Asymptotic Properties of Laguerre-Sobolev type Orthogonal Polynomials

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Abstract

In this contribution we consider the asymptotic behavior of sequences of monic polynomials orthogonal with respect to a Sobolev-type inner product

$$\langle p,q\rangle_S=\int_0^\infty p(x)q(x)x^\alpha e^{-x}dx+Np'(a)q'(a),\ \ \alpha>-1$$

where $N\in\mathbb{R}_+$, and $a\in\mathbb{R}_-$.

We study the outer relative asymptotics of these polynomials with respect to the standard Laguerre polynomials. The analogue of the Mehler-Heine formula as well as a Plancherel-Rotach formula for the rescaled polynomials are given. The behavior of their zeros is also analyzed in terms of their dependence on N.

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1 Introduction

Let $\{\widehat{L}_n^\alpha(x)\}_{n\geq 0}$ be the sequence of Laguerre monic polynomials, orthogonal with respect to the inner product in the linear space $\mathbb{P}$ of polynomials of real coefficients

$$\langle p,q\rangle_\alpha=\int_0^\infty pqx^\alpha e^{-x}dx,\ \ \alpha>-1,\ \ p,q\in\mathbb{P}.$$

We introduce the Sobolev-type inner product

$$\langle p,q\rangle_S=\int_0^\infty pqx^\alpha e^{-x}dx+Np'(a)q'(a),\ \ N\in\mathbb{R}_+,\ \ a<0. \tag{1}$$

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In the sequel, we will denote by $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$ the sequence of monic polynomials orthogonal with respect to the above inner product. These polynomials are said to be Laguerre-Sobolev-type polynomials. It is worth pointing out that many properties of the standard Laguerre polynomials are lost when (1) is considered. In a more general framework, for measures supported on the interval $[0,\infty]$ the zeros can be complex or, if real, they can be located outside $[a,+\infty]$. Analytic properties of these families of orthogonal polynomials have been studied in [15], and some properties concerning the behavior of their zeros have been obtained in [14].

The case $a=0$ has been studied extensively in [8], where a representation of these polynomials in terms of standard Laguerre polynomials is given. The distribution of their zeros, their dependence on the mass N as well as their interlacing properties have been analyzed in [5] and [8]. Their asymptotic behavior and a Mehler-Heine-type formula are also deduced. In [9], the authors obtain a second order linear differential equation that they satisfy. An electrostatic interpretation of their zeros as equilibrium points with respect to a logarithmic potential under the action of an external field is given.

When the mass point is located outside the support of the measure ($a\in\mathbb{R}_-$), the study of their analytic properties has been treated for the standard inner product

$$\langle p,q\rangle=\int_0^{\infty} pqx^{\alpha}e^{-x}dx+Mp(a)q(a),\ \ M\in\mathbb{R}_+,\ \ a<0, \tag{2}$$

in [10], [7] and [11]. Indeed they constitute an extension of the so-called Laguerre-Krall orthogonal polynomials.

The aim of this paper is to bring together different types of asymptotic results for the Laguerre-Sobolev-type polynomials orthogonal with respect to (1), and establish some results about the behavior and monotonicity of their zeros.

In Section 2 we set up the notation and basic background concerning classical Laguerre orthogonal polynomials. In Section 3 we prove a very useful result concerning the rate of convergence of the ratio of two classical Laguerre polynomials of different parameter and degree. Section 4 is focused on the study of the outer relative asymptotics of the polynomials $\{\widehat{Q}_n^{\alpha}(x)\}_{n\geq 0}$ with respect to the Laguerre polynomials. A Mehler-Heine type formula is presented in Section 5. In Section 6 we study the Plancherel Rotach asymptotics of the scaled Laguerre-Sobolev-type orthogonal polynomials. Finally, in Section 7, the distribution and monotonicity of their zeros is analyzed in terms of their dependence on N.

2 Preliminaries

The Laguerre monic orthogonal polynomials are the monic polynomials orthogonal with respect to the inner product

$$\langle p,q\rangle_{\alpha}=\int_0^{\infty} pqx^{\alpha}e^{-x}dx,\ \alpha>-1,\ p,q\in\mathbb{P}.$$

The expression of these polynomials as an ${}_1F_1$ hypergeometric function is very well known in the literature (see [1], [12], [16], [18], among others) and, in fact, they constitute a family of classical orthogonal polynomials (see [13] and [16]). The connection between these two facts follows from a characterization of such orthogonal polynomials as eigenfunctions of a second order linear differential operator with polynomial coefficients. Nevertheless, we are interested in other structural properties of Laguerre polynomials which will be useful in the sequel (see [4] and [13]).

Proposition 1 *Let $\{L_n^{(\alpha)}(x)\}_{n\geq 0}$ be the sequence of Laguerre orthogonal polynomials with leading coeficients $\frac{(-1)^n}{n!}$, i.e.*

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}\widehat{L}_n^{\alpha}(x) \tag{3}$$

where $\widehat{L}_n^{\alpha}(x)$ denotes the monic Laguerre polynomial of degree n. Then the following statements hold.

1. *Three term recurrence relation.*

$$nL_n^{(\alpha)}(x) = (-x+2n+\alpha-1)L_{n-1}^{(\alpha)}(x) - (n+\alpha-1)L_{n-2}^{(\alpha)}(x), \quad n\geq 2, \tag{4}$$

   with $L_0^{(\alpha)}(x)=1$, $L_1^{(\alpha)}(x) = -x+\alpha+1$.

2. *Structure relation. For every $n\in\mathbb{N}$,*

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x). \tag{5}$$

3. *Norm. For every $n\in\mathbb{N}$,*

$$||L_n^{(\alpha)}||_\alpha^2 = \Gamma(\alpha+1)\binom{n+\alpha}{n}. \tag{6}$$

4. *Hahn condition (see [18], Formula 5.1.14). For every $n\in\mathbb{N}$,*

$$(L_n^{(\alpha)})'(x) = -L_{n-1}^{(\alpha+1)}(x) = x^{-1}\{nL_n^{(\alpha)}(x) - (n+\alpha)L_{n-1}^{(\alpha)}(x)\}. \tag{7}$$

5. *Outer strong asymptotics (Perron's asymptotics formula on $\mathbb{C}\setminus\mathbb{R}_+$). For $\alpha\in\mathbb{R}$ we have*

$$\begin{aligned} L_n^{(\alpha)}(x) &= \frac{1}{2}\pi^{-1/2}e^{x/2}(-x)^{-\alpha/2-1/4}n^{\alpha/2-1/4}e^{2(-nx)^{1/2}} \\ &\quad\times\left\{\sum_{k=0}^{p-1}C_k(\alpha;x)n^{-k/2} + \mathcal{O}(n^{-p/2})\right\}. \end{aligned} \tag{8}$$

   Here $C_k(\alpha;x)$ is independent of n. This relation holds for x in the complex plane with a cut along the positive real semiaxis, and it also holds if x is in the cut plane mentioned. $(-x)^{-\alpha/2-1/4}$ and $(-x)^{1/2}$ must be taken real and positive if $x<0$. The bound for the remainder holds uniformly in every compact subset of the complex plane with empty intersection with $\mathbb{R}_+$ (see [18], Theorem 8.22.3).

6. *Mehler-Heine type formula. Let $j\in N\cup\{0\}$ and J_α be the Bessel function of the first kind. Then*

$$\lim_{n\to\infty}\frac{L_n^{(\alpha)}(x/(n+j))}{n^\alpha} = x^{-\alpha/2}J_\alpha\left(2\sqrt{x}\right), \tag{9}$$

   uniformly on compact subsets of $\mathbb{C}$ (see [18], Theorem 8.1.3).

