Zeros of Orthogonal Polynomials Generated by Canonical Perturbations of Measures

Edmundo J. Huertas [a,1] Francisco Marcellán [a,1]
Fernando R. Rafaeli [b,1]

[a] *Departamento de Matemáticas, Escuela Politécnica Superior, Universidad Carlos III, Leganés-Madrid, Spain*

[b] *Departamento de Matemática, Estatística e Computação, FCT, Universidade Estadual Paulista - Unesp, Brazil*

Abstract

In this paper we analyze the behaviour of the zeros of polynomials orthogonal with respect to the Uvarov perturbation of a positive Borel measure $d\mu(x)$. When the measure is semiclassical, then its electrostatic interpretation is given.

Key words: Orthogonal polynomials, interlacing, monotonicity, asymptotic behavior, electrostatics interpretation

1 Introduction and statement of the main results

This paper deals with the behavior of the zeros of the sequence of monic polynomials $\{p_n(\lambda, c; x)\}_{n\geq 0}$ orthogonal with respect to the Uvarov perturbation $d\mu(\lambda, c; x) = d\mu(x) + \lambda\delta(x - c)$, where $d\mu(x)$ is a positive Borel measure supported in a finite or infinite interval (a, b), $\delta(x-c)$ is the Dirac delta functional at c, with $c \notin (a, b)$, and λ is a nonnegative real number. In other words this means that this sequence of polynomials is orthogonal with respect to the

[1] This work has been done in the framework of a joint project of Dirección General de Investigación, Ministerio de Educación y Ciencia of Spain and the Brazilian Science Foundation CAPES, Project CAPES/DGU 160/08. The work of the first and second authors has been supported by Dirección General de Investigación, Ministerio de Ciencia e Innovación of Spain, grant MTM2009-12740-C03-01. The work of the third author has been supported by CNPq, FAPESP and FUNDUNESP.

inner product

$$\langle p, q \rangle_{\lambda,c} = \int_a^b p(x)q(x)d\mu(x) + \lambda p(c)q(c). \tag{1}$$

Let $x_{n,k}(\lambda, c)$, $k = 1, \ldots, n$, be the zeros of $p_n(\lambda, c; x)$. If λ is a nonnegative real number then $d\mu(\lambda, c; x)$ is a positive measure, and, as a consequence, the zeros of $p_n(\lambda, c; x)$ are real, simple, and lie in (c, b) (resp. in (a, c)) if $c \leq a$ (resp. if $c \geq b$), that is,

$$\min\{a, c\} < x_{n,1}(\lambda, c) < \cdots < x_{n,n}(\lambda, c) < \max\{b, c\}.$$

In particular, if $\lambda = 0$ we denote by $x_{n,k} := x_{n,k}(0, c)$ the zeros of $p_n(x) := p_n(0, c; x)$, the sequence of monic polynomials orthogonal with respect to the measure $d\mu(x)$.

Now, some natural questions arise: Are there values of the parameter λ for which the zeros of $p_n(\lambda, c; x)$ interlace with the zeros of $p_n(x)$?. Are the zeros $x_{n,k}(\lambda, c)$ monotonic functions with respect to the parameter λ?. Do the zeros $x_{n,k}(\lambda, c)$ converge when λ goes to infinity?. If so, what the speed of convergence is?.

One of our main contributions regards the questions posed above. We provide an interlacing property as well as the monotonicity and asymptotic behaviour of the zeros of the polynomial $p_n(\lambda, c; x)$ with respect to λ.

Theorem 1 *Let $\lambda > 0$ and $z_{n,1}(c), ..., z_{n,n}(c)$ be the zeros of the polynomial $r_n(c; x)$ defined below.*

(i) If $c \leq a$, then

$$c < x_{n,1}(\lambda, c) < x_{n,1} < z_{n-1,1}(c) < x_{n,2}(\lambda, c) < x_{n,2} < \cdots$$
$$\cdots < z_{n-1,n-1}(c) < x_{n,n}(\lambda, c) < x_{n,n}.$$

Moreover, each $x_{n,k}(\lambda, c)$ is a decreasing function of λ and, for each $k = 1, \ldots, n-1$,

$$\lim_{\lambda\to\infty} x_{n,1}(\lambda, c) = c, \quad \lim_{\lambda\to\infty} x_{n,k+1}(\lambda, c) = z_{n-1,k}(c),$$

as well as

$$\lim_{\lambda\to\infty} \lambda[x_{n,1}(\lambda, c) - c] = \frac{-p_n(c)}{K_{n-1}(c, c)r_{n-1}(c; c)},$$
$$\lim_{\lambda\to\infty} \lambda[x_{n,k+1}(\lambda) - z_{n-1,k}(c)] = \frac{-p_n(z_{n-1,k}(c))}{K_{n-1}(c, c)(z_{n-1,k}(c) - c)[r_{n-1}(c; x)]'_{x=z_{n-1,k}(c)}}. \tag{2}$$

(*ii*) *If* $c \geq b$, *then*

$$x_{n,1} < x_{n,1}(\lambda, c) < z_{n-1,1}(c) < \cdots$$

$$\cdots < x_{n,n-1} < x_{n,n-1}(\lambda, c) < z_{n-1,n-1}(c) < x_{n,n} < x_{n,n}(\lambda, c) < c.$$

Moreover, each $x_{n,k}(\lambda, c)$ *is an increasing function of* λ *and, for each* $k = 1, \dots, n-1$,

$$\lim_{\lambda\to\infty} x_{n,n}(\lambda, c) = c, \quad \lim_{\lambda\to\infty} x_{n,k}(\lambda, c) = z_{n-1,k}(c),$$

and

$$\lim_{\lambda\to\infty} \lambda[c - x_{n,n}(\lambda, c)] = \frac{p_n(c)}{K_{n-1}(c,c) r_{n-1}(c;c)},$$

$$\lim_{\lambda\to\infty} \lambda[z_{n-1,k}(c) - x_{n,k}(\lambda, c)] = \frac{p_n(z_{n-1,k}(c))}{K_{n-1}(c,c)(z_{n-1,k}(c) - c)[r_{n-1}(c;x)]'_{x=z_{n-1,k}(c)}}.$$

Note that the mass point c attracts one zero of $p_n(\lambda, c; x)$, that is, when λ goes to infinity, it captures either the smallest or the largest zero, according to the location of the point c with respect to the interval (a, b). In addition, when either $c < a$ or $c > b$, at most one of the zeros of $p_n(\lambda, c; x)$ is located outside (a, b).

We point out that Theorem 1 is general in two aspects and uses new approaches to the analyses of zeros: $d\mu(x)$ is any positive Borel measure and c is any value outside (a, b).

Some particular cases of these polynomials appear in the seminal papers by H. L. Krall [25] and A. M. Krall [24] devoted to the spectral analysis of fourth order linear differential operators with polynomial coefficients. T. H. Koornwinder [23] analyzed a general situation for Jacobi weights when two masses are added at the end points of the interval [-1,1]. Later on, in [15], Krall-Hermite and Krall-Bessel polynomials are studied in the framework of Darboux transformations.

In [22] analytic properties of orthogonal polynomials with respect to a perturbation of the Laguerre weight when a mass is added at $x = 0$ are considered. In [29], the holonomic equation for such perturbations when the mass point is located in the negative real semi-axis is deduced. In the framework of the spectral theory of higher order linear differential operators, in [20] and [21] the authors obtain infinite order differential operators such that the Krall-Laguerre and Krall-Jacobi are their eigenfunctions, respectively. In particular, for some choices of the parameters of Laguerre and Jacobi weights they prove that the differential operator has a finite order.

On the other hand, in [26] the authors deduced the invariance of the semiclassical character of semiclassical linear functionals under Uvarov transformations independently of the location of the mass point.

All the above questions concerning the behaviour of the zeros of the polynomials $p_n(\lambda, c; x)$ were answered for two important and particular cases in [6] and [7]. The authors considered the cases when $d\mu(x) = x^\alpha e^{-x} dx$ with $(a, b) = (0, \infty)$ and $c = 0$, and $d\mu(x) = (1-x)^\alpha (1+x)^\beta dx$ with $(a, b) = (-1, 1)$ and $c = 1$, respectively.

We also provide the second order linear differential equation that the polynomial $p_n(\lambda, c; x)$ satisfies when the measure $d\mu(x)$ in (1) is semiclassical (for definition of a semiclassical measure see [30]). This is the main tool for the electrostatic interpretation of zeros.

Theorem 2 *The monic orthogonal polynomial sequence $\{p_n(\lambda, c; x)\}_{n\geq 0}$ satisfies the holonomic equation (second order linear differential equation)*

$$\mathcal{A}(x;n)(p_n(\lambda,c;x))'' + \mathcal{B}(x;n)(p_n(\lambda,c;x))' + \mathcal{C}(x;n)p_n(\lambda,c;x) = 0, \tag{3}$$

where

$$\begin{aligned}
\mathcal{A}(x;n) &= \frac{c_n\,[\phi(x)]^2}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}, \\
\mathcal{B}(x;n) &= \frac{\phi(x)\left[B(x,n) - \tilde{B}(x,n) + c_n(\phi'(x) - A(x,n))\right]}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} \\
&\qquad - \frac{c_n\phi(x)^2\left(\tilde{B}(x,n) - c_n\tilde{A}(x,n)\right)'}{\left(\tilde{B}(x,n) - c_n\tilde{A}(x,n)\right)^2}, \\
\mathcal{C}(x;n) &= \frac{A(x,n)\tilde{B}(x,n) - B(x,n)\tilde{A}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} - \phi(x)D\left(\frac{\tilde{B}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\right).
\end{aligned}$$

In order to justify our approach, we must point out that the behavior of the zeros of orthogonal polynomials has been extensively studied because of their applications in many areas of physics and engineering. First, the zeros of orthogonal polynomials are the nodes of the Gaussian quadrature rules and also play an important role in some of their extensions like Gauss-Radau, Gauss-Lobatto, and Gauss-Kronrod rules, among others (see [5], [12] [32]). Second, the zeros of classical orthogonal polynomials are the electrostatic equilibrium points of positive unit charges interacting according to a logarithmic potential under the action of an external field, see Stieltjes' papers [36], [37], [38] and [39], Szegő's book [40, Section 6.7], and some recent works like D. K. Dimitrov and W. Van Assche [8], A. Grunbaum [13] and [14], M. E. H. Ismail

[17], and F. Marcellán, A. Martínez-Finkelshtein and P. Martínez-González [27] among others. Third, in a more general framework, the counting measure of zeros weakly converges to the equilibrium measure associated with a logarithmic potential (see [35]). Fourth, zeros of orthogonal polynomials are used in collocation methods for boundary value problems of second order linear differential operators (see [2]). Fifth, global properties of zeros of orthogonal polynomials can be analyzed when they satisfy second order differential equations with polynomial coefficients using the WKB method (see [1]). Finally, zeros of orthogonal polynomials are eigenvalues of Jacobi matrices and its role in Numerical Linear Algebra is very well known.

