S_{n,k}\}_{n\in\mathbb{N}}$ satisfy the three-term recurrence relations (13), (14), respectively, with initial conditions $R_{0,k}(x)=G_{0,k}(x)=D_k^{1/2}$ and $R_{-1,k}(x)=G_{-1,k}(x)=\mathbf{0}_N$. Then, the matrix polynomial sequences $\{R_{n,k}\}_{n\in\mathbb{N}}$, $\{S_{n,k}\}_{n\in\mathbb{N}}$ are orthogonal with respect to a certain varying matrix of measures W_k. Under the assumptions (10) it is easy to see that the limit conditions (15) are satisfied for $n_m=k_m=m$. Then, Theorem 8 holds.

Finally, from Theorem 7 we have that if the matrix sequence $(D_n)_{n\in\mathbb{N}}$ is increasing, then the zeros of $V_n^{D_n}(x)$ and $G_n^{D_n}(x)$ are bounded (and so Γ is a compact set), as we wanted to prove. $\square$

Example 2. In Example 1 let us take the scalar measure $d\mu=x^\alpha e^{-x}dx$, $\alpha>-1$, supported on $(0,\infty)$ and let $\{L_n^\alpha\}_{n\in\mathbb{N}}$ be the corresponding sequence of monic orthogonal polynomial (Laguerre polynomials of parameter α). The biorthogonal sequences with respect to the new measure $\left(\begin{smallmatrix} x & -1\\ 0 & x\end{smallmatrix}\right)x^\alpha e^{-x}dx$ are

$$V_n(x)=\begin{pmatrix} L_n^{\alpha+1}(x) & \frac{L_n^{\alpha+1}(x)-\frac{\alpha+n+1}{\alpha+1}L_n^\alpha(x)}{x}\\ 0 & L_n^{\alpha+1}(x)\end{pmatrix}=\begin{pmatrix} L_n^{\alpha+1}(x) & -\frac{n}{\alpha+1}L_{n-1}^{\alpha+2}(x)\\ & L_n^{\alpha+1}(x)\end{pmatrix},$$

$$G_n(x)=-\begin{pmatrix} (-1)^n\frac{\Gamma(\alpha+1)K_n^\alpha(x,0)}{\Gamma(\alpha+n+1)} & 0\\ \frac{nK_n^\alpha(x,0)}{(\alpha+1)}+(-1)^n\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+n+1)}\frac{\partial K_n^\alpha(x,0)}{\partial y} & (-1)^n\frac{\Gamma(\alpha+1)K_n^\alpha(x,0)}{\Gamma(\alpha+n+1)}\end{pmatrix}.$$

On the other hand, the sequences $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ satisfy the three-term recurrence relations (4) and (5) with $A_n=I_N$,

$$B_n=\begin{pmatrix} 2n+\alpha+2 & -2/(\alpha+1)\\ 0 & 2n+\alpha+2\end{pmatrix},\quad C_n=\begin{pmatrix} n(\alpha+n+1) & 0\\ 0 & n(\alpha+n+1)\end{pmatrix}.$$

On the other hand, taking the sequence of positive definite matrices $(D_n)_{n\in\mathbb{N}}$, where $D_n = \begin{pmatrix} n^2 & 0 \\ 0 & n^2 \end{pmatrix}$, we find that $D_n^{1/2} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}$, $D_n^{-1/2} = \begin{pmatrix} 1/n & 0 \\ 0 & 1/n \end{pmatrix}$ and $D_n^{-1/2} D_{n-1}^{1/2} = I_N$. Thus,

$$D_n^{-1/2} B_n D_n^{-1/2} = \begin{pmatrix} (2n+\alpha+2)/n^2 & -2/((\alpha+1)n^2) \\ 0 & (2n+\alpha+2)/n^2 \end{pmatrix}, \text{ and so } D_n^{-1/2} B_n D_n^{-1/2} \to \mathbf{0}_N ,$$

$$D_n^{-1/2} C_n D_n^{-1/2} = \begin{pmatrix} \frac{\alpha+n+1}{n} & 0 \\ 0 & \frac{\alpha+n+1}{n} \end{pmatrix}, \text{ and so } D_n^{-1/2} C_n D_n^{-1/2} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From Theorem 8 we get

$$\lim_{n\to\infty} D_n^{1/2} V_{n-1}^{D_n}(z) \left(V_n^{D_n}(z)\right)^{-1} D_n^{1/2} = \int \frac{dW_{I,0,I}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

$$\lim_{n\to\infty} D_n^{1/2} C_n^{-1} \left(G_n^{D_n}(z)\right)^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{dW_{I,0,I}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

where $W_{I,0,I}$ is the matrix of measures associated with the sequence of orthogonal polynomials $\{U_n^{C,0,I}\}_{n\in\mathbb{N}}$ satisfying

$$x U_n^{I,0,I}(x) = U_{n+1}^{I,0,I}(x) + U_{n-1}^{I,0,I}(x), \quad n \in \mathbb{N},$$

with initial conditions $U_0^{I,0,I}(x) = I_N$, $U_{-1}^{I,0,I}(x) = \mathbf{0}_N$.