7. *Plancherel-Rotach type formula. Let $\varphi(x) = x+\sqrt{x^2-1}$ with $\sqrt{x^2-1}>0$ if $|x|>1$, be the conformal mapping of $\mathbb{C}\setminus[-1,1]$ onto the exterior of the unit circle. Then*

$$\lim_{n\to\infty}\frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} = \frac{-1}{\varphi((x-2)/2)}, \tag{10}$$

   uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$.

On the other hand, if

$$K_n(x,y)=\sum_{k=0}^{n}\frac{\widehat{L}_k^{\alpha}(x)\widehat{L}_k^{\alpha}(y)}{||\widehat{L}_k^{\alpha}||_{\alpha}^2} \tag{11}$$

denotes the n-th kernel polynomial associated with the Laguerre orthogonal polynomials and according to the Christoffel-Darboux formula, then, for every $n\in\mathbb{N}$, we derive

$$K_n(x,y)=\frac{\widehat{L}_{n+1}^{\alpha}(x)\widehat{L}_n^{\alpha}(y)-\widehat{L}_{n+1}^{\alpha}(y)\widehat{L}_n^{\alpha}(x)}{x-y}\frac{1}{||\widehat{L}_n^{\alpha}||_{\alpha}^2}.$$

In the sequel we will use the following notation

$$\frac{\partial^{k+j}K_n\left(x,y\right)}{\partial x^k\partial y^j}=K_n^{(k,j)}\left(x,y\right). \tag{12}$$

From the Christoffel-Darboux formula, an easy computation shows that (see [8])

$$K_{n-1}^{(0,j)}(x,y)=\frac{j!}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2(x-y)^{j+1}}\left(\widehat{L}_n^{\alpha}(x)Q_j(x,y;L_{n-1}^{\alpha})-\widehat{L}_{n-1}^{\alpha}(x)Q_j(x,y;L_n^{\alpha})\right), \tag{13}$$

where $Q_j\left(x,y,\widehat{L}_n^{\alpha}\right)$ denotes the Taylor polynomial of degree j of the polynomial $\widehat{L}_n^{\alpha}\left(x\right)$ around $x=y$.

Let q be a polynomial with $\deg q\leq n$. An interesting property of $K_n^{(0,j)}\left(x,y\right)$ is

$$\int_0^{\infty}K_n^{(0,j)}\left(x,y\right)q\left(x\right)x^{\alpha}e^{-x}dx=q^{(j)}\left(y\right). \tag{14}$$

Notice that for $j=0$, the well-known "reproducing property" of the n-th kernel polynomial

$$\int_0^{\infty}K_n\left(x,y\right)q\left(x\right)x^{\alpha}e^{-x}dx=q\left(y\right)$$

appears. We can also express the confluent form of (12) as

$$K_{n-1}^{(j,j)}\left(a,a\right)=\frac{\left(j!\right)^2}{(2j+1)!||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}\times$$

$$\sum_{k=0}^{j}\binom{2j+1}{k}\left[(\widehat{L}_{n-1}^{\alpha})^{(k)}\left(a\right)(\widehat{L}_n^{\alpha})^{(2j+1-k)}\left(a\right)-(\widehat{L}_n^{\alpha})^{(k)}\left(a\right)(\widehat{L}_{n-1}^{\alpha})^{(2j+1-k)}\left(a\right)\right].$$

3 Rate of Convergence

In this section, from the Perron's formula (8) we prove two lemmas concerning the rate of convergence of the quotient of two standard Laguerre polynomials as $n\to\infty$. Throughout this work, we will need to compute several ratios of Laguerre polynomials with different parameters and degrees, and that is where these Lemmas will be useful. Taking $p=3$ in (8) we have

$$\begin{aligned}L_n^{(\alpha)}\left(x\right) &= \frac{1}{2}\pi^{-1/2}e^{x/2}\left(-x\right)^{-\alpha/2-1/4}n^{\alpha/2-1/4}e^{2(-nx)^{1/2}}\\ &\times\left\{C_0(\alpha;x)+C_1(\alpha;x)n^{-1/2}+C_2(\alpha;x)n^{-1}+\mathcal{O}\left(n^{-3/2}\right)\right\},\end{aligned}$$

where every $C_\nu(\alpha;x)$ is independent of n, but depends on α. Thus,

$$\begin{aligned}\frac{L_{n+j}^{(\alpha)}(x)}{L_{n+k}^{(\beta)}(x)} &= (-x)^{-\alpha/2+\beta/2}\left[\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}}\right] e^{\left[2(-(n+j)x)^{1/2}-\left(2(-(n+k)x)^{1/2}\right)\right]} \\ &\quad\times\frac{C_0(\alpha;x)+C_1(\alpha;x)(n+j)^{-1/2}+C_2(\alpha;x)\,(n+j)^{-1}+\mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\beta;x)+C_1(\beta;x)(n+k)^{-1/2}+C_2(\beta;x)\,(n+k)^{-1}+\mathcal{O}\left((n+k)^{-3/2}\right)}.\end{aligned} \tag{15}$$

First, we study the term inside the square brackets. From $(n+j)^{\frac{\alpha}{2}-\frac{1}{4}} = n^{\frac{\alpha}{2}-\frac{1}{4}}\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}$ and $(n+k)^{\frac{\beta}{2}-\frac{1}{4}} = n^{\frac{\beta}{2}-\frac{1}{4}}\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}$, we obtain

$$\left[\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}}\right] = \frac{n^{\frac{\alpha}{2}-\frac{1}{4}}\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}}{n^{\frac{\beta}{2}-\frac{1}{4}}\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\frac{\left(1+\frac{j}{n}\right)^{\frac{\alpha}{2}-\frac{1}{4}}}{\left(1+\frac{k}{n}\right)^{\frac{\beta}{2}-\frac{1}{4}}}. \tag{16}$$

Next, using the expansion

$$(1+z)^y = 1+yz+\mathcal{O}\left(z^2\right),\quad |z|<1,$$

for both $y=\frac{\alpha}{2}-\frac{1}{4}$ with $z=\frac{j}{n}$ and $y=-\left(\frac{\beta}{2}-\frac{1}{4}\right)$ with $z=\frac{k}{n}$, we obtain

$$\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\left[1+\frac{j}{n}\left(\frac{\alpha}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right]\times\left[1-\frac{k}{n}\left(\frac{\beta}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

The product of the last two square brackets yields

$$1+\frac{j}{n}\left(\frac{\alpha}{2}-\frac{1}{4}\right)-\frac{k}{n}\left(\frac{\beta}{2}-\frac{1}{4}\right)+\mathcal{O}\left(\frac{1}{n^2}\right).$$

As a conclusion, (16) becomes

$$\frac{(n+j)^{\alpha/2-1/4}}{(n+k)^{\beta/2-1/4}} = n^{\frac{\alpha}{2}-\frac{\beta}{2}}\left(1+\left[j\left(\frac{\alpha}{2}-\frac{1}{4}\right)-k\left(\frac{\beta}{2}-\frac{1}{4}\right)\right]\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right). \tag{17}$$

On the other hand, in (15)

$$e^{\left[2(-(n+j)x)^{1/2}-\left(2(-(n+k)x)^{1/2}\right)\right]} = e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)}. \tag{18}$$

But

$$\begin{aligned}(n+j)^{1/2}-(n+k)^{1/2} &= n^{1/2}\left(\left(1+\frac{j}{n}\right)^{1/2}-\left(1+\frac{k}{n}\right)^{1/2}\right) \\ &= \frac{1}{2}\,(j-k)\,n^{-1/2}+\mathcal{O}(n^{-3/2}).\end{aligned}$$

Therefore, (18) becomes

$$e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)} = \exp\left[\frac{\sqrt{-x}}{\sqrt{n}}\,(j-k)+\mathcal{O}(n^{-3/2})\right],$$

and using

$$\exp z = 1+z+\frac{1}{2}z^2+\mathcal{O}\left(z^3\right),$$

we can write

$$e^{2\sqrt{-x}\left((n+j)^{1/2}-(n+k)^{1/2}\right)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}}\,(j-k) - \frac{x}{2n}\,(j-k)^2 + \mathcal{O}(n^{-3/2}). \tag{19}$$

Multiplying (17) and (19) we can rewrite (15) as

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_{n+k}^{(\beta)}(x)} = (-x)^{-\frac{\alpha}{2}+\frac{\beta}{2}} n^{\frac{\alpha}{2}-\frac{\beta}{2}} \times$$

$$\left(1 + \frac{\sqrt{-x}}{\sqrt{n}}\,(j-k) + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \left(\frac{\beta}{2}-\frac{1}{4}\right)k - \frac{x}{2}\,(j-k)^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \tag{20}$$
$$\times \frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)\,(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\beta;x) + C_1(\beta;x)(n+k)^{-1/2} + C_2(\beta;x)\,(n+k)^{-1} + \mathcal{O}\left((n+k)^{-3/2}\right)}.$$

Next we will state two useful lemmas considering some particular values of the parameters α, β, j and k.