The structure of the manuscript is as follows. In Section 2 we prove Theorem 1. It follows from the Christoffel formula, from connections formula for the perturbed polynomials in terms of the initial ones and from a lemma concerning the behavior of the zeros of a linear combination of two polynomials. In addition, we obtain new connection formulas for orthogonal polynomials obtained from Uvarov and Christoffel transformations and some results about their zeros. In section 3, we check these results for the Jacobi-type and Laguerre-type orthogonal polynomials introduced by T. H. Koornwinder [23]. In Section 4 we obtain the holonomic equations that the polynomials $p_n(\lambda, c; x)$ satisfy using an alternative approach to the standard ones. We focus our attention on the electrostatic interpretation of the zeros as equilibrium points in a logarithmic potential interaction of positive unit charges under the presence of an external field. We analyze such an equilibrium problem when the mass point is located either on the boundary or in the exterior of the support of the measure, respectively, for the Laguerre and Jacobi weights transformations.

2 Proof of Theorem 1

In this section, we prove Theorem 1 and present some new results.

Proof of Theorem 1: Le $y_{n,k}(c)$ be the zeros of the monic polynomials $q_n(c;x)$ orthogonal with respect to the perturbed measure

$$d\mu_1(c;x) = |x-c|d\mu(x),$$

where $c \notin (a,b)$. This perturbation is the so-called Christoffel perturbation (see [42] and [43]). It is well known that $q_n(c;x)$ is the monic kernel polynomial which can be represented as (see [5, (7.3)])

$$q_n(c;x) = \frac{1}{x-c}\left[p_{n+1}(x) - \frac{p_{n+1}(c)}{p_n(c)}p_n(x)\right] = \frac{\|p_n\|_\mu^2}{p_n(c)}K_n(c,x), \tag{4}$$

where $K_n(c,x)$ is the n-th kernel polynomial defined by

$$K_n(c,x) = \sum_{j=0}^{n} \frac{p_j(c)p_j(x)}{\|p_j\|_\mu^2}. \tag{5}$$

In Chihara's book [2, Theorem 7.2] we found the following interlacing property involving the zeros of $q_n(c;x)$, $p_{n+1}(x)$ and $p_n(x)$:

- If $c \le a$, then

$$x_{n+1,1} < x_{n,1} < y_{n,1}(c) < x_{n+1,2} < \cdots < x_{n,n} < y_{n,n}(c) < x_{n+1,n+1};$$

- If $c \ge b$, then

$$x_{n+1,1} < y_{n,1}(c) < x_{n,1} < \cdots < x_{n+1,n} < y_{n,n}(c) < x_{n,n} < x_{n+1,n+1}.$$

We introduce the monic polynomials $r_n(c;x)$ orthogonal with respect to the measure

$$d\mu_2(c;x) = |x-c|d\mu_1(c;x) = (x-c)^2 d\mu(x).$$

Using (4) we deduce that

$$\begin{aligned} r_n(c;x) &= \frac{1}{x-c}\left[q_{n+1}(c;x) - \frac{q_{n+1}(c;c)}{q_n(c;c)}q_n(c;x)\right] \\ &= \frac{1}{(x-c)^2}\left[p_{n+2}(x) - d_n p_{n+1}(x) + e_n p_n(x)\right], \end{aligned} \tag{6}$$

where

$$\begin{aligned} d_n &= \frac{p_{n+2}(c)}{p_{n+1}(c)} + \frac{q_{n+1}(c;c)}{q_n(c;c)} = \frac{p_{n+2}(c)+p_n(c)}{p_{n+1}(c)}e_n, \\ e_n &= \frac{q_{n+1}(c;c)}{q_n(c;c)}\frac{p_{n+1}(c)}{p_n(c)} = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2}\frac{K_{n+1}(c,c)}{K_n(c,c)} > 0. \end{aligned}$$

Notice that $r_n(c;c) \neq 0$. If we denote by $z_{n,k}(c)$ the zeros of $r_n(c;x)$, then using the three term recurrence relation

$$p_{n+1}(x) = (x-\beta_n)p_n(x) - \gamma_n p_{n-1}(x),$$

where

$$\beta_n = \frac{\langle xp_n, p_n\rangle_\mu}{\|p_n\|_\mu^2},\ n \ge 0, \quad \text{and} \quad \gamma_n = \frac{\|p_n\|_\mu^2}{\|p_{n-1}\|_\mu^2} > 0,\ n \ge 1,$$

in (6) we obtain

$$r_n(c;x) = \frac{1}{(x-c)^2}\left[(x-\beta_{n+1}-d_n)p_{n+1}(x) + (e_n-\gamma_{n+1})p_n(x)\right]. \tag{7}$$

On the other hand,

$$e_n - \gamma_{n+1} = \frac{\|p_{n+1}\|_\mu^2}{\|p_n\|_\mu^2}\left(\frac{K_{n+1}(c,c)}{K_n(c,c)} - 1\right) > 0. \tag{8}$$

Thus, evaluating $r_n(c;x)$ at the zeros $x_{n+1,k}$, from (7) and (8),

$$Sign\left[r_n(c;x_{n+1,k})\right] = Sign\left[p_n(x_{n+1,k})\right],\ k = 1,\dots,n+1.$$

Since the zeros of $p_{n+1}(x)$ and $p_n(x)$ interlace, we conclude that

Theorem 3 *The inequalities*

$$x_{n+1,1} < z_{n,1}(c) < x_{n+1,2} < z_{n,2}(c) < \cdots < x_{n+1,n} < z_{n,n}(c) < x_{n+1,n+1} \tag{9}$$

hold for every $n \in \mathbb{N}$.

In [1, (8)] (see also [41]) the authors show that

$$p_n(\lambda,c;x) = p_n(x) - \frac{\lambda p_n(c)}{1+\lambda K_{n-1}(c,c)} K_{n-1}(c,x). \tag{10}$$

In [16] another connection formula was obtained . Using the similar idea as in Proposition 4 in [16] we obtain

Theorem 4 (Connection Formula) *The polynomials $\widehat{p}_n(\lambda,c;x)$, with the normalization $\widehat{p}_n(\lambda,c;x) = k_n p_n(\lambda,c;x)$, can be represented as*

$$\widehat{p}_n(\lambda,c;x) = p_n(x) + \lambda K_{n-1}(c,c)\,(x-c) r_{n-1}(c;x), \tag{11}$$

where $k_n = 1 + \lambda K_{n-1}(c,c)$.

Using the interlacing property (9) and the connection formula (11), we get

$$Sign\left[\widehat{p}_n(\lambda,c;x_{n,k})\right] = Sign\left[\lambda(x_{n,k}-c) r_{n-1}(c;x)\right],\ k = 1,\dots,n,$$

and

$$Sign\left[\widehat{p}_n(\lambda,c;z_{n-1,k}(c))\right] = Sign\left[p_n(z_{n-1,k}(c))\right],\ k = 1,\dots,n-1,$$

which yield the inequalities stated in Theorem 1. It remains to show the monotonicity, asymptotics and the speed of the convergence of the zeros $x_{n,k}(\lambda,c)$ with respect to λ. Indeed, it follows from the technique developed in [3, Lemma 1] and [7, Lemmas 1 and 2] (see also [33, Theorem 3.9]) concerning the zeros of a linear combination of two polynomials with interlacing zeros.

We also derive a representation for the monic polynomial $p_n(\lambda,c;x)$ as a combination of two Christoffel polynomials it will be very useful for obtain the holomonic equation for $p_n(\lambda,c;x)$.

Corollary 1 *The monic polynomials* $p_n(\lambda, c; x)$ *can be also represented as*

$$p_n(\lambda, c; x) = q_n(c; x) + c_n\, q_{n-1}(c; x), \tag{12}$$

where

$$c_n = \frac{1 + \lambda K_n(c,c)}{1 + \lambda K_{n-1}(c,c)} \frac{p_{n-1}(c)}{p_n(c)} \gamma_n \quad \textit{and} \quad \gamma_n = \frac{\|p_n\|_\mu^2}{\|p_{n-1}\|_\mu^2}. \tag{13}$$

Using (5), we obtain

$$p_n(x) = \frac{\|p_n\|_\mu^2}{p_n(c)} \left[K_n(c,x) - K_{n-1}(c,x)\right]. \tag{14}$$

From (4) and (14), we have

$$p_n(x) = q_n(c; x) - \gamma_n \frac{p_{n-1}(c)}{p_n(c)} q_{n-1}(c; x). \tag{15}$$

Therefore, substituting (4) and (15) in (10) we get (12).

In the following we deduce the value λ_0 of the mass such that for $\lambda > \lambda_0$ one of the zeros of $p_n(\lambda, c; x)$ is located outside (a, b).

Corollary 2 *Let* $\lambda > 0$.

(*i*) *If* $c < a$, *then the smallest zero* $x_{n,1}(\lambda, c)$ *satisfies*

$$x_{n,1}(\lambda, c) > a, \quad \text{for} \quad \lambda < \lambda_0,$$

$$x_{n,1}(\lambda, c) = a, \quad \text{for} \quad \lambda = \lambda_0,$$

$$x_{n,1}(\lambda, c) < a, \quad \text{for} \quad \lambda > \lambda_0,$$

where

$$\lambda_0 = \lambda_0(n, a, c) = \frac{-p_n(a)}{K_{n-1}\,(c,c)\,(a-c) r_{n-1}(c; a)} > 0.$$

(*ii*) *If* $c > b$, *then the largest zero* $x_{n,n}(\lambda, c)$ *satisfies*

$$x_{n,n}(\lambda, c) < b, \quad \text{for} \quad \lambda < \lambda_0,$$

$$x_{n,n}(\lambda, c) = b, \quad \text{for} \quad \lambda = \lambda_0,$$

$$x_{n,n}(\lambda, c) > b, \quad \text{for} \quad \lambda > \lambda_0,$$

where $\lambda_0 = \lambda_0(n, b, c)$.

The proofs are a consequence of the connection formula (11).