Notice that it is an interesting situation, because $W_{I,0,I}(x)$ is a positive definite matrix of measures. Moreover, using Corollary 2.3 in [8] we obtain

$$\int \frac{dW_{I,0,I}(t)}{z-t} = \frac{z\, I}{2} - \frac{\sqrt{(z^2-4)\, I}}{2}.$$

Example 3. Let $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ be matrix polynomial sequences satisfying the recurrence relations (4) and (5), respectively, with

$$A_n = \begin{pmatrix} 2n^2 & 0 \\ 7n^2+1 & 5n^2 \end{pmatrix}, \quad B_n = \begin{pmatrix} 3/n & 4/n \\ n & 8n^2 \end{pmatrix}, \quad C_n = \begin{pmatrix} 2n^2 & 2n \\ 0 & n \end{pmatrix}.$$

If we consider the sequence of positive definite matrices, $(D_n)_{n\in\mathbb{N}}$,

$$D_n = \frac{n^8}{(n^2-1)^2} \begin{pmatrix} 1/n^2 + 1/n^4 & -2/n^3 \\ -2/n^3 & 1/n^2 + 1/n^4 \end{pmatrix},$$

then $D_n^{1/2} = \frac{n^4}{n^2-1} \begin{pmatrix} 1/n & -1/n^2 \\ -1/n^2 & 1/n \end{pmatrix}$, $D_n^{-1/2} = \begin{pmatrix} 1/n & 1/n^2 \\ 1/n^2 & 1/n \end{pmatrix}$, $D_n^{-1/2} D_{n-1}^{1/2} \to I_N$. From here

$$D_n^{-1/2} A_n D_n^{-1/2} = \begin{pmatrix} 2 + 1/n^3 + 5/n^2 + 7/n & (7n^3+7n^2+1)/n^4 \\ 7 + 1/n^2 + 7/n & 5 + 1/n^3 + 2/n^2 + 7/n \end{pmatrix},$$

and so, $D_n^{-1/2} A_n D_n^{-1/2} \to \begin{pmatrix} 2 & 0 \\ 7 & 5 \end{pmatrix}$,

$$D_n^{-1/2} B_n D_n^{-1/2} = \begin{pmatrix} (9n^2+3n+4)/n^4 & (8n^3+5n+3)/n^4 \\ (9n^4+3n+4)/n^5 & (8n^5+n^3+4n+3)/n^5 \end{pmatrix},$$

$$D_n^{-1/2} B_n D_n^{-1/2} \to \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix},$$

$$D_n^{-1/2} C_n D_n^{-1/2} = \begin{pmatrix} 2+3/n^2 & 5/n \\ (3n^2+2)/n^3 & 1+4/n^2 \end{pmatrix}, \quad D_n^{-1/2} C_n D_n^{-1/2} \to \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Theorem 8 yields

$$\lim_{n\to\infty} D_n^{1/2} V_{n-1}^{D_n}(z) \left(V_n^{D_n}(z)\right)^{-1} A_{n-1}^{-1} D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

$$\lim_{n\to\infty} D_n^{1/2} C_n^{-1} \left(G_n^{D_n}(z)\right)^{-1} G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{dW_{C,B,A}(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma,$$

where $dW_{C,B,A}(x)$ is the matrix of measures of the sequence of biorthonormal polynomials satisfying for all $n \in \mathbb{N}$,

$$x U_n^{C,B,A}(x) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} U_{n+1}^{C,B,A}(x) + \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix} U_n^{C,B,A}(x) + \begin{pmatrix} 2 & 0 \\ 7 & 5 \end{pmatrix} U_{n-1}^{C,B,A}(x),$$

$$x T_n^{A,B,C}(x) = T_{n+1}^{A,B,C}(x) \begin{pmatrix} 2 & 0 \\ 7 & 5 \end{pmatrix} + T_n^{A,B,C}(x) \begin{pmatrix} 0 & 0 \\ 0 & 8 \end{pmatrix} + T_{n-1}^{A,B,C}(x) \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix},$$

with initial conditions $U_0^{C,B,A}(x) = T_0^{C,B,A}(x) = I_N$, $U_{-1}^{C,B,A}(x) = T_{-1}^{C,B,A}(x) = \mathbf{0}_N$.