Lemma 1 *Given two standard Laguerre polynomials of the same parameter α and different degree, the following statement holds. For $x \in \mathbb{C}\setminus\mathbb{R}_+$*

$$\frac{L_{n+j}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} = 1 + \frac{\sqrt{-x}}{\sqrt{n}} j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})$$

where $\sqrt{-x}$ must be taken real and positive if $x < 0$.

Proof. Letting $\alpha = \beta$ and $k = 0$ in (20) yields

$$\begin{aligned}\frac{L_{n+j}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} &= \left(1 + \frac{\sqrt{-x}}{\sqrt{n}} j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \\ &\quad \times \frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)\,(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2} + C_2(\alpha;x)n^{-1} + \mathcal{O}\left(n^{-3/2}\right)}.\end{aligned} \tag{21}$$

In the numerator of (21) we have

$$\begin{aligned}& C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)\,(n+j)^{-1} + \mathcal{O}((n+j)^{-3/2}) \\ =\ & C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2}\left(1+\frac{j}{n}\right)^{-1/2} + C_2(\alpha;x)n^{-1}\left(1+\frac{j}{n}\right)^{-1} + \mathcal{O}\left(n^{-3/2}\right)\end{aligned}$$

Thus

$$\frac{C_0(\alpha;x) + C_1(\alpha;x)(n+j)^{-1/2} + C_2(\alpha;x)\,(n+j)^{-1} + \mathcal{O}\left((n+j)^{-3/2}\right)}{C_0(\alpha;x) + C_1(\alpha;x)n^{-1/2} + C_2(\alpha;x)n^{-1} + \mathcal{O}\left(n^{-3/2}\right)} = 1 + \mathcal{O}(n^{-3/2}) \tag{22}$$

This shows that, under these conditions, there are no terms of order either $\mathcal{O}\left(n^{-1/2}\right)$ or $\mathcal{O}\left(n^{-1}\right)$ in the expansion (22). Thus, we can rewrite (21) as

$$\begin{aligned}\frac{L_{n+j}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} &= \left(1 + \frac{\sqrt{-x}}{\sqrt{n}} j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2})\right) \times \left(1 + \mathcal{O}\left(n^{-3/2}\right)\right) \\ &= 1 + \frac{\sqrt{-x}}{\sqrt{n}} j + \left[\left(\frac{\alpha}{2}-\frac{1}{4}\right)j - \frac{x}{2}j^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).\end{aligned}$$

■

Lemma 2 *Given two standard Laguerre polynomials of equal degree n and different parameter, the following statements hold. For* $x \in \mathbb{C}\setminus\mathbb{R}_+$

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+1)}(x)} = \frac{\sqrt{-x}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2}+\frac{1}{4}\right)+\frac{x}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}),$$
$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+2)}(x)} = \frac{-x}{n} + \mathcal{O}(n^{-3/2}),$$

where $\sqrt{-x}$ *must be taken real and positive if* $x < 0$.

Proof. Using (5) and proceeding by induction, it is easy to see that

$$L_n^{(\alpha)}(x) = \sum_{\nu=0}^{\ell}(-1)^{\nu}\binom{\ell}{\nu}L_{n-\nu}^{(\alpha+\ell)}(x), \quad \ell = 1, 2, \ldots$$

and therefore

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+\ell)}(x)} = \sum_{\nu=0}^{\ell}(-1)^{\nu}\binom{\ell}{\nu}\frac{L_{n-\nu}^{(\alpha+\ell)}(x)}{L_n^{(\alpha+\ell)}(x)}.$$

Next we can use Lemma 1 with parameter $\alpha+\ell$ and $j = -\nu$ in order to evaluate the last ratios as follows

$$\frac{L_{n-\nu}^{(\alpha+\ell)}(x)}{L_n^{(\alpha+\ell)}(x)} = 1 - \frac{\sqrt{-x}}{\sqrt{n}}\nu - \left[\left(\frac{\alpha+\ell}{2}-\frac{1}{4}\right)\nu + \frac{x}{2}\nu^2\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).$$

Hence

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+\ell)}(x)} = \tag{23}$$

$$\sum_{\nu=0}^{\ell}(-1)^{\nu}\binom{\ell}{\nu}\left(1 - \frac{\sqrt{-x}}{\sqrt{n}}\nu - \left[\left(\frac{\alpha+\ell}{2}-\frac{1}{4}\right)\nu + \frac{x}{2}\nu^2\right]\frac{1}{n}\right) + \mathcal{O}(n^{-3/2}).$$

Taking $\ell = 1$ in (23)

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+1)}(x)} = \frac{\sqrt{-x}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2}+\frac{1}{4}\right)+\frac{x}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).$$

If $\ell = 2$ we have

$$\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha+2)}(x)} = \frac{-x}{n} + \mathcal{O}(n^{-3/2}).$$

■

4 Relative Outer Asymptotics

In this section, we first express the Laguerre-Sobolev-type polynomials in terms of the standard Laguerre orthogonal polynomials $\widehat{L}_n^{\alpha}(x)$ and the Kernel polynomial (13). Next we find the relative asymptotic behavior of Laguerre-Sobolev-type orthogonal polynomials in the exterior of the positive real semi-axis. Taking into account the Fourier expansion

$$\widehat{Q}_n^{\alpha}(x) = \widehat{L}_n^{\alpha}(x) + \sum_{k=0}^{n-1} a_{n,k}\widehat{L}_k^{\alpha}(x),$$

we obtain, for $k=0,1,2,...,n-1$,

$$a_{n,k}=\frac{-N(\widehat{Q}_n^\alpha)'(a)(\widehat{L}_k^\alpha)'(a)}{||\widehat{L}_k^\alpha||_\alpha^2}.$$

Thus

$$\widehat{Q}_n^\alpha(x)=\widehat{L}_n^\alpha(x)-N(\widehat{Q}_n^\alpha)'(a)\sum_{k=0}^{n-1}\frac{(\widehat{L}_k^\alpha)'(a)\widehat{L}_k^\alpha(x)}{||\widehat{L}_k^\alpha||_\alpha^2}$$

and, from (12),

$$\widehat{Q}_n^\alpha(x)=\widehat{L}_n^\alpha(x)-N(\widehat{Q}_n^\alpha)'(a)K_{n-1}^{(0,1)}(x,a). \tag{24}$$

Our next step is to find $(\widehat{Q}_n^\alpha)'(a)$. In order to do that, we take the derivative in the former expression and evaluate it at $x=a$

$$(\widehat{Q}_n^\alpha)'(a)=(\widehat{L}_n^\alpha)'(a)-N(\widehat{Q}_n^\alpha)'(a)K_{n-1}^{(1,1)}(a,a).$$