3 Application to classical measures

3.1 Jacobi type (Jacobi-Koornwinder) orthogonal polynomials

Let $\{p_n^{(\alpha,\beta)}(x)\}_{n\geq 0}$ be the monic Jacobi polynomial sequence which is orthogonal with respect to the measure $d\mu_{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$, $\alpha, \beta > -1$, supported on $(-1, 1)$. We consider the following Uvarov perturbations of $d\mu_{\alpha,\beta}(x)$ where either $a = -1$ or $a = 1$, and $\lambda \geq 0$.

$$d\mu(\lambda, -1; x) = d\mu_{\alpha,\beta}(x) + \lambda\delta(x + 1), \tag{16}$$

$$d\mu(\lambda, 1; x) = d\mu_{\alpha,\beta}(x) + \lambda\delta(x - 1). \tag{17}$$

Such orthogonal polynomials were first studied by T. H. Koornwinder (see [23]), in 1984. There, he adds simultaneously two Dirac delta functions at the end points $x = -1$ and $x = 1$, that is,

$$d\mu_{M,N}(x) = d\mu_{\alpha,\beta}(x) + M\delta(x + 1) + N\delta(x - 1).$$

Denote by $\{\widehat{p}_n^{(\alpha,\beta)}(\lambda, -1; x)\}_{n\geq 0}$ and $\{\widehat{p}_n^{(\alpha,\beta)}(\lambda, 1; x)\}_{n\geq 0}$ the sequences of orthogonal polynomials with respect (16) and (17), with the normalization introduced in Theorem 2, respectively. Then, the connection formulas are

$$\widehat{p}_n^{(\alpha,\beta)}(\lambda, -1; x) = p_n^{(\alpha,\beta)}(x) + \lambda K_{n-1}(-1, -1)(x + 1)p_{n-1}^{(\alpha,\beta+2)}(x)$$

and

$$\widehat{p}_n^{(\alpha,\beta)}(\lambda, 1; x) = p_n^{(\alpha,\beta)}(x) + \lambda K_{n-1}(1, 1)(x - 1)p_{n-1}^{(\alpha+2,\beta)}(x).$$

It is straightforward to see that

$$K_{n-1}(-1, -1) = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n + \beta + 1)\Gamma(n + \alpha + \beta + 1)}{\Gamma(n)\Gamma(\beta + 1)\Gamma(\beta + 2)\Gamma(n + \alpha)}$$

and

$$K_{n-1}(1, 1) = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n + \alpha + 1)\Gamma(n + \alpha + \beta + 1)}{\Gamma(n)\Gamma(\alpha + 1)\Gamma(\alpha + 2)\Gamma(n + \beta)}.$$

Recently, several authors ([1], [7], [11]) have been contributed to the analysis of the behavior of the zeros of $\widehat{p}_n^{(\alpha,\beta)}(\lambda, -1; x)$ and $\widehat{p}_n^{(\alpha,\beta)}(\lambda, 1; x)$.

Let us denote by $x_{n,k}(\alpha, \beta; \lambda) := x_{n,k}(\alpha, \beta; -1, \lambda)$ and $x_{n,k}(\alpha, \beta)$, $k = 1, \ldots, n$, the zeros of $\widehat{p}_n^{(\alpha,\beta)}(\lambda, -1; x)$ and $p_n^{(\alpha,\beta)}(x)$, respectively, in an increasing order. Then, applying Theorem 1, we obtain

Theorem 5 ([7]) *The inequalities*

$$-1 < x_{n,1}(\alpha,\beta;\lambda) < x_{n,1}(\alpha,\beta) < x_{n-1,1}(\alpha,\beta+2) < x_{n,2}(\alpha,\beta;\lambda) < x_{n,2}(\alpha,\beta) < \cdots$$
$$< x_{n-1,n-1}(\alpha,\beta+2) < x_{n,n}(\alpha,\beta;\lambda) < x_{n,n}(\alpha,\beta)$$

hold for every $\alpha,\beta > -1$. Moreover, each $x_{n,k}(\alpha,\beta;\lambda)$ is a decreasing function of λ and, for each $k = 1,\dots,n-1$,

$$\lim_{\lambda\to\infty} x_{n,1}(\alpha,\beta;\lambda) = -1, \quad \lim_{\lambda\to\infty} x_{n,k+1}(\alpha,\beta;\lambda) = x_{n-1,k}(\alpha,\beta+2),$$

and

$$\lim_{\lambda\to\infty} \lambda[x_{n,1}(\alpha,\beta;\lambda)+1] = h_n(\alpha,\beta),$$
$$\lim_{\lambda\to\infty} \lambda[x_{n,k+1}(\alpha,\beta;\lambda) - x_{n-1,k}(\alpha,\beta+2)] = \frac{[1-x_{n-1,k}(\alpha,\beta+2)]\,h_n(\alpha,\beta)}{2(\beta+2)},$$

where

$$h_n(\alpha,\beta) = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\beta+2)\Gamma(\beta+3)\Gamma(n+\alpha)}{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)}.$$

From (2)

$$\lim_{\lambda\to\infty} \lambda[x_{n,1}(\alpha,\beta;\lambda)+1] = \frac{-p_n^{(\alpha,\beta)}(-1)}{K_{n-1}(-1,-1)p_{n-1}^{(\alpha,\beta+2)}(-1)}.$$

Since

$$p_n^{(\alpha,\beta)}(-1) = \frac{(-1)^n 2^n \Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\beta+1)\Gamma(2n+\alpha+\beta+1)}$$

and

$$K_{n-1}(-1,-1) = \frac{1}{2^{\alpha+\beta+1}}\frac{\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n)\Gamma(\beta+1)\Gamma(\beta+2)\Gamma(n+\alpha)}$$

we obtain

$$\frac{-p_n^{(\alpha,\beta)}(-1)}{K_{n-1}(-1,-1)p_{n-1}^{(\alpha,\beta+2)}(-1)} = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\beta+2)\Gamma(\beta+3)\Gamma(n+\alpha)}{\Gamma(n+\beta+2)\Gamma(n+\alpha+\beta+2)} = h_n(\alpha,\beta).$$

It also follows from (2) that

$$\lim_{\lambda\to\infty} \lambda[x_{n,k+1}(\alpha,\beta;\lambda) - x_{n-1,k}(\alpha,\beta+2)]$$
$$= \frac{-p_n^{(\alpha,\beta)}(x_{n-1,k}(\alpha,\beta+2))}{K_{n-1}(-1,-1)(x_{n-1,k}(\alpha,\beta+2)+1)[p_{n-1}^{(\alpha,\beta+2)}(x)]'\big|_{x=x_{n-1,k}(\alpha,\beta+2)}}.$$

On the other hand, from

$$n(n+\alpha)(1+x)p_{n-1}^{(\alpha,\beta+2)}(x) = n(n+\alpha+\beta+1)p_n^{(\alpha,\beta)}(x)+(\beta+1)(1-x)[p_n^{(\alpha,\beta)}(x)]'$$

we derive

$$\begin{aligned} &n(n+\alpha+\beta+1)p_n^{(\alpha,\beta)}(x_{n-1,k}(\alpha,\beta+2)) \\ &\quad = -(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))[p_n^{(\alpha,\beta)}(x)]'\Big|_{x=x_{n-1,k}(\alpha,\beta+2)} \end{aligned}$$

as well as

$$\begin{aligned} &n(n+\alpha)(1+x_{n-1,k}(\alpha,\beta+2))[p_n^{(\alpha,\beta)}(x)]'\Big|_{x=x_{n-1,k}(\alpha,\beta+2)} \\ &\quad = [n(n+\alpha+\beta+1)-(\beta+1)][p_n^{(\alpha,\beta)}(x)]'\Big|_{x=x_{n-1,k}(\alpha,\beta+2)} \\ &\qquad +(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))[p_n^{(\alpha,\beta)}(x)]''\Big|_{x=x_{n-1,k}(\alpha,\beta+2)}. \end{aligned}$$

Using the last two equalities and the differential equation for the Jacobi polynomials

$$(1-x^2)[p_n^{(\alpha,\beta)}(x)]''+[\beta-\alpha-(\alpha+\beta+1)x][p_n^{(\alpha,\beta)}(x)]'+n(n+\alpha+\beta+1)p_n^{(\alpha,\beta)}(x) = 0$$

we obtain

$$\begin{aligned} &(1+x_{n-1,k}(\alpha,\beta+2))[p_n^{(\alpha,\beta+2)}(x)]'\Big|_{x=x_{n-1,k}(\alpha,\beta+2)} \\ &\quad = \frac{-(n+\beta+1)(n+\alpha+\beta+1)}{(\beta+1)(1-x_{n-1,k}(\alpha,\beta+2))}p_n^{(\alpha,\beta)}(x_{n-1,k}(\alpha,\beta+2)). \end{aligned}$$

Therefore,

$$\begin{aligned} &\lim_{\lambda\to\infty}\lambda[x_{n,k+1}(\alpha,\beta;\lambda)-x_{n-1,k}(\alpha,\beta+2)] \\ &\quad = \frac{-p_n^{(\alpha,\beta)}(x_{n-1,k}(\alpha,\beta+2))}{K_{n-1}(-1,-1)(x_{n-1,k}(\alpha,\beta+2)+1)[p_{n-1}^{(\alpha,\beta+2)}(x)]'\Big|_{x=x_{n-1,k}(\alpha,\beta+2)}} \\ &\quad = \frac{[1-x_{n-1,k}(\alpha,\beta+2)]\,h_n(\alpha,\beta)}{2(\beta+2)}. \end{aligned}$$

Let $x_{n,k}(\alpha,\beta;\lambda) := x_{n,k}(\alpha,\beta;1,\lambda)$ be the zeros of $\widehat{p}_n^{(\alpha,\beta)}(\lambda,1;x)$. Then

Theorem 6 ([7]) *The inequalities*

$$x_{n,1}(\alpha,\beta) < x_{n,1}(\alpha,\beta;\lambda) < x_{n-1,1}(\alpha+2,\beta) < \cdots <$$

$$x_{n,n-1}(\alpha,\beta) < x_{n,n-1}(\alpha,\beta;\lambda) < x_{n-1,n-1}(\alpha+2,\beta) < x_{n,n}(\alpha,\beta) < x_{n,n}(\alpha,\beta;\lambda) < 1$$

hold for every $\alpha,\beta > -1$*. Moreover, each* $x_{n,k}(\alpha,\beta;\lambda)$ *is an increasing function of* λ *and, for each* $k = 1,\ldots,n-1$*,*

$$\lim_{\lambda\to\infty} x_{n,n}(\alpha,\beta;\lambda) = 1, \quad \lim_{\lambda\to\infty} x_{n,k}(\alpha,\beta;\lambda) = x_{n-1,k}(\alpha+2,\beta),$$

and

$$\lim_{\lambda\to\infty} \lambda[1 - x_{n,n}(\alpha,\beta;\lambda)] = g_n(\alpha,\beta),$$

$$\lim_{\lambda\to\infty} \lambda[x_{n-1,k}(\alpha+2,\beta) - x_{n,k}(\alpha,\beta;\lambda)] = \frac{[1 + x_{n-1,k}(\alpha+2,\beta)]\, g_n(\alpha,\beta)}{2(\alpha+2)},$$

where

$$g_n(\alpha,\beta) = \frac{2^{\alpha+\beta+2}\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+3)\Gamma(n+\beta)}{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}.$$

We can proceed as in the proof of Theorem 5. We only observe that

$$\frac{n+\beta}{2n}(x-1)p_{n-1}^{(\alpha+2,\beta)}(x) = p_n^{(\alpha,\beta)}(x) - \frac{\alpha+1}{n(n+\alpha+\beta+1)}(1+x)[p_n^{(\alpha,\beta)}(x)]'.$$

To illustrate the results of Theorem 6, the graphs of $\widehat{p}_3^{(\alpha,\beta)}(\lambda+\varepsilon,1;x)$, for $\alpha=\beta=0$ and some values of $\varepsilon > 0$ appear in Figure 1 where the monotonicity of the zeros of $\widehat{p}_3^{(\alpha,\beta)}(\lambda,1;x)$ as a function of the mass λ is clarified.