5. The singular case

In this section, we study the case when the limit matrices A or C are singular. In [11], for the case of symmetric recurrence coefficients the authors proved that the outer ratio asymptotics also exists in the singular case, although they cannot compute explicitly the degenerate positive definite matrix of measures appearing in the limit. A similar argument can be applied for obtaining the existence of outer ratio asymptotics for matrix polynomials satisfying recurrence relations with nonsymmetric coefficients.

Theorem 9. *Let $\{V_n\}_{n\in\mathbb{N}}$, $\{G_n\}_{n\in\mathbb{N}}$ be the sequences of biorthogonal matrix polynomials with respect to a quasidefinite matrix of measures, W, satisfying the recurrence relations (4) and (5). Let us suppose that $(D_n)_{n\in\mathbb{N}}$ is an increasing sequence of matrices. Under the hypotheses of Theorem 8, if we assume the limit matrix A to be singular, then there exists a matrix of measures ν_1 for which*

$$\lim_{n\to\infty} D_n^{1/2} V_{n-1}^{D_n}(z) \left(V_n^{D_n}(z)\right)^{-1} A_{n-1}^{-1} D_n^{1/2} = \int \frac{d\,\nu_1(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma.$$

Moreover, if the matrix C is singular, then there exists a matrix of measures ν_2 such that

$$\lim_{n\to\infty} D_n^{1/2} C_n^{-1} \left(G_n^{D_n}(z)\right)^{-1}(z) G_{n-1}^{D_n}(z) D_n^{1/2} = \int \frac{d\,\nu_2(t)}{z-t}, \quad z \in \mathbb{C} \setminus \Gamma.$$

Moreover, if we write $F_A(z) = \int \frac{d\,\nu_1(t)}{z-t}$ and $F_C(z) = \int \frac{d\,\nu_2(t)}{z-t}$, with $z \notin \operatorname{supp}(\nu)$, then these analytic matrix functions satisfy the matrix equation

$$C\,F(z) A\,F(z) + (B - zI)\,F(z) + I = \mathbf{0}_N. \tag{23}$$

Proof. Following the techniques given in Section 4 of [11], it is enough to reduce the result to the case of varying recurrence coefficients and to use the matrix polynomials $x^l I_N$ instead of $U_n^{C,B,A}(x)$ and $T_n^{A,B,C}(x)$ in the proof of Theorem 6.

In order to prove that the Hilbert transforms of the measures ν_1 and ν_2 satisfy the matrix equation (23), let us remind that the polynomials $V_n^{D_k}(x)$ and $G_n^{D_k}(x)$ satisfy (21), (22), i.e.

$$xD_k^{1/2}V_n^{D_k}(x) = D_k^{-1/2}A_n\,V_{n+1}^{D_k}(x) + D_k^{-1/2}B_n\,V_n^{D_k}(x) + D_k^{-1/2}C_n\,V_{n-1}^{D_k}(x),$$
$$xG_n^{D_k}(x)D_k^{1/2} = G_{n-1}^{D_k}(x)A_{n-1}D_k^{-1/2} + G_n^{D_k}(x)B_nD_k^{-1/2} + G_{n+1}^{D_k}(x)C_{n+1}D_k^{-1/2}.$$

Let us multiply the right hand side of the first one by $\left(V_n^{D_n}\right)^{-1}(x)\,D_n^{-1/2}$, and the left hand side of the second one by $D_n^{-1/2}\left(G_n^{D_n}\right)^{-1}(x)$. Now, we put $k=n$ and take limit as n tends to infinity. Then, from Theorem 8

$$\lim_{n\to\infty} D_n^{-1/2}A_n\,V_{n+1}^{D_k}(x)\left(V_n^{D_n}(x)\right)^{-1}D_n^{-1/2} = F_A^{-1},$$
$$\lim_{n\to\infty} D_n^{-1/2}\left(G_n^{D_n}\right)^{-1}(x)\,G_{n+1}^{D_n}(x)\,C_{n+1}\,D_n^{-1/2} = F_C^{-1},$$

and the result follows. □

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