Thus, using (7),

$$(\widehat{Q}_n^\alpha)'(a)=\frac{(\widehat{L}_n^\alpha)'(a)}{1+NK_{n-1}^{(1,1)}(a,a)},$$

so that

$$\widehat{Q}_n^\alpha(x)=\widehat{L}_n^\alpha(x)-N\frac{(\widehat{L}_n^\alpha)'(a)}{1+NK_{n-1}^{(1,1)}(a,a)}K_{n-1}^{(0,1)}(x,a). \tag{25}$$

Taking in account the normalization (3), (24) yields

$$Q_n^{(\alpha)}(x)=L_n^{(\alpha)}(x)+\frac{NL_{n-1}^{(\alpha+1)}(a)}{1+NK_{n-1}^{(1,1)}(a,a)}K_{n-1}^{(0,1)}(x,a) \tag{26}$$

where $Q_n^{(\alpha)}(x)=\frac{(-1)^n}{n!}\widehat{Q}_n^\alpha(x)$. From (13)

$$\begin{aligned}K_{n-1}^{(0,1)}(x,a) &= \frac{1}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}\left[\frac{\widehat{L}_n^\alpha(x)\widehat{L}_{n-1}^\alpha(a)-\widehat{L}_{n-1}^\alpha(x)\widehat{L}_n^\alpha(a)}{(x-a)^2}\right.\\ &\qquad\left.+\frac{\widehat{L}_n^\alpha(x)(\widehat{L}_{n-1}^\alpha)'(a)-\widehat{L}_{n-1}^\alpha(x)(\widehat{L}_n^\alpha)'(a)}{(x-a)}\right],\end{aligned} \tag{27}$$

and using the normalization (3), the above expression becomes

$$K_{n-1}^{(0,1)}(x,a)=\frac{n!(n-1)!}{||\widehat{L}_{n-1}^\alpha||_\alpha^2}L_{n-1}^{(\alpha)}(x)L_{n-2}^{(\alpha+1)}(a)\times \tag{28}$$

$$\left[\frac{1}{(x-a)^2}\left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}-1\right)\left(\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}-\frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}\right)+\frac{1}{(x-a)}\left(\frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)\right].$$

On the other hand, to compute $K_{n-1}^{(1,1)}(a,a)$ it suffices to consider the coefficient of $(x-a)$ in (27),

$$K_{n-1}^{(1,1)}(a,a)$$

$$= \frac{1}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}} \left[\frac{(\widehat{L}_{n}^{\alpha})'''(a)\widehat{L}_{n-1}^{\alpha}(a) - (\widehat{L}_{n-1}^{\alpha})'''(a)\widehat{L}_{n}^{\alpha}(a)}{3!} + \frac{(\widehat{L}_{n}^{\alpha})''(a)(\widehat{L}_{n-1}^{\alpha})'(a) - (\widehat{L}_{n-1}^{\alpha})''(a)(\widehat{L}_{n}^{\alpha})'(a)}{2!} \right].$$

Thus, taking into account the normalization (3) and after a cumbersome computation, we have

$$K_{n-1}^{(1,1)}(a,a)$$

$$= \frac{n!(n-1)!}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2}} \left[\frac{L_{n-4}^{(\alpha+3)}(a)L_{n-1}^{(\alpha)}(a)}{3!} \left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)} - \frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} \right) + \frac{L_{n-3}^{(\alpha+2)}(a)L_{n-2}^{(\alpha+1)}(a)}{2!} \left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right) \right]. \tag{29}$$

Next, substituting (28) and (29) in (26), we obtain

$$\frac{Q_{n}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} = 1+$$

$$\frac{N \frac{L_{n-1}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} \left(\frac{1}{(x-a)^{2}} \left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - 1 \right) \left(\frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} - \frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} \right) + \frac{1}{(x-a)} \left(\frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right) \right)}{\frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2} / n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)L_{n-2}^{(\alpha+1)}(a)} + N \left(\frac{1}{3!} \frac{L_{n-4}^{(\alpha+3)}(a)L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)L_{n-1}^{(\alpha+1)}(a)} \left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)} - \frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} \right) + \frac{1}{2!} \frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+1)}(a)} \left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right) \right)}.$$

Due to the presence $n!(n-1)!$ in the denominator, the expression $\frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2} / n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)L_{n-2}^{(\alpha+1)}(a)} \to 0$ much faster than $\mathcal{O}(n^{-3/2})$ when $n \to \infty$, and therefore we can remove it from the computations. Thus,

$$\frac{Q_{n}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} \sim 1 + \frac{L_{n-1}^{(\alpha)}(x)}{L_{n}^{(\alpha)}(x)} \times$$

$$\frac{\frac{1}{(x-a)^{2}} \left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - 1 \right) \left(\frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} - \frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} \right) + \frac{1}{(x-a)} \left(\frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right)}{\frac{1}{3!} \frac{L_{n-4}^{(\alpha+3)}(a)L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)L_{n-1}^{(\alpha+1)}(a)} \left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)} - \frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} \right) + \frac{1}{2!} \frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+1)}(a)} \left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right)}. \tag{30}$$

Next, taking into account Lemmas 1 and 2, we estimate the behavior of the numerator in (30). Taking into account

$$\begin{aligned}
\left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - 1 \right) &= \frac{\sqrt{|a|}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2} + \frac{1}{4} \right) + \frac{|a|}{2} \right] \frac{1}{n} + \mathcal{O}(n^{-3/2}), \\
\left(\frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} - \frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} \right) &= \frac{-\left(\sqrt{-x} - \sqrt{|a|} \right)}{\sqrt{n}} + \frac{x + |a|}{2n} + \mathcal{O}(n^{-3/2}), \\
\left(\frac{L_{n}^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)} - \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)} \right) &= \frac{\left(\sqrt{-x} - \sqrt{|a|} \right)}{\sqrt{n}} - \frac{x + |a| + 1}{2n} + \mathcal{O}(n^{-3/2}),
\end{aligned}$$

we conclude

$$\begin{aligned}
&\frac{1}{(x-a)^2}\left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}-1\right)\left(\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}-\frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}\right)+\frac{1}{(x-a)}\left(\frac{L_n^{(\alpha)}(x)}{L_{n-1}^{(\alpha)}(x)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)\\
=\ &\frac{1}{(x-a)}\frac{\sqrt{-x}-\sqrt{|a|}}{\sqrt{n}}+\mathcal{O}\left(n^{-1}\right). \qquad (31)
\end{aligned}$$

We next turn to estimating the denominator in (30). Using the Lemmas of Section 3 we deduce that

$$\begin{aligned}
\frac{L_{n-4}^{(\alpha+3)}(a)}{L_{n-2}^{(\alpha+1)}(a)} &= \frac{\frac{L_{n-4}^{(\alpha+3)}(a)}{L_{n-3}^{(\alpha+3)}(a)}\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-2}^{(\alpha+3)}(a)}}{\frac{L_{n-2}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+3)}(a)}} = \frac{1-\frac{2\sqrt{|a|}}{\sqrt{n}}-\left[\left(\frac{2\alpha}{2}+\frac{5}{2}\right)-\frac{4|a|}{2}\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})}{\frac{|a|}{n}+\mathcal{O}(n^{-3/2})}\\
&= \frac{n}{|a|}+\mathcal{O}\left(n^{1/2}\right)
\end{aligned}$$

and so

$$\begin{aligned}
\frac{L_{n-4}^{(\alpha+3)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-1}^{(\alpha+1)}(a)} &= \left(\frac{n}{|a|}+\mathcal{O}\left(n^{1/2}\right)\right)\left(\frac{\sqrt{|a|}}{\sqrt{n}}+\left[\left(\frac{\alpha}{2}+\frac{1}{4}\right)-\frac{|a|}{2}\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})\right)\\
&= \frac{\sqrt{n}}{\sqrt{|a|}}+\mathcal{O}(1).
\end{aligned}$$