In Table 1 we describe the zeros of $\widehat{p}_3^{(\alpha,\beta)}(\lambda,1;x)$, with $\alpha=\beta=0$, for several choices of λ. Notice that the largest zero converges to 1 and the other two

Table 1
Zeros of $\widehat{p}_3^{(\alpha,\beta)}(\lambda,1;x)$ for some values of λ.

λ	$x_{3,1}(0,0;\lambda)$	$x_{3,2}(0,0;\lambda)$	$x_{3,3}(0,0;\lambda)$
0	−0.774597	0	0.774597
1	−0.757872	0.0753429	0.955257
10	−0.755305	0.0868168	0.994575
100	−0.755004	0.0881528	0.999446
1000	−0.754974	0.0882886	0.999944

zeros converge to the zeros of the Jacobi polynomial $p_2^{(2,0)}(x)$, that is, they

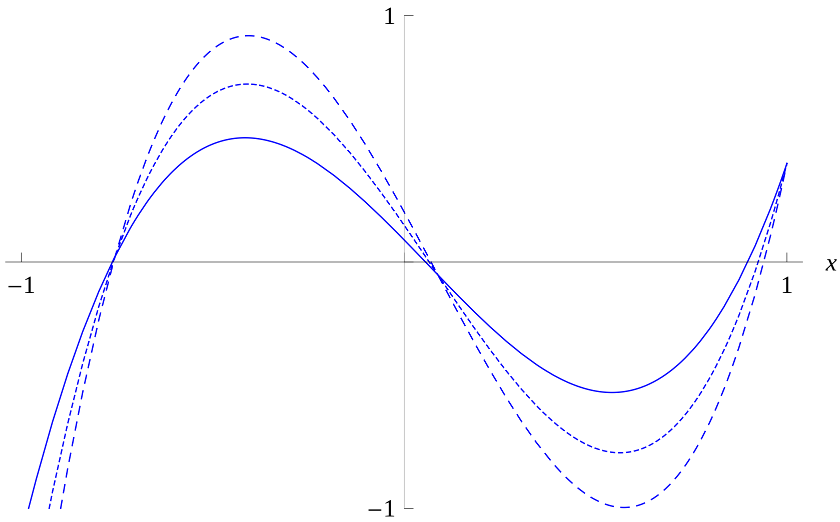

Fig. 1. The graphs of $\widehat{p}_3^{(\alpha,\beta)}(\lambda+\epsilon,1;x)$ for some values of ε.

converge to $x_{2,1}(2,0) = -0.75497$ and $x_{2,2}(4) = 0.0883037$. Also note that all the zeros increase when λ increase.

3.2 Laguerre type (Laguerre-Koornwinder) orthogonal polynomials

Let $\{p_n^{(\alpha)}(x)\}_{n\geq 0}$ be the monic Laguerre polynomial which are orthogonal with respect to the measure $d\mu(x) = x^\alpha e^{-x}dx$, $\alpha > -1$, supported on $(0,+\infty)$. We will consider the Uvarov perturbation on $d\mu_\alpha(x)$ with $c = 0$

$$d\mu(\lambda, 0; x) = d\mu(x) + \lambda\delta(x), \ \ \lambda \geq 0. \tag{18}$$

The polynomial $p_n^{(\alpha)}(\lambda;x) := p_n^{(\alpha)}(\lambda,0;x)$ orthogonal with respect to (18) was also obtained by T. H. Koornwinder [23] as a special limit case of the Jacobi-Koornwinder (Jacobi type) orthogonal polynomial. Analytic properties of these polynomials have been studied in the last years (see [1], [6], [10], [22], among others). The connection formula (11) reads

$$\widehat{p}_n^{(\alpha)}(\lambda,x) = p_n^{(\alpha)}(x) + \lambda K_{n-1}(0,0)\, x\, p_{n-1}^{(\alpha+2)}(x),$$

where

$$K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)}.$$

Now, we will analyze the behavior of their zeros. Let denote by $x_{n,k}(\alpha;\lambda)$ and $x_{n,k}(\alpha)$, $k = 1,\ldots,n$, the zeros of the Laguerre type and the classical Laguerre orthogonal polynomials, respectively. Applying the results of Theorem 1 we obtain

Theorem 7 ([6]) *The inequalities*

$$0 < x_{n,1}(\alpha;\lambda) < x_{n,1}(\alpha) < x_{n-1,1}(\alpha+2) < x_{n,2}(\alpha;\lambda) < \\ x_{n,2}(\alpha) < \cdots < x_{n-1,n-1}(\alpha+2) < x_{n,n}(\alpha;\lambda) < x_{n,n}(\alpha)$$

hold for every $\alpha > -1$. Moreover, each $x_{n,k}(\alpha;\lambda)$ is a decreasing function of λ and, for each $k = 1,\dots,n-1$,

$$\lim_{\lambda\to\infty} x_{n,1}(\alpha;\lambda) = 0, \quad \lim_{\lambda\to\infty} x_{n,k+1}(\alpha;\lambda) = x_{n-1,k}(\alpha+2),$$

as well as

$$\lim_{\lambda\to\infty} \lambda x_{n,1}(\alpha;\lambda) = g_n(\alpha),$$

$$\lim_{\lambda\to\infty} \lambda[x_{n,k+1}(\alpha;\lambda) - x_{n-1,k}(\alpha+2)] = \frac{g_n(\alpha)}{\alpha+2},$$

where

$$g_n(\alpha) = \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)}.$$

From (2)

$$\lim_{\lambda\to\infty} \lambda x_{n,1}(\alpha;\lambda) = \frac{-p_n^{(\alpha)}(0)}{K_{n-1}(0,0)p_{n-1}^{(\alpha+2)}(0)}.$$

Since

$$p_n^{(\alpha)}(0) = \frac{(-1)^n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)} \quad \text{and} \quad K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n)\Gamma(\alpha+1)\Gamma(\alpha+2)},$$

we obtain

$$\frac{-p_n^{(\alpha)}(0)}{K_{n-1}(0,0)p_{n-1}^{(\alpha+2)}(0)} = \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+3)}{\Gamma(n+\alpha+2)} = g_n(\alpha).$$

From (2)

$$\lim_{\lambda\to\infty} \lambda[x_{n,k+1}(\alpha;\lambda) - x_{n-1,k}(\alpha+2)] \\ = \frac{-p_n^{(\alpha)}(x_{n-1,k}(\alpha+2))}{K_{n-1}(0,0)x_{n-1,k}(\alpha+2)[p_{n-1}^{(\alpha+2)}(x)]'\big|_{x=x_{n-1,k}(\alpha+2)}}.$$

On the other hand, It is easy to verify that

$$xp_{n-1}^{(\alpha+2)}(x) = p_n^{(\alpha)}(x) + \frac{\alpha+1}{n}[p_n^{(\alpha)}(x)]'.$$

Thus,

$$[p_n^{(\alpha)}(x)]'\big|_{x=x_{n-1,k}(\alpha+2)} = -\frac{n}{\alpha+1}p_n^{(\alpha)}(x_{n-1,k}(\alpha+2))$$

and

$$x_{n-1,k}(\alpha+2)[p_n^{(\alpha)}(x)]'\Big|_{x=x_{n-1,k}(\alpha+2)}$$
$$= [p_n^{(\alpha)}(x)]'\Big|_{x=x_{n-1,k}(\alpha+2)} + \frac{\alpha+1}{n}[p_n^{(\alpha)}(x)]''\Big|_{x=x_{n-1,k}(\alpha+2)}.$$

Using the last two equalities and the differential equation for the Laguerre polynomials

$$x[p_n^{(\alpha)}(x)]'' + (\alpha+1-x)[p_n^{(\alpha)}(x)]' + np_n^{(\alpha)}(x) = 0$$

we obtain

$$x_{n-1,k}(\alpha+2)[p_n^{(\alpha)}(x)]'\Big|_{x=x_{n-1,k}(\alpha+2)} = \frac{-(n+\alpha+1)}{\alpha+1}p_n^{(\alpha)}(x_{n-1,k}(\alpha+2)).$$

Therefore,

$$\lim_{\lambda\to\infty}\lambda[x_{n,k+1}(\alpha;\lambda) - x_{n-1,k}(\alpha+2)]$$
$$= \frac{-p_n^{(\alpha)}(x_{n-1,k}(\alpha+2))}{K_{n-1}(0,0)x_{n-1,k}(\alpha+2)[p_{n-1}^{(\alpha+2)}(x)]'\Big|_{x=x_{n-1,k}(\alpha+2)}}$$
$$= \frac{\Gamma(n)\Gamma(\alpha+2)\Gamma(\alpha+2)}{\Gamma(n+\alpha+2)} = \frac{g_n(\alpha)}{\alpha+2}.$$

To illustrate the results of Theorem 7 we enclose the graphs of $p_3^{(\alpha)}(\lambda+\varepsilon;x)$, for $\alpha = 2$ and some values of $\varepsilon > 0$. The Figure 2 shows the monotonicity of the zeros of $p_3^{(\alpha)}(\lambda;x)$ as a function of the mass λ.

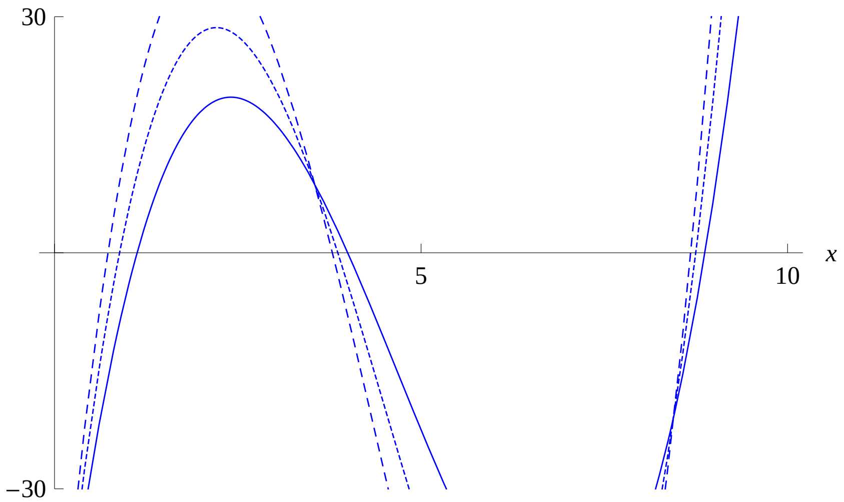

Fig. 2. The graphs of $p_3^{(\alpha)}(\lambda+\varepsilon;x)$ for some values of ε.

The table 2 describes the zeros of $p_3^{(\alpha)}(\lambda;x)$, with $\alpha=2$, for several choices of λ. Observe that the smallest zero converges to 0 and the other two zeros converge to the zeros of the Laguerre polynomial $p_2^{(4)}(x)$, that is, they converge to $x_{2,1}(4)=3.55051$ and $x_{2,2}(4)=8.44949$. Note that all the zeros decrease when λ increases.

Table 2
Zeros of $p_3^{(\alpha)}(\lambda;x)$ for some values of λ.

λ	$x_{3,1}(2;\lambda)$	$x_{3,2}(2;\lambda)$	$x_{3,3}(2;\lambda)$
0	1.51739	4.31158	9.17103
1	0.321731	3.64053	8.53774
10	0.0390611	3.5604	8.45936
100	0.00399042	3.55151	8.45049
1000	0.00039990	3.55061	8.44959

4 Semiclassical orthogonal polynomials and spectral transformations

We assume that $d\mu(x)=\omega(x)dx$, where $\omega(x)$ is a weight function supported on the real line. We can associate with $\omega(x)$ an external potential $\upsilon(x)$ such that $\omega(x)=\exp(-\upsilon(x))$. Notice that if $\upsilon(x)$ is assumed to be differentiable on the support of $d\mu(x)=\omega(x)\,dx$ then

$$\frac{\omega'(x)}{\omega(x)}=-\upsilon'(x).$$

If $\upsilon'(x)$ is a rational function on (a,b), then the weight function $\omega(x)$ is said to be semiclassical (see [30], [34]). The linear functional u associated with $\omega(x)$,

$$\langle u,p(x)\rangle=\int_a^b p(x)\omega(x)\,dx,$$

satisfies a distributional equation (which is known in the literature as Pearson equation)

$$D(\sigma(x)u)=\tau(x)u,$$

where $\sigma(x)$ and $\tau(x)$ are non-zero polynomials such that $\sigma(x)$ is monic and $\deg(\tau(x))\geq 1$. Notice that, in terms of the weight function, the above relation reads

$$\frac{\omega'(x)}{\omega(x)}=\frac{\tau(x)-\sigma'(x)}{\sigma(x)},\text{ or, equivalently } \upsilon'(x)=-\frac{\tau(x)-\sigma'(x)}{\sigma(x)}.$$

Let consider the linear functional u_1 associated with the measure $d\mu_1(c,x)$. In order to find the Pearson equation that u_1 satisfies, we analyze two situations:

- If $\sigma(c) \neq 0$, then

$$\begin{aligned} D\left((x-c)\sigma(x)u_1\right) &= D\left((x-c)^2\sigma(x)u\right) = 2(x-c)\sigma(x)u + (x-c)^2 D(\sigma(x)u) \\ &= 2\sigma(x)u_1 + (x-c)^2\tau(x)u = \left[2\sigma(x) + (x-c)\tau(x)\right] u_1. \end{aligned}$$

  Thus,

$$D\left(\phi(x)u_1\right) = \psi(x)u_1,$$

  where

$$\left| \begin{aligned} &\phi(x) = (x-c)\sigma(x) \\ &\psi(x) = 2\sigma(x) + (x-c)\tau(x). \end{aligned} \right. \tag{19}$$

- If $\sigma(c) = 0$, i.e., $\sigma(x) = (x-c)\bar{\sigma}(x)$, then

$$\begin{aligned} D\left(\sigma(x)u_1\right) &= D\left((x-c)\bar{\sigma}(x)u_1\right) = D\left((x-c)^2\bar{\sigma}(x)u\right) = D\left((x-c)\sigma(x)u\right) \\ &= \sigma(x)u + (x-c)D\left(\sigma(x)u\right) = \sigma(x)u + (x-c)\tau(x)u = \left(\bar{\sigma}(x) + \tau(x)\right) u_1. \end{aligned}$$

  In this case,

$$D\left(\phi(x)u_1\right) = \psi(x)u_1,$$

  with

$$\left| \begin{aligned} &\phi(x) = \sigma(x) \\ &\psi(x) = \bar{\sigma}(x) + \tau(x). \end{aligned} \right. \tag{20}$$

It is well known that the sequence of monic polynomials $\{q_n(c;x)\}_{n\geq 0}$, orthogonal with respect to $d\mu_1(c,x)$, satisfies the structure relation (see [9] and [30])

$$\phi(x)D\left(q_n(c;x)\right) = A(x,n)q_n(c;x) + B(x,n)q_{n-1}(c;x), \tag{21}$$

where $A(x,n)$ and $B(x,n)$ are polynomials of a fixed degree, and the three term recurrence relation (see [4])

$$x\, q_n(c;x) = q_{n+1}(c;x) + \tilde{\beta}_n\, q_n(c;x) + \tilde{\gamma}_n\, q_{n-1}(c;x), \quad n \geq 0, \tag{22}$$

with initial conditions $q_0(c;x) = 1$ and $q_{-1}(c;x) = 0$, and

$$\tilde{\beta}_n = \beta_{n+1} + \frac{p_{n+2}(c)}{p_{n+1}(c)} - \frac{p_{n+1}(c)}{p_n(c)}, \quad n \geq 0, \tag{23}$$

and

$$\tilde{\gamma}_n = \frac{p_{n+1}(c)p_{n-1}(c)}{\left[p_n(c)\right]^2}\gamma_n > 0 \quad n \geq 1. \tag{24}$$

Lemma 1 *[17] We have*

$$A(x,n)+A(x,n-1)+\frac{(x-\tilde{\beta}_{n-1})}{\tilde{\gamma}_{n-1}}B(x,n-1)=\phi'(x)-\psi(x). \qquad (25)$$

According to a result by Ismail ([17], (1.12)) which must be adapted to our situation since we use monic polynomials, we get

$$\begin{aligned} A(x,n)+A(x,n-1)+\frac{(x-\tilde{\beta}_{n-1})}{\tilde{\gamma}_{n-1}}B(x,n-1)= \\ -\phi(x)\frac{[\omega_1(x)]'}{\omega_1(x)}-\phi(x)\frac{\psi(x)-\phi'(x)}{\phi(x)}\phi'(x)-\psi(x), \end{aligned}$$

where $\omega_1(x)=(x-c)\omega(x)$.

4.1 Uvarov transformations and holonomic equation

We consider the Uvarov transformation of the semiclassical measure $d\mu(x)=\omega(x)dx$. We stablish the holonomic equation of these polynomials.

Proof of Theorem 2: Applying the derivative operator in (12) and multiplying it by $\phi(x)$, we obtain

$$\phi(x)D\left(p_n(\lambda,c;x)\right)=\phi(x)D\left(q_n(c;x)\right)+c_n\phi(x)D\left(q_{n-1}(c;x)\right). \qquad (26)$$

Thus, substituting (21) in (26), yields

$$\begin{aligned} \phi(x)D\left(p_n(\lambda,c;x)\right)=A(x,n)q_n(c;x)+ \\ \left[B(x,n)+c_nA(x,n-1)\right]q_{n-1}(c;x)+c_nB(x,n-1)q_{n-2}(c;x). \end{aligned}$$

Using (22) in the above expression, we obtain

$$\phi(x)\left(p_n(\lambda,c;x)\right)'=\tilde{A}(x,n)q_n(c;x)+\tilde{B}(x,n)q_{n-1}(c;x), \qquad (27)$$

where

$$\tilde{A}(x,n)=A(x,n)-\frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1) \quad \text{and} \qquad (28)$$

$$\tilde{B}(x,n)=B(x,n)+c_nA(x,n-1)+\frac{c_n}{\tilde{\gamma}_{n-1}}(x-\tilde{\beta}_{n-1})B(x,n-1). \qquad (29)$$

Therefore, from (12) and (27), it follows that

$$\begin{bmatrix} 1 & c_n \\ \tilde{A}(x,n) & \tilde{B}(x,n) \end{bmatrix} \begin{bmatrix} q_n(c;x) \\ q_{n-1}(c;x) \end{bmatrix} = \begin{bmatrix} p_n(\lambda,c;x) \\ \phi(x) D\left(p_n(\lambda,c;x)\right) \end{bmatrix},$$

that is,

$$q_n(c;x) = \frac{\tilde{B}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\, p_n(\lambda,c;x) - \frac{c_n\phi(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\, D\left(p_n(\lambda,c;x)\right)$$

and

$$q_{n-1}(c;x) = \frac{-\tilde{A}(x,n)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} p_n(\lambda,c;x) + \frac{\phi(x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} D\left(p_n(\lambda,c;x)\right).$$

Substituting the above two expressions in (21), we deduce

$$\begin{aligned}
\phi(x)D&\left(\frac{\tilde{B}(x,n)p_n(\lambda,c;x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} - \frac{c_n\phi(x)D\left(p_n(\lambda,c;x)\right)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\right) \\
&= A(x,n)\left(\frac{\tilde{B}(x,n)p_n(\lambda,c;x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} - \frac{c_n\phi(x)D\left(p_n(\lambda,c;x)\right)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\right) \\
&\qquad + B(x,n)\left(\frac{-\tilde{A}(x,n)p_n(\lambda,c;x)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)} + \frac{\phi(x)D\left(p_n(\lambda,c;x)\right)}{\tilde{B}(x,n) - c_n\tilde{A}(x,n)}\right).
\end{aligned}$$

Then the statement of Theorem 2 follows in a straightforward way.

A different approach to this differential equation appears in [26] using the fact that the Uvarov transform of a semiclassical linear functional is again a semiclassical linear functional.