We also need to consider

$$\begin{aligned}
\left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)}-\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}\right) &= \frac{3}{2n}+\mathcal{O}(n^{-3/2}),\\
\left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right) &= \frac{1}{2n}+\mathcal{O}(n^{-3/2})
\end{aligned}$$

and

$$\begin{aligned}
\frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+1)}(a)} &= \frac{\frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-2}^{(\alpha+2)}(a)}\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+2)}(a)}}{\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-1}^{(\alpha+2)}(a)}} = \frac{1-2\frac{\sqrt{|a|}}{\sqrt{n}}-\left[\left(\frac{2\alpha}{2}+\frac{3}{2}\right)-\frac{4|a|}{2}\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})}{\frac{\sqrt{|a|}}{\sqrt{n}}+\left[\left(\frac{\alpha}{2}+\frac{3}{4}\right)-\frac{|a|}{2}\right]\frac{1}{n}+\mathcal{O}(n^{-3/2})}\\
&= \frac{\sqrt{n}}{\sqrt{|a|}}-\frac{2\alpha+6\,|a|+3}{4\,|a|}+\mathcal{O}(n^{-1/2}) = \frac{\sqrt{n}}{\sqrt{|a|}}+\mathcal{O}(1).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\frac{1}{3!}\frac{L_{n-4}^{(\alpha+3)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-1}^{(\alpha+1)}(a)}\left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)}-\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}\right)+\frac{1}{2!}\frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+1)}(a)}\left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)\\
=\ &\left(\frac{\sqrt{n}}{\sqrt{|a|}}+\mathcal{O}(1)\right)\left(\frac{1}{3!}\left(\frac{3}{2n}+\mathcal{O}(n^{-3/2})\right)+\frac{1}{2!}\left(\frac{1}{2n}+\mathcal{O}(n^{-3/2})\right)\right)\\
=\ &\left(\frac{\sqrt{n}}{\sqrt{|a|}}+\mathcal{O}(1)\right)\left(\frac{1}{2n}+\mathcal{O}(n^{-3/2})\right)\\
=\ &\frac{1}{2\sqrt{n}}\,|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right). \qquad (32)
\end{aligned}$$

Now, combining (31) with (32), we conclude that (30) in $\mathbb{C}\diagdown\mathbb{R}_+$ behaves

$$\frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = 1 + \frac{L_{n-1}^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} \frac{\frac{\sqrt{-x}-\sqrt{|a|}}{(x-a)} + \mathcal{O}\left(n^{-1/2}\right)}{\frac{1}{2}|a|^{-1/2} + \mathcal{O}\left(n^{-1/2}\right)}.$$

Finally,

Theorem 1

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}(x)}{L_n^{(\alpha)}(x)} = \frac{\sqrt{-x}-\sqrt{|a|}}{\sqrt{-x}+\sqrt{|a|}},$$

uniformly on compact subsets of $\mathbb{C}\diagdown\mathbb{R}_+$.

5 Mehler-Heine type Formula

Concerning the Mehler-Heine formula, notice that we can rewrite (28) as

$$K_{n-1}^{(0,1)}(x,a) = \frac{n!\,(n-1)!L_{n-2}^{(\alpha+1)}(a)}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}\times$$

$$\left[\left(\frac{1}{(x-a)^2}\frac{L_n^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - \frac{1}{(x-a)}\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)L_{n-1}^{(\alpha)}(x) + \left(\frac{1}{(x-a)} - \frac{1}{(x-a)^2}\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)L_n^{(\alpha)}(x)\right]$$

and combining the above expression with (26) we have

$$Q_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) + \frac{NL_{n-1}^{(\alpha+1)}(a)}{1+NK_{n-1}^{(1,1)}(a,a)}\frac{n!\,(n-1)!L_{n-2}^{(\alpha+1)}(a)}{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2}\times$$

$$\left[\left(\frac{1}{(x-a)^2}\frac{L_n^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - \frac{1}{(x-a)}\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)L_{n-1}^{(\alpha)}(x) + \left(\frac{1}{(x-a)} - \frac{1}{(x-a)^2}\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)L_n^{(\alpha)}(x)\right].$$

Next, the change of variable $x \to x/n$, and division by n^{α} yield

$$\frac{Q_n^{(\alpha)}(x/n)}{n^{\alpha}} = \frac{L_n^{(\alpha)}(x/n)}{n^{\alpha}} + \frac{N}{\frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2/n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)L_{n-2}^{(\alpha+1)}(a)} + N\frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^2K_{n-1}^{(1,1)}(a,a)/n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)L_{n-2}^{(\alpha+1)}(a)}}\times$$

$$\left[\left(\frac{1}{(x/n-a)^2}\frac{L_n^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - \frac{1}{(x/n-a)}\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)\frac{L_{n-1}^{(\alpha)}(x/n)}{n^{\alpha}} + \left(\frac{1}{(x/n-a)} - \frac{1}{(x/n-a)^2}\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)\frac{L_n^{(\alpha)}(x/n)}{n^{\alpha}}\right]. \quad (33)$$

Notice that the denominator in the coefficient of the second term is exactly (32). So we have

$$\frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2} \diagup n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)\, L_{n-2}^{(\alpha+1)}(a)} + N \frac{||\widehat{L}_{n-1}^{\alpha}||_{\alpha}^{2} K_{n-1}^{(1,1)}(a,a) \diagup n!(n-1)!}{L_{n-1}^{(\alpha+1)}(a)\, L_{n-2}^{(\alpha+1)}(a)} \frac{1}{2\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right).$$

A trivial verification shows that

$$\begin{aligned}
\frac{L_{n}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} &= \frac{L_{n}^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)} \frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha)}(a)} \frac{L_{n-2}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}, \\
\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} &= \frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha)}(a)} \frac{L_{n-2}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}
\end{aligned}$$

where, using again Lemmas 1 and 2 we see that

$$\begin{aligned}
\frac{L_{n}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} &= \frac{\sqrt{|a|}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2}+\frac{1}{4}\right) + \frac{3\,|a|}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}), \\
\frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} &= \frac{\sqrt{|a|}}{\sqrt{n}} + \left[\left(\frac{\alpha}{2}+\frac{1}{4}\right) + \frac{|a|}{2}\right]\frac{1}{n} + \mathcal{O}(n^{-3/2}).
\end{aligned}$$

Therefore, the limit of the expressions in the square brackets in (33) is

$$\begin{aligned}
&\left(\frac{1}{(x/n-a)^2} \frac{L_{n}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)} - \frac{1}{(x/n-a)} \frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right) \frac{L_{n-1}^{(\alpha)}(x/n)}{n^{\alpha}} \\
= &\left(-\frac{1}{|a|} - \frac{|a|^{-3/2}(|a|-1)}{\sqrt{n}} + \mathcal{O}(n^{-1})\right) x^{-\alpha/2} J_{\alpha}(2\sqrt{x})
\end{aligned}$$

and

$$\begin{aligned}
&\left(\frac{1}{(x/n-a)} - \frac{1}{(x/n-a)^2} \frac{L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right) \frac{L_{n}^{(\alpha)}(x/n)}{n^{\alpha}} \\
= &\left(\frac{1}{|a|} - \frac{|a|^{-3/2}}{\sqrt{n}} + \mathcal{O}(n^{-1})\right) x^{-\alpha/2} J_{\alpha}(2\sqrt{x}).
\end{aligned}$$

In (33) we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{Q_{n}^{(\alpha)}(x/n)}{n^{\alpha}} &= \lim_{n\to\infty} \left(1 + \frac{N}{\frac{1}{2\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right)}\right) \times \\
&\left[\left(-\frac{1}{|a|} - \frac{|a|^{-3/2}(|a|-1)}{\sqrt{n}} + \mathcal{O}(n^{-1})\right) + \left(\frac{1}{|a|} - \frac{|a|^{-3/2}}{\sqrt{n}} + \mathcal{O}(n^{-1})\right)\right] x^{-\alpha/2} J_{\alpha}(2\sqrt{x}) \\
= \lim_{n\to\infty} &\left(1 + \frac{\frac{-1}{\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right)}{\frac{1}{2\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right)}\right) x^{-\alpha/2} J_{\alpha}\left(2\sqrt{x}\right).
\end{aligned}$$

Thus

$$\lim_{n\to\infty} \frac{\frac{-1}{\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right)}{\frac{1}{2\sqrt{n}} |a|^{-1/2} + \mathcal{O}\left(n^{-1}\right)} = -2.$$

As a conclusion,

Theorem 2
$$\lim_{n\to\infty}\frac{Q_n^{(\alpha)}(x/n)}{n^{\alpha}}=-x^{-\alpha/2}J_{\alpha}\left(2\sqrt{x}\right),$$
uniformly on every compact subset of the complex plane.

Notice that this Mehler-Heine type formula is the same as in the case of Laguerre-type orthogonal polynomials obtained in [10] and [7]. In [10] there is a misprint in the sign of Proposition 2.5 (b) and, thus, that Mehler-Heine type formula (Proposition 3.5 (b) in [10]) appears with the wrong sign.

6 Plancherel-Rotach type Formula

Our next goal is to determine the Plancherel-Rotach type formula of the Laguerre-Sobolev-type polynomials. Scaling the variable $x\to nx$, from (30) we know that

$$\frac{Q_n^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}=1+\frac{L_{n-1}^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}\times$$

$$\frac{\frac{1}{(nx-a)^2}\left(\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}-1\right)\left(\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}-\frac{L_n^{(\alpha)}(nx)}{L_{n-1}^{(\alpha)}(nx)}\right)+\frac{1}{(nx-a)}\left(\frac{L_n^{(\alpha)}(nx)}{L_{n-1}^{(\alpha)}(nx)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)}{\frac{1}{3!}\frac{L_{n-4}^{(\alpha+3)}(a)L_{n-1}^{(\alpha)}(a)}{L_{n-2}^{(\alpha+1)}(a)L_{n-1}^{(\alpha+1)}(a)}\left(\frac{L_{n-3}^{(\alpha+3)}(a)}{L_{n-4}^{(\alpha+3)}(a)}-\frac{L_n^{(\alpha)}(a)}{L_{n-1}^{(\alpha)}(a)}\right)+\frac{1}{2!}\frac{L_{n-3}^{(\alpha+2)}(a)}{L_{n-1}^{(\alpha+1)}(a)}\left(\frac{L_{n-2}^{(\alpha+2)}(a)}{L_{n-3}^{(\alpha+2)}(a)}-\frac{L_{n-1}^{(\alpha+1)}(a)}{L_{n-2}^{(\alpha+1)}(a)}\right)}.$$

According to (10), (32) and the Lemmas 1 and 2, the above expression, when $n\to\infty$ behaves like

$$\frac{Q_n^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}=1+\frac{\left(\frac{-1}{\varphi((x-2)/2)}\right)}{\frac{1}{2\sqrt{n}}|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right)}\times$$

$$\begin{aligned}&\left(\frac{1}{(nx-a)^2}\left(\frac{\sqrt{|a|}(1+\varphi\left((x-2)/2\right))}{\sqrt{n}}+\mathcal{O}(n^{-1})\right)\right.\\
&\qquad\left.-\frac{1}{(nx-a)}\left(\varphi\left((x-2)/2\right)+1+\frac{\sqrt{|a|}}{\sqrt{n}}+\mathcal{O}(n^{-1})\right)\right)\\
=\;&1+\frac{\left(\frac{-1}{\varphi((x-2)/2)}\right)}{\frac{1}{2\sqrt{n}}|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right)}\times\left(\mathcal{O}(n^{-5/2})-\frac{1}{(nx-a)}\left(1+\varphi\left((x-2)/2\right)+\mathcal{O}(n^{-1/2})\right)\right)\\
=\;&1+\frac{\frac{1}{(nx-a)}\left(\frac{1+\varphi((x-2)/2)}{\varphi((x-2)/2)}\right)+\mathcal{O}(n^{-3/2})}{\frac{1}{2\sqrt{n}}|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right)}.\end{aligned}$$

After some computations we obtain

$$\frac{Q_n^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)}=1+\frac{\frac{1}{(nx-a)}\left(\frac{1+\varphi((x-2)/2)}{\varphi((x-2)/2)}\right)+\mathcal{O}(n^{-3/2})}{\frac{1}{2\sqrt{n}}|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right)},$$

and a straightforward computation gives

$$\lim_{n\to\infty} \frac{\frac{1}{(nx-a)}\left(\frac{1+\varphi((x-2)/2)}{\varphi((x-2)/2)}\right)+\mathcal{O}(n^{-3/2})}{\frac{1}{2\sqrt{n}}\,|a|^{-1/2}+\mathcal{O}\left(n^{-1}\right)} = 0.$$

Therefore,

Theorem 3

$$\lim_{n\to\infty} \frac{Q_n^{(\alpha)}(nx)}{L_n^{(\alpha)}(nx)} = 1$$

holds uniformly on compact subsets of $\mathbb{C}\setminus[0,4]$.

7 The Zeros

Next, we will analyze the behavior of the zeros of a polynomial $f(x) = h_n(x) + cg_n(x)$, that is a linear combination of two polynomials of the same degree. Our goal is to obtain results concerning the monotonicity and speed of convergence of the zeros of $Q_n^{(\alpha)}(x)$. For this purpose we need the following lemma concerning the behavior and the asymptotics of the zeros of linear combinations of two polynomials with interlacing zeros, whose proof we omit (see [2, Lemma 1] or [6, Lemma 3])

Lemma 3 *Let* $h_n(x) = a(x-x_1)\cdots(x-x_n)$ *and* $g_n(x) = b(x-y_1)\cdots(x-y_n)$ *be polynomials with real and simple zeros, where* a *and* b *are positive real constants.*

If

$$y_1 < x_1 < \cdots < y_n < x_n,$$

then, for any real constant $c > 0$, *the polynomial*

$$f(x) = h_n(x) + cg_n(x)$$

has n *real zeros* $\eta_1 < \cdots < \eta_n$ *which interlace with the zeros of* $h_n(x)$ *and* $g_n(x)$ *as follows*

$$y_1 < \eta_1 < x_1 < \cdots < y_n < \eta_n < x_n.$$

Moreover, each $\eta_k = \eta_k(c)$ *is a decreasing function of* c *and, for each* $k = 1, \ldots, n$,

$$\lim_{c\to\infty} \eta_k = y_k \quad \text{and} \quad \lim_{c\to\infty} c[\eta_k - y_k] = \frac{-h_n(y_k)}{g_n'(y_k)}.$$

Let us introduce the nth-degree monic polynomial,

$$\begin{aligned}\widehat{G}_{n,a}^{\alpha}(x) &= \lim_{N\to\infty}\left(\widehat{L}_n^{\alpha}(x) - N\frac{(\widehat{L}_n^{\alpha})'(a)}{1+NK_{n-1}^{(1,1)}(a,a)}K_{n-1}^{(0,1)}(x,a)\right) \\ &= \widehat{L}_n^{\alpha}(x) - \frac{(\widehat{L}_n^{\alpha})'(a)}{K_{n-1}^{(1,1)}(a,a)}K_{n-1}^{(0,1)}(x,a), \qquad (34)\end{aligned}$$