If we evaluate the second-order linear differential equation (3) at $x_{n,k}(\lambda) := x_{n,k}(\lambda, c)$, then we obtain

$$\mathcal{A}(x_{n,k}(\lambda);n)(p_n(\lambda,c;x_{n,k}(\lambda)))'' + \mathcal{B}(x_{n,k}(\lambda);n)(p_n(\lambda,c;x_{n,k}(\lambda)))' = 0.$$

Hence,

$$\frac{(p_n(\lambda,c;x_{n,k}(\lambda)))''}{(p_n(\lambda,c;x_{n,k}(\lambda)))'} = -\frac{\mathcal{B}(x_{n,k}(\lambda);n)}{\mathcal{A}(x_{n,k}(\lambda);n)}. \tag{30}$$

Substituting $\mathcal{A}(x_{n,k}(\lambda);n)$ and $\mathcal{B}(x_{n,k}(\lambda);n)$ in the right hand side of (30), we get

$$\begin{aligned}
\frac{(p_n(\lambda,c;x_{n,k}(\lambda)))''}{(p_n(\lambda,c;x_{n,k}(\lambda)))'} &= \frac{(\tilde{B}(x_{n,k}(\lambda),n) - c_n\tilde{A}(x_{n,k}(\lambda),n))'}{\tilde{B}(x_{n,k}(\lambda),n) - c_n\tilde{A}(x_{n,k}(\lambda),n)} + \\
&\frac{\tilde{B}(x_{n,k}(\lambda),n) - B(x_{n,k}(\lambda),n) + c_nA(x_{n,k}(\lambda),n) - c_n\phi'(x_{n,k}(\lambda))}{c_n\phi(x_{n,k}(\lambda))}.
\end{aligned}$$

If we denote $Q(x,n) := \tilde{B}(x,n) - c_n\tilde{A}(x,n)$ and using (25), (28), and (29), then

$$Q(x,n) = B(x,n) + c_n\left[-2A(x,n) + \phi'(x) - \psi(x) + \frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1)\right]. \quad (31)$$

On the other hand, from (31) and (28), we obtain

$$\tilde{B}(x,n) - B(x,n) + c_nA(x,n) - c_n\phi'(x) = -c_n\psi(x).$$

Thus

$$\frac{(p_n(\lambda,c;x_{n,k}(\lambda)))''}{(p_n(\lambda,c;x_{n,k}(\lambda)))'} = D\left[\ln Q(x,n)\right]|_{x=x_{n,k}(\lambda)} - \frac{\psi(x_{n,k}(\lambda))}{\phi(x_{n,k}(\lambda))}.$$

We consider two external fields

$$-\int\frac{\psi(x)}{\phi(x)}dx \quad \text{and} \quad \ln Q(x,n).$$

Thus the total external potential $V(x)$ is given by

$$V(x) = -\int\frac{\psi(x)}{\phi(x)}dx + \ln Q(x,n). \quad (32)$$

Let us consider a system of n movable unit charges in (c,b) or (a,c), depending on the location of the point c with respect to (a,b), in the presence of the external potential $V(x)$ given in (32). Let $\mathbf{x} := (x_1,\ldots,x_n)$, where $x_1,\ldots,x_n$ denote the location of the charges. The total energy of the system is

$$E(\mathbf{x}) = \sum_{k=1}^{n} V(x_k) - 2\sum_{1\le j<k\le n}\ln|x_j - x_k|.$$

In order to find the critical points of $E(\mathbf{x})$ we set

$$-\frac{\partial}{\partial x_j}E(\mathbf{x}) = 0 \Leftrightarrow \frac{\psi(x_j)}{\phi(x_j)} - \frac{Q'(x_j,n)}{Q(x_j,n)} + 2\sum_{1\le k\le n, k\ne j}\frac{1}{x_j - x_k} = 0, \quad j = 1,\ldots,n. \quad (33)$$

Let $f(y) := (y-x_1)\cdots(y-x_n)$. Thus,

$$\frac{\psi(x_j)}{\phi(x_j)} - \frac{Q'(x_j,n)}{Q(x_j,n)} + \frac{f''(x_j)}{f'(x_j)} = 0, \quad j = 1,\ldots,n,$$

or, equivalently,

$$f''(y) + \frac{\mathcal{B}(y;n)}{\mathcal{A}(y;n)}f'(y) = 0, \quad y = x_1,\ldots,x_n.$$

Therefore

$$f''(y) + \frac{\mathcal{B}(y;n)}{\mathcal{A}(y;n)}f'(y) + \frac{\mathcal{C}(y;n)}{\mathcal{A}(y;n)}f(y) = 0, \quad y = x_1,\ldots,x_n. \quad (34)$$

On the other hand, from (3) and (34) we obtain $f(y) = p_n(\lambda, c; y)$, which means that the zeros of $p_n(\lambda, c; x)$ satisfy (33).

4.2 Electrostatic interpretation of the zeros of Laguerre type orthogonal polynomials

We give an electrostatic interpretation for the zeros of Laguerre type polynomials $p_n^{(\alpha)}(\lambda, c; x)$ which are orthogonal with respect to the measure $d\mu(\lambda, c; x) = x^\alpha e^{-x}dx + \lambda\delta(x-c)$, where $c \le 0$ and $\lambda \ge 0$.

We analyze two cases:

1. First, we consider $c = 0$. Thus, the polynomials $p_n^{(\alpha)}(\lambda, 0; x)$ are orthogonal with respect to

$$d\mu(\lambda, 0; x) = x^\alpha e^{-x}dx + \lambda\delta(x).$$

The measure

$$d\mu_1(x) = x^{\alpha+1}e^{-x}dx$$

satisfies a Pearson equation with (see (20))

$$\phi(x) = \sigma(x) = x, \quad \psi(x) = \bar{\sigma}(x) + \tau(x) = \alpha + 2 - x.$$

On the other hand, the structure relation (21) reads (see [40])

$$\phi(x)D\left(p_n^{(\alpha+1)}(x)\right) = A(x,n)p_n^{(\alpha+1)}(x) + B(x,n)p_{n-1}^{(\alpha+1)}(x),$$

where

$$\phi(x) = x, \quad A(x,n) = n, \quad B(x,n) = n + \alpha + 1.$$

The coefficients (13) and (22) are

$$\begin{aligned}
\tilde{\gamma}_n &= n(n+\alpha+1), \\
c_n &= -\frac{1+\lambda K_n(0,0)}{1+\lambda K_{n-1}(0,0)} \frac{p_{n-1}^{(\alpha)}(0)}{p_n^{(\alpha)}(0)} n(n+\alpha) \\
&= \frac{n!\Gamma(\alpha+1)\Gamma(\alpha+2) + \lambda\Gamma(n+\alpha+2)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2) + \lambda\Gamma(n+\alpha+1)}
\end{aligned}$$

As a conclusion, $Q(x,n)$ in (31) becomes

$$Q(x,n) = n(n+\alpha+1) - c_n(2n+1+\alpha-c_n) + c_n x$$

with zero

$$u_n = (2n+1+\alpha-c_n) - \frac{n(n+\alpha+1)}{c_n}.$$

It is easy to see that $0 < c_n < n+\alpha+1$. Thus, $Q(0,n) < 0$ and it implies that $u_n > 0$.

The electrostatic interpretation of the distribution of zeros means that we have an equilibrium position under the presence of an external potential

$$\ln Q(x,n) - \ln x^{\alpha+2}e^{-x},$$

where the first term represents a short range potential corresponding to a unit charge located at u_n and the second one is a long range potential associated with the weight function (see also [18] and [19]).

2. Now, we take $c < 0$. In this case $d\mu_1(x) = (x-c)x^{\alpha}e^{-x}dx$. Thus,

the structure relation *(21)* for $d\mu_1(x)$ is

$$\phi(x)D(q_n(c;x)) = A(x,n)q_n(c;x) + B(x,n)q_{n-1}(c;x),$$

where

$$\begin{aligned}
\phi(x) &= (x-c)x,\\
A(x,n) &= n\left[x - (n+1+a_n)\left(1+\frac{n+\alpha}{a_{n-1}}\right)\right],\\
B(x,n) &= \frac{n\,(n+\alpha)}{a_{n-1}}\left[a_n x - (n+1+a_n)\,(n+1+a_n+\alpha)\right].
\end{aligned}$$

From (4) we obtain

$$(x-c)\,q_n(c;x) = p_{n+1}^{(\alpha)}(x) - \frac{p_{n+1}^{(\alpha)}(c)}{p_n^{(\alpha)}(c)}p_n^{(\alpha)}(x).$$

Taking derivatives with respect to x in both hand sides of the above expression, and multiplying the resulting expression by x, we derive

$$xq_n(c;x) + x\,(x-c)\,D\,(q_n(c;x)) = xD\left(p_{n+1}^{(\alpha)}(x)\right) - \frac{p_{n+1}^{(\alpha)}(c)}{p_n^{(\alpha)}(c)}xD\left(p_n^{(\alpha)}(x)\right).$$

Using the structure relation and the three therm recurrence relation for Laguerre polynomials we obtain

$$\begin{aligned}
xq_n(c;x) + x\,(x-c)\,D\,(q_n(c;x)) = &\\
&(n+1)\,p_{n+1}^{(\alpha)}(x) + (n+1)\,(n+1+\alpha)\,p_n^{(\alpha)}(x)\\
&\qquad - \frac{p_{n+1}^{(\alpha)}(c)}{p_n^{(\alpha)}(c)}\left[np_n^{(\alpha)}(x) + n\,(n+\alpha)\,p_{n-1}^{(\alpha)}(x)\right],
\end{aligned}$$

or

$$xq_n(c;x) + x\,(x-c)\,Dq_n(c;x) = \left[(n+1)\,(x-n) - \frac{np_{n+1}^{(\alpha)}(c)}{p_n^{(\alpha)}(c)}\right] p_n^{(\alpha)}(x)$$
$$- \left(n\,(n+1)\,(n+\alpha) + n\,(n+\alpha)\,\frac{p_{n+1}^{(\alpha)}(c)}{p_n^{(\alpha)}(c)}\right) p_{n-1}^{(\alpha)}(x).$$

Put

$$a_n = \frac{p_{n+1}^{(\alpha)}\,(c)}{p_n^{(\alpha)}\,(c)}$$

and, from (15),

$$p_n^{(\alpha)}(x) = q_n(c;x) - \frac{\gamma_n}{a_{n-1}} q_{n-1}(c;x),$$

we have

$$xq_n(c;x) + x\,(x-c)\,D\,(q_n(c;x)) = ((x-n)\,(n+1) - na_n)\,q_n(c;x)$$
$$- \left[((n+1)\,(x-n) - na_n)\,\frac{n\,(n+\alpha)}{a_{n-1}} + n\,(n+\alpha)\,(n+1+a_n)\right] q_{n-1}(c;x)$$
$$+ n\,(n-1)\,(n+\alpha)\,(n-1+\alpha)\,(n+1+a_n)\,\frac{q_{n-2}(c;x)}{a_{n-2}}.$$

Using (22) and (23)

$$\frac{\tilde{\gamma}_{n-1}}{\gamma_{n-1}} = \frac{p_n^{(\alpha)}(c)}{p_{n-1}^{(\alpha)}(c)}\frac{p_{n-2}^{(\alpha)}(c)}{p_{n-1}^{(\alpha)}(c)}$$

we obtain

$$\frac{a_{n-1}}{a_{n-2}} = \frac{\tilde{\gamma}_{n-1}}{(n-1)\,(n-1+\alpha)}\frac{(n-1)\,(n-1+\alpha)}{a_{n-2}} = \frac{\tilde{\gamma}_{n-1}}{a_{n-1}}$$

and, then,

$$xq_n(c;x) + x\,(x-c)\,D\,(q_n(c;x)) =$$
$$\left[((n+1)\,(x-n) - na_n) - \frac{n\,(n+\alpha)\,(n+1+a_n)}{a_{n-1}}\right] q_n(c;x)$$
$$+ n\,(n+\alpha)\left[\frac{1}{a_{n-1}}\,(n+1+a_n)\left(x - \tilde{\beta}_{n-1}\right)\right.$$
$$\left. - \frac{1}{a_{n-1}}\,((n+1)\,(x-n) - na_n) - (n+1+a_n)\right] q_{n-1}(c;x).$$

Therefore,

$$\phi(x)D(q_n(c;x)) = A(x,n)q_n(c;x) + B(x,n)q_{n-1}(c;x),$$

where

$$
\begin{aligned}
\phi(x) &= x\,(x-c) \\
A(x,n) &= ((n+1)\,(x-n) - n a_n) - \frac{n\,(n+\alpha)\,(n+1+a_n)}{a_{n-1}} - x \\
B(x,n) &= n\,(n+\alpha)\left[\frac{1}{a_{n-1}}\,(n+1+a_n)\left(x-\tilde{\beta}_{n-1}\right)\right. \\
&\qquad \left. - \frac{1}{a_{n-1}}\,((n+1)\,(x-n) - n a_n) - (n+1+a_n)\right].
\end{aligned}
$$

Simplifying these expressions we derive

$$
\begin{aligned}
A(x,n) &= n\left[x - (n+1+a_n)\left(1+\frac{n+\alpha}{a_{n-1}}\right)\right], \\
B(x,n) &= n\,(n+\alpha)\left[\frac{a_n}{a_{n-1}}x + \frac{n+1+a_n}{a_{n-1}}\left(n-\tilde{\beta}_{n-1}\right) - (n+1+a_n)\right].
\end{aligned}
$$

In the last expression, using again (23)

$$
\tilde{\beta}_n = \beta_{n+1} + a_{n+1} - a_n = 2n + \alpha + 3 + a_{n+1} - a_n
$$

we obtain

$$
B(x,n) = \frac{n\,(n+\alpha)}{a_{n-1}}\left[a_n x - (n+1+a_n)\,(n+1+a_n+\alpha)\right].
$$

This is an alternative approach to the method described in [29]. Notice that the Pearson equation for the linear functional associated with the measure becomes

$$
D\left(\phi u_1\right) = \psi u_1
$$

where (see (19))

$$
\begin{aligned}
\phi(x) &= (x-c)\sigma(x) = (x-c)x, \\
\psi(x) &= 2\sigma(x) + (x-c)\tau(x) = 2x + (x-c)\,(\alpha+1-x).
\end{aligned}
$$

According to (23) and (24),

$$
\tilde{\beta}_{n-1} = \beta_n + a_n - a_{n-1}, \;\; \text{and} \;\; \tilde{\gamma}_{n-1} = \frac{a_{n-1}}{a_{n-2}}\gamma_{n-1}.
$$

This means that $Q(x,n)$ in (31) is the following quadratic polynomial

$$
Q\,(x,n) = c_n x^2 + r_n x + s_n
$$

with

$$
\begin{aligned}
r_n &= n\,(n+\alpha)\,\frac{a_n}{a_{n-1}} + c_n^2 - c_n\,(c+\alpha+1+2n) \\
&= (c_n+a_n)\,(c_n-a_n) - (c_n-a_n)\,c - (c_n+a_n)\,(2n+\alpha+1)
\end{aligned}
$$

and

$$
\begin{aligned}
s_n = (n+1+a_n)\,[(n+1+a_n+\alpha)\,(2n+1+a_n+\alpha-c-2c_n)+2cc_n] \\
+ c\alpha c_n + c_n^2\,(a_n-a_{n-1}+1-c)\,.
\end{aligned}
$$

The zeros of this polynomial are

$$
\begin{aligned}
z_{1,n} &= -\frac{1}{2c_n}\left(r_n+\sqrt{r_n^2-4s_nc_n}\right), \\
z_{2,n} &= -\frac{1}{2c_n}\left(r_n-\sqrt{r_n^2-4s_nc_n}\right).
\end{aligned}
$$

Taking into account

$$
\frac{\psi(x)}{\phi(x)} = \frac{2}{x-c} + \frac{\alpha+1}{x} - 1,
$$

the electrostatic interpretation means that the equilibrium position for the zeros under the presence of an external potential

$$
\ln\, Q(x,n) - \ln\, (x-c)^2\, x^{\alpha+1} e^{-x},
$$

where the first one is a short range potential corresponding to two unit charges located at $z_{1,n}$ and $z_{2,n}$ and the second one is a long range potential associated with a polynomial perturbation of the weight function.

4.3 Electrostatic interpretation for the zeros of Jacobi type orthogonal polynomials

We furnish an electrostatic interpretation for the zeros of Jacobi type polynomials $p_n^{(\alpha,\beta)}(\lambda, c; x)$ which are orthogonal with respect to the measure

$$
d\mu(\lambda, c; x) = (1-x)^{\alpha}(1+x)^{\beta}dx + \lambda\delta(x-c),
$$

with $c \notin (-1,1)$ and $\lambda \geq 0$.

For this propose we separate in two cases:

1. First, we consider $c=-1$. Thus, the polynomials $p_n^{(\alpha,\beta)}(\lambda, -1; x)$ are orthogonal with respect to

$$
d\mu(\lambda, -1; x) = (1-x)^{\alpha}(1+x)^{\beta}dx + \lambda\delta(x+1).
$$

The measure

$$d\mu_1(x) = (x-(-1))d\mu(x) = (1-x)^{\alpha}(1+x)^{\beta+1}dx$$

satisfies a Pearson equation with (see (20))

$$\phi(x) = \sigma(x) = 1-x^2, \quad \psi(x) = \bar{\sigma}(x) + \tau(x) = (\beta-\alpha+1) - (\alpha+\beta+3)\,x.$$

On the other hand, the structure relation (21) reads

$$\phi(x)D\left(p_n^{(\alpha,\beta+1)}(x)\right) = A(x,n)p_n^{(\alpha,\beta+1)}(x) + B(x,n)p_{n-1}^{(\alpha,\beta+1)}(x),$$

where

$$\begin{aligned} A(x,n) &= \frac{-n[\beta-\alpha+1+(2n+\alpha+\beta+1)x]}{2n+\alpha+\beta+1}, \\ B(x,n) &= \frac{4n(n+\alpha)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)^2(2n+\alpha+\beta)}. \end{aligned}$$

The coefficient $\tilde{\gamma}_n$ in (22) when $q_n(-1;x) = p_n^{(\alpha,\beta+1)}(x)$ is

$$\tilde{\gamma}_n = \gamma_n^{\alpha,\beta+1} = \frac{4n(n+\alpha)(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)^2(2n+\alpha+\beta+2)}$$

and

$$c_n = \frac{1+\lambda K_n(-1,-1)}{1+\lambda K_{n-1}(-1,-1)}\,\frac{2n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} > 0,$$

Thus,

$$\begin{aligned} Q(x,n) &= B(x,n) + \\ & c_n\left[(2n+\alpha+\beta)c_n - \frac{(\alpha+\beta+1)(\beta-\alpha+1)}{2n+\alpha+\beta+1}\right] + (2n+\alpha+\beta+1)c_n x. \end{aligned}$$

Observe that the zero of $Q(x,n)$ belongs to $(-1,1)$. In fact, after some tedious calculations we see that

$$\begin{aligned} & Q(1) = \\ & B(1,n) + c_n\left[(2n+\alpha+\beta)c_n + \frac{2(2n(n+\alpha+\beta+1)+\alpha(\alpha+\beta+1))}{2n+\alpha+\beta+1}\right] > 0, \end{aligned}$$

and

$$\begin{aligned} Q(-1) &= \frac{-2^{\alpha+\beta+3}(\beta+1)\Gamma(n)\Gamma(n+\alpha)\Gamma(\beta+2)^2\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)\lambda}{2n+\alpha+\beta} \\ &\times \frac{1}{2^{\alpha+\beta+1}\Gamma(n)\Gamma(n+\alpha)\Gamma(\beta+1)\Gamma(\beta+2)+\lambda\Gamma(n+\beta+1)\Gamma(n+\alpha+\beta+1)} < 0. \end{aligned}$$

Using some known properties of the Jacobi polynomials we conclude that

$$u_n = -1 + 2\lambda \frac{n(n+\alpha)\left(p_n^{(\alpha,\beta)}(-1)\right)^2 \Big/ \left\|p_n^{(\alpha,\beta)}\right\|_\mu^2}{(2n+\alpha+\beta+1)^2\left(1+\lambda K_n(-1,-1)\right)}$$
$$\times \left[\frac{\left(p_n^{(\alpha,\beta)}(-1)\right)^2 \Big/ \left\|p_n^{(\alpha,\beta)}\right\|_\mu^2}{K_{n-1}(-1,-1)\left(1+\lambda K_{n-1}(-1,-1)\right)}\right].$$

The electrostatic interpretation means that the equilibrium position for the zeros under the presence of an external potential

$$\ln Q(x,n) - \ln\,(1-x)^{\alpha+1}(1+x)^{\beta+2},$$

where the first one is a short range potential corresponding to a unit charge located at the zero of $Q(x,n)$ and the other one is a long range potential associated with the weight function.

2. We take $c < -1$. Then,

$$d\mu_1(x) = (x-c)(1-x)^\alpha(1+x)^\beta dx$$

and the structure relation *(21)* for the above measure is

$$\phi(x)D(q_n(c;x)) = A(x,n)q_n(c;x) + B(x,n)q_{n-1}(c;x),$$

where

$$\begin{aligned}
\phi(x) \quad &= (x-c)(1-x^2),\\
A(x,n) &= a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{1}{\lambda_{n-1}} - 1 + x^2,\\
B(x,n) &= \left(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n\right)\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\\
&\quad -(a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}.
\end{aligned}$$

Put

$$\lambda_n = \lambda_n^{\alpha,\beta}(c) = \frac{p_{n+1}^{(\alpha,\beta)}(c)}{p_n^{(\alpha,\beta)}(c)}. \tag{35}$$

From (4) we obtain

$$(x-c)\,q_n(c;x) = p_{n+1}^{(\alpha,\beta)}(x) - \lambda_n p_n^{(\alpha,\beta)}(x).$$

Taking derivatives with respect to x in both hand sides of the above expression, and multiplying them by $1-x^2$, we see that

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\left(q_n(c;x)\right)\\
&= (1-x^2)D\left(p_{n+1}^{(\alpha,\beta)}(x)\right) - \lambda_n(1-x^2)D\left(p_n^{(\alpha,\beta)}(x)\right).
\end{aligned}
$$

Since

$$
(1-x^2)D\left(p_n^{(\alpha,\beta)}(x)\right) = a_n p_n^{(\alpha,\beta)}(x) + b_n p_{n-1}^{(\alpha,\beta)}(x),
$$

where

$$
\begin{aligned}
a_n &= a_n^{\alpha,\beta}(x) = \frac{-n[\beta-\alpha+(2n+\alpha+\beta)x]}{2n+\alpha+\beta},\\
b_n &= b_n^{\alpha,\beta} = \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)},
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\left(q_n(c;x)\right)\\
&= a_{n+1}p_{n+1}^{(\alpha,\beta)}(x) + \left(b_{n+1}-\lambda_n a_n\right)p_n^{(\alpha,\beta)}(x) - \lambda_n b_n p_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}
$$