To characterize these polynomials, first we will observe that they are quasi-orthogonal of order 2 (see [3, Definition 1]) with respect to the measure

$$d\mu^{**} = (x-a)^2 x^{\alpha} e^{-x} dx = (x-a)^2 d\mu_{\alpha},$$

which is an iterated Christoffel perturbation of the Laguerre weight, as in ([11, Subsection 2.2]). It means that $\widehat{G}_{n,a}^{\alpha}(x)$ is a linear combination of three consecutive polynomials of the sequence $\{\widehat{R}_n\}_{n\geq 0}$ of monic polynomials orthogonal with respect to μ^{**}. Indeed, for $n \geq 2$,

$$\widehat{G}_{n,a}^{\alpha}(x) = \widehat{R}_n(x) + b_n\widehat{R}_{n-1}(x) + c_n\widehat{R}_{n-2}(x),$$

where b_n and $c_n \neq 0$ are real numbers. Since

$$c_n = \frac{\int_0^{\infty} \widehat{G}_{n,a}^{\alpha}(x)\widehat{R}_{n-2}(x)d\mu^{**}}{||\widehat{R}_{n-2}||^2_{\mu^{**}}}$$

we have

Proposition 2 *If a is a real negative number, then c_n is positive for every $n \geq 2$.*

Proof. We only need to study the sign of the numerator. According to (34)

$$\begin{aligned}\int_0^{\infty} \widehat{G}_{n,a}^{\alpha}(x)\widehat{R}_{n-2}(x)d\mu^{**} &= \int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha} \\ &\quad - \frac{(\widehat{L}_n^{\alpha})'(a)}{K_{n-1}^{(1,1)}(a,a)} \int_0^{\infty} K_{n-1}^{(0,1)}(x,a)\widehat{R}_{n-2}(x)d\mu^{**} \\ &= ||\widehat{L}_n^{\alpha}||^2_{\alpha} - \frac{(\widehat{L}_n^{\alpha})'(a)}{K_{n-1}^{(1,1)}(a,a)} \int_0^{\infty} K_{n-1}^{(0,1)}(x,a)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha}.\end{aligned}$$

From (11) and (12) it follows immediately that

$$K_{n-1}^{(0,1)}(x,a) = K_n^{(0,1)}(x,a) - \frac{\widehat{L}_n^{\alpha}(x)(\widehat{L}_n^{\alpha})'(a)}{||\widehat{L}_n^{\alpha}||^2_{\alpha}},$$

hence

$$\begin{aligned}\int_0^{\infty} K_{n-1}^{(0,1)}(x,a)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha} &= \int_0^{\infty} K_n^{(0,1)}(x,a)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha} \\ &\quad - \frac{(\widehat{L}_n^{\alpha})'(a)}{||\widehat{L}_n^{\alpha}||^2_{\alpha}} \int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha}.\end{aligned}$$

The second integral on the right-hand side is

$$\int_0^{\infty} \widehat{L}_n^{\alpha}(x)\widehat{R}_{n-2}(x)(x-a)^2 d\mu_{\alpha} = ||\widehat{L}_n^{\alpha}||^2_{\alpha},$$

while the first one vanishes because $\deg K_n^{(0,1)}(x,a) = \deg \widehat{R}_{n-2}(x)(x-a)^2 = n$, and therefore we can apply the property (14)

$$\int_0^{\infty} K_n^{(0,1)}(x,a)\, q(x)\, d\mu_{\alpha} = q'(a).$$

If we denote $q(x) = \widehat{R}_{n-2}(x)(x-a)^2$, then $q'(x) = [\widehat{R}_{n-2}(x)]'(x-a)^2 + 2(x-a)\widehat{R}_{n-2}(x)$ and consequently $q'(a) = 0$. Therefore

$$\int_0^{\infty} \widehat{G}_{n,a}^{\alpha}(x)\widehat{R}_{n-2}(x)d\mu^{**} = ||\widehat{L}_n^{\alpha}(x)||^2_{\alpha} + \frac{\left((\widehat{L}_n^{\alpha})'(a)\right)^2}{K_{n-1}^{(1,1)}(a,a)}.$$

Thus,

$$c_n = \frac{\int_0^\infty \widehat{G}^\alpha_{n,a}(x)\widehat{R}_{n-2}(x)d\mu^{**}}{||\widehat{R}_{n-2}||^2_{\mu^{**}}} > 0, \ \ \forall n \geq 2.$$

■

On the other hand, let $\{\eta_{n,k}\} \equiv \eta_{n,1} < \eta_{n,2} < ... < \eta_{n,n}$ be the zeros of $\widehat{Q}^\alpha_n(x)$ and $\{x_{n,k}\} \equiv x_{n,1} < x_{n,2} < ... < x_{n,n}$ be the zeros of $\widehat{L}^\alpha_n(x)$, with $1 \leq k \leq n$. Notice that these zeros are real and simple (see [14], Proposition 3.2). Thus

Proposition 3 *([14], Proposition 6.2) The polynomial $\widehat{G}^\alpha_{n,a}(x)$ has n real and simple zeros $y_{n,1} < y_{n,2} < ... < y_{n,n}$. The inequalities*

$$y_{n,1} < a < x_{n,1} < y_{n,2} < x_{n,2} < \cdots < y_{n,n} < x_{n,n} \tag{35}$$

hold for every $n \geq 2, \ \ n \in \mathbb{N}$.

Notice that $\widehat{Q}^\alpha_1(x) = \widehat{L}^\alpha_1(x)$.

Next, we normalize the connection formula (25) in a more useful way, in order to apply Lemma 3 and obtain some results concerning monotonicity, asymptotics, and speed of convergence for the zeros in terms of the mass N.

Proposition 4 *The polynomials $\{\tilde{Q}^\alpha_n(x)\}_{n\geq 0}$, with $\tilde{Q}^\alpha_n(x) = \lambda_{n-1}\widehat{Q}^\alpha_n(x)$, can be represented as*

$$\tilde{Q}^\alpha_n(x) = \widehat{L}^\alpha_n(x) + NK^{(1,1)}_{n-1}(a,a)\,\widehat{G}^\alpha_{n,a}(x)\,, \tag{36}$$

where

$$\lambda_{n-1} = 1 + NK^{(1,1)}_{n-1}(a,a)\,.$$

Proof. Replacing (34) in (36)

$$\begin{aligned}
\lambda_{n-1}\widehat{Q}^\alpha_n(x) &= \widehat{L}^\alpha_n(x) + NK^{(1,1)}_{n-1}(a,a)\left(\widehat{L}^\alpha_n(x) - \frac{(\widehat{L}^\alpha_n)'(a)}{K^{(1,1)}_{n-1}(a,a)}K^{(0,1)}_{n-1}(x,a)\right), \\
\left(1 + NK^{(1,1)}_{n-1}(a,a)\right)\widehat{Q}^\alpha_n(x) &= \left(1 + NK^{(1,1)}_{n-1}(a,a)\right)\widehat{L}^\alpha_n(x) - N(\widehat{L}^\alpha_n)'(a)K^{(0,1)}_{n-1}(x,a), \\
\widehat{Q}^\alpha_n(x) &= \widehat{L}^\alpha_n(x) - N\frac{(\widehat{L}^\alpha_n)'(a)}{1 + NK^{(1,1)}_{n-1}(a,a)}K^{(0,1)}_{n-1}(x,a),
\end{aligned}$$

which is the connection formula (25). ■

We point out the fact the Laguerre-Sobolev-type polynomial $\widehat{Q}^\alpha_n(x)$ appears as a linear combination of two polynomials of degree n. Thus, from (36), (35), and Lemma 3, we immediately conclude

Theorem 4 *If $a \in \mathbb{R}_-$, then*

$$y_{n,1} < \eta_{n,1} < x_{n,1} < y_{n,2} < \eta_{n,2} < x_{n,2} < \cdots < y_{n,n} < \eta_{n,n} < x_{n,n}.$$

Moreover, each $\eta_{n,k}$ is a decreasing function of N and, for each $k = 1, \ldots, n$

$$\lim_{N\to\infty} \eta_{n,k} = y_{n,k},$$

as well as

$$\lim_{N\to\infty} N[\eta_{n,k} - y_{n,k}] = \frac{-\widehat{L}^\alpha_n(y_{n,k})}{[\widehat{G}^\alpha_{n,a}(x)]'_{x=y_{n,k}}}.$$

Notice that the mass point a does not attract any zero of $\widehat{Q}_n^{\alpha}(x)$ when $N \to \infty$, as in the standard case. By standard we mean the case of the polynomials orthogonal with respect to the inner product (2) (see [11]).

7.1 The Minimum Mass

When $a \in \mathbb{R}_-$, at most one of the zeros of $\widehat{Q}_n^{\alpha}(x)$ is located outside $[0, +\infty)$. Next we provide the explicit value N_0 of the mass such that for $N > N_0$ this situation appears, i.e, one of the zeros is located outside $[0, +\infty)$.

Corollary 1 *If $a \in \mathbb{R}_-$, then the smallest zero $\eta_{n,1} = \eta_{n,1}(a)$ satisfies*

$$\begin{aligned} \eta_{n,1} &> 0, \quad \text{for} \quad N < N_0, \\ \eta_{n,1} &= 0, \quad \text{for} \quad N = N_0, \\ \eta_{n,1} &< 0, \quad \text{for} \quad N > N_0, \end{aligned}$$

where

$$N_0 = N_0(n, \alpha, a) = \left(\frac{(\widehat{L}_n^{\alpha})'(a)}{\widehat{L}_n^{\alpha}(0)} K_{n-1}^{(0,1)}(0, a) - K_{n-1}^{(1,1)}(a, a) \right)^{-1} > 0. \tag{37}$$

Proof. It suffices to use (25) together with the fact that $Q_n^{\alpha}(0) = 0$ if and only if $N = N_0$

$$\begin{aligned} \widehat{Q}_n^{\alpha}(0) &= \widehat{L}_n^{\alpha}(0) - N_0 \frac{(\widehat{L}_n^{\alpha})'(a)}{1 + N_0 K_{n-1}^{(1,1)}(a, a)} K_{n-1}^{(0,1)}(0, a) = 0, \\ \widehat{L}_n^{\alpha}(0) &= N_0 \frac{(\widehat{L}_n^{\alpha})'(a)}{1 + N_0 K_{n-1}^{(1,1)}(a, a)} K_{n-1}^{(0,1)}(0, a), \\ N_0 &= N_0(n, \alpha, a) = \left(\frac{(\widehat{L}_n^{\alpha})'(a)}{\widehat{L}_n^{\alpha}(0)} K_{n-1}^{(0,1)}(0, a) - K_{n-1}^{(1,1)}(a, a) \right)^{-1}. \end{aligned}$$

■

Notice that, according to the Hurwitz theorem, for n large enough, only one zero of $\widehat{Q}_n^{\alpha}$ is located outside of $[0, +\infty)$ and it is attracted by a.

Next we show some numerical experiments using Mathematica software, dealing with the least zero of Laguerre-Sobolev-type polynomials. We are interested to show the location and behavior of this least zero. In the first two tables we recover the results in [17] when the mass point is located at $x = 0$, for $n = 2,\ 3$ and $\alpha = -1/2,\ 1,\ 5$. (Notice that in this work the authors label the zeros in a reverse order).

N	$\eta_{2,1}(-1/2)$	N	$\eta_{2,1}(1)$	N	$\eta_{2,1}(5)$
$1/2$	0.115964	$3/2$	0.271499	710	0.0419159
$N_0 = \sqrt{\pi}/2$	0	$N_0 = 2$	0	$N_0 = 720$	0
1	−0.0313955	$5/2$	−0.230139	730	−0.0414199

N	$\eta_{3,1}(-1/2)$	N	$\eta_{3,1}(1)$	N	$\eta_{3,1}(5)$
$1/4$	0.00211646	$1/5$	0.407703	79	0.0251697
$N_0 = \sqrt{\pi}/7$	0	$N_0 = 2/5$	0	$N_0 = 80$	0
$1/2$	−0.133233	$3/2$	−0.275762	81	−0.0248324

In the next two tables, we show the position for the first and second zeros of Laguerre-Sobolev-type polynomial of degree $n = 15$ and $\alpha = 0$, for some choices of the mass N. For $N = 0$ obviously we recover the least zero and the second zero of the classical Laguerre polynomials (in bold). When the mass point is located at $a = 0$ we get

$\eta_{15,k}$	$N = 0$	$N = 5.0 \cdot 10^{-12}$	$N = 5.0 \cdot 10^{-8}$	$N = 5.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	0.0933078	0.0933046	0.0620821	-0.146205
$k = 2$	**0.492692**	0.492692	0.492682	0.417657	0.263754

as well as when the mass point is located at $a = -1$.

$\eta_{15,k}$	$N = 0$	$N = 5.0 \cdot 10^{-12}$	$N = 5.0 \cdot 10^{-8}$	$N = 5.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.0933078**	0.0933076	0.0915341	-1.35377	-1.36544
$k = 2$	**0.492692**	0.492691	0.485200	0.148587	0.148434

In the next two tables we provide numerical evidences in support of Corollary 1, where the exact values of N_0 are calculated for two specific cases. For this purpose we begin by analyzing the smallest zero of the the Laguerre-Sobolev-type polynomials of degree $n = 7$, $\alpha = 2$ and with the mass point located at $a = -2$. Calculations show that for the values of N_0 given by (37), we have $N_0 = 3.21582 \cdot 10^{-4} \in (3.0 \cdot 10^{-4}, 4.0 \cdot 10^{-4})$.

$\eta_{7,k}$:	$N = 0$	$N = 5.0 \cdot 10^{-5}$	$N = 3.0 \cdot 10^{-4}$	$N = 4.0 \cdot 10^{-4}$	$N = 5.0 \cdot 10^{-3}$
$k = 1$	**0.783096**	0.705892	0.0636699	$-$**0.775950**	-2.70450

The table below shows that, with the mass point located at $a = -1$, we need larger values of N_0 to get the least zero as a negative real number. Now the estimate is $1.0 \cdot 10^{-3} < N_0 < 2 \cdot 10^{-3}$, according to the exact value $N_0(7, 2, -1) = 1.88442 \cdot 10^{-3}$

$\eta_{7,k}$:	$N = 0$	$N = 5.0 \cdot 10^{-4}$	$N = 1.0 \cdot 10^{-3}$	$N = 2.0 \cdot 10^{-3}$	$N = 5.0 \cdot 10^{-2}$
$k = 1$	**0.783096**	0.603763	0.384610	$-$**0.0452617**	-1.81059

Finally, another interesting question is to study, for a fixed value N, the behavior of zeros of Laguerre–Sobolev-type polynomials in terms of the parameter α. Notice that, for a fixed value of α we can loose its negative zero, as it occurs in the standard case (see [7]). We show the behavior of the first two zeros to give more information about their relative spacing. For instance, let us show the first two zeros of the Laguerre-Sobolev-type polynomials of degree $n = 12$, when $N = 1.5 \cdot 10^{-7}$ and the mass point is located at $a = -3$

$\eta_{12,k}$:	$\alpha = -1/2$	$\alpha = 0$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
$k = 1$	-2.81937	-2.52014	-0.0397219	**0.625246**	**1.29029**
$k = 2$	0.0716143	0.164964	0.855437	1.54668	2.57453

and again, the first two zeros when $N = 5.0 \cdot 10^{-5}$ and $a = -1$

$\eta_{12,k}$:	$\alpha = -1/2$	$\alpha = 0$	$\alpha = 2$	$\alpha = 3$	$\alpha = 5$
$k = 1$	-0.16977	-0.185167	**0.242738**	**0.600667**	**1.27787**
$k = 2$	0.137987	0.272018	1.0244	1.53932	2.55799

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