The three terms recurrence relation of monic Jacobi polynomials implies

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\left(q_n(c;x)\right)\\
&= \left[a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n\right]p_n^{(\alpha,\beta)}(x) - (a_{n+1}\gamma_n + \lambda_n b_n)p_{n-1}^{(\alpha,\beta)}(x).
\end{aligned}
$$

From (15) and (35),

$$
p_n^{(\alpha,\beta)}(x) = q_n(c;x) - \frac{\gamma_n}{\lambda_{n-1}}q_{n-1}(c;x).
$$

Then

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\left(q_n(c;x)\right)\\
&= \left[a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n\right]q_n(c;x)\\
&\quad - \left[\left(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n\right)\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\right]q_{n-1}(c;x)\\
&\quad (a_{n+1}\gamma_n + \lambda_n b_n)\frac{\gamma_{n-1}}{\lambda_{n-2}}q_{n-2}(c;x).
\end{aligned}
$$

From (22) for monic kernels,

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\,(q_n(c;x))\\
&= [a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n]\, q_n(c;x)\\
&\quad - \left[(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\right] q_{n-1}(c;x)\\
&\quad (a_{n+1}\gamma_n + \lambda_n b_n)\frac{\gamma_{n-1}}{\lambda_{n-2}}\,\frac{(x-\tilde{\beta}_{n-1})q_{n-1}(c;x) - q_n(c;x)}{\tilde{\gamma}_{n-1}}.
\end{aligned}
$$

According to (23) and (35), we obtain

$$
\frac{\tilde{\gamma}_{n-1}}{\gamma_{n-1}} = \frac{\lambda_{n-1}}{\lambda_{n-2}}.
$$

Therefore

$$
\begin{aligned}
&(1-x^2)q_n(c;x) + (x-c)\,(1-x^2)D\,(q_n(c;x))\\
&= \left[a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\tfrac{1}{\lambda_{n-1}}\right] q_n(c;x)\\
&\quad - \Big[(a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\tfrac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\\
&\qquad\qquad - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}\Big]\, q_{n-1}(c;x).
\end{aligned}
$$

Thus

$$
\phi(x)D(q_n(c;x)) = A(x,n)q_n(c;x) + B(x,n)q_{n-1}(c;x),
$$

where

$$
\begin{aligned}
\phi(x) &= (x-c)(1-x^2),\\
A(x,n) &= a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{1}{\lambda_{n-1}} - 1 + x^2,\\
B(x,n) &= (a_{n+1}(x-\beta_n) + b_{n+1} - \lambda_n a_n)\,\frac{\gamma_n}{\lambda_{n-1}} + a_{n+1}\gamma_n + \lambda_n b_n\\
&\qquad - (a_{n+1}\gamma_n + \lambda_n b_n)\frac{x-\tilde{\beta}_{n-1}}{\lambda_{n-1}}.
\end{aligned}
$$

Simplifying these expressions we have

$$
A(x,n) = A_{n,0} + A_{n,1}x + A_{n,2}x^2
$$

and

$$
B(x,n) = B_{n,0} + B_{n,1}x + B_{n,2}x^2.
$$

Notice that the Pearson equation for the linear functional associated with the measure

$$d\mu_1(x) = (x-c)(1-x)^{\alpha}(1+x)^{\beta}dx$$

becomes

$$D(\phi u_1) = \psi u_1,$$

with (see (19))

$$\phi(x) = (x-c)\sigma(x) = (x-c)(1-x^2),$$

$$\psi(x) = 2\sigma(x) + (x-c)\tau(x) = 2(1-x^2) + (x-c)(\beta-\alpha-(\alpha+\beta+2)x),$$

which means that $Q(x,n)$ is the following quadratic polynomial

$$\begin{aligned} Q(x,n) &= B(x,n) + c_n\left[-2A(x,n) + \phi'(x) - \psi(x) + \frac{c_n}{\tilde{\gamma}_{n-1}}B(x,n-1)\right] \\ &= \left[B_{n,2} + \left(\alpha+\beta+1-2A_{n,2} + \frac{c_n B_{n-1,2}}{\tilde{\gamma}_{n-1}}\right)c_n\right]x^2 \\ &\quad + \left\{\frac{c_n^2 B_{n-1,1}}{\tilde{\gamma}_{n-1}} + B_{n,1} - [\alpha(c-1)+\beta(c+1)+2A_{n,1}]c_n\right\}x \\ &\quad + B_{n,0} - [2A_{n,0}+1+c(\alpha-\beta)]c_n + \frac{c_n^2 B_{n-1,0}}{\tilde{\gamma}_{n-1}}. \end{aligned}$$

Taking into account

$$\frac{\psi(x)}{\phi(x)} = \frac{2}{x-c} - \frac{\alpha+1}{1-x} + \frac{\beta+1}{1+x},$$

the electrostatic interpretation means that the equilibrium position for the zeros under the presence of an external potential

$$\ln Q(x,n) - \ln (x-c)^2 (1-x)^{\alpha+1}(1+x)^{\beta+1},$$

where the first one is a short range potential corresponding to two unit charges located at the zeros of $Q(x,n)$ and the second one is a long range potential associated with a polynomial perturbation of the weight function.

References

[1] R. Álvarez-Nodarse, F. Marcellán, and J. Petronilho, *WKB Approximation and Krall-type Orthogonal Polynomials*, Acta Appl. Math. **54** (1998), 27–58.

[2] C. Bernardi and Y. Maday, *Approximations spectrales de problèmes aux limites elliptiques*. Mathematics and Applications Vol. **10**, Springer Verlag, Paris. 1992. In French.

[3] C. F. Bracciali, D. K. Dimitrov, and A. Sri Ranga, *Chain sequences and symmetric generalized orthogonal polynomials*, J. Comput. Appl. Math. **143** (2002), 95–106.

[4] M. I. Bueno and F. Marcellán, *Darboux transformations and perturbation of linear functionals*, Lin. Alg. Appl., **384**, 2004, 215-242.

[5] T. S. Chihara, *An Introduction to Orthogonal Polynomials.* Mathematics and its Applications Series, Gordon and Breach, New York, 1978.

[6] D. K. Dimitrov, F. Marcellán, and F. R. Rafaeli, *Monotonicity of zeros of Laguerre-Sobolev type orthogonal polynomials*, J. Math. Anal. Appl. **368** (2010), 80–89.

[7] D. K. Dimitrov, M. V. Mello, and F. R. Rafaeli, *Monotonicity of zeros of Jacobi-Sobolev type orthogonal polynomials*, Appl. Numer. Math. **60** (2010), 263–276.

[8] D. K. Dimitrov and W Van Assche, *Lamé differential equations and electrostatics*, Proc. Amer. Math. Soc. **128** (2000), 3621—3628.

[9] J. Dini and P. Maroni, *La multiplication d' une forme linéaire par une forme rationnelle. Application aux polynômes de Laguerre-Hahn*, Ann. Polon. Math. **52** (1990), 175–185.

[10] H. Dueñas and F. Marcellán, *Laguerre-Type orthogonal polynomials. Electrostatic interpretation*, Int. J. Pure and Appl. Math. **38** (2007), 345–358.

[11] H. Dueñas and F. Marcellán, *Jacobi-Type orthogonal polynomials: holonomic equation and electrostatic interpretation*, Comm. Anal. Theory Cont. Frac. **15** (2008), 4–19.

[12] W. Gautschi, *Orthogonal Polynomials: Computation and Approximation*, in Numerical Mathematics and Scientific Computation Series, Oxford University Press. New York. 2004.

[13] F. A. Grünbaum, *Variations on a theme of Heine and Stieltjes: an electrostatic interpretation of the zeros of certain polynomials*, J. Comput. Appl. Math. **99** (1998), 189–194.

[14] F. A. Grünbaum, *Electrostatic interpretation for the zeros of certain polynomials and the Darboux process*, J. Comput. Appl. Math. **133** (2001), 397—412 .

[15] F. A. Grünbaum, L. Haine, and E. Horozov, *On the Krall-Hermite and Krall-Bessel polynomials*, Internat. Math. Res. Notices **19** (1997), 956–966.

[16] J. J. Guadalupe , M. Pérez, F. J. Ruiz, and J. L. Varona, *Asymptotic behaviour of ortoghonal polynomials relative to measures with mass points*, Mathematika **40** (1993), 331–344.

[17] M. E. H. Ismail, *An electrostatics model for zeros of general orthogonal polynomials*, Pacific J. Math. **193** (2000), 355–369.

[18] M. E. H. Ismail, *More on electrostatic models for zeros of orthogonal polynomials*, Numer. Funct. Anal. Optimiz. **21** (2000), 191–204.

[19] M. E. H. Ismail, *Classical and Quantum Orthogonal Polynomials in One Variable*, Encyclopedia of Mathematics and its Applications, Vol. **98**. Cambridge University Press, Cambridge UK. 2005.

[20] J. Koekoek and R. Koekoek, *On a differential equation for Koornwinder's generalized Laguerre polynomials*, Proc. Amer. Math. Soc. **112** (1991), 1045–1054.

[21] J. Koekoek and R. Koekoek, *Differential equations for generalized Jacobi polynomials*, J. Comput. Appl. Math. **126** (2000), 1–31.

[22] R. Koekoek, *Generalizations of classical Laguerre polynomials and some q-analogues*, Doctoral Dissertation, Techn. Univ. of Delft, The Netherlands, 1990.

[23] T. H. Koornwinder, *Orthogonal polynomials with weight function* $(1-x)^{\alpha}(1+x)^{\beta}+M\delta(x+1)+N\delta(x-1)$, Canad. Math. Bull. **27** (1984), 205–214.

[24] A. M. Krall, *Orthogonal Polynomials satisfying fourth order differential equations*, Proc. Roy. Soc. Edinburgh Sect A **87** (1980/81), 271–288.

[25] H. L. Krall, *On orthogonal polynomials satisfying a certain fourth order differential equation*, Pennsylvania State College Studies **6** (1940). 24 pp.

[26] F. Marcellán and P. Maroni, *Sur l' adjonction d' une masse de Dirac à une forme réguliére et semi-classique* Annal. Mat. Pura Appl. **162** (1992), 1–22.

[27] F. Marcellán, A. Martínez-Finkelshtein and P. Martínez-González, *Electrostatic models for zeros of polynomials: Old, new and some open problems. J. Comput. Appl. Math.* **207** (2007), 258—272 .

[28] F. Marcellán and F. R. Rafaeli, *A Note on monotonicity of zeros of generalized Hermite-Sobolev type orthogonal polynomials.* Integral Transforms Spec. Funct. **21** (2010), 831–838.

[29] F. Marcellan and A. Ronveaux, *Differential equations for classical type orthogonal polynomials*, Canad. Math. Bull. **32** (1989), 404–411.

[30] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*, in *Orthogonal Polynomials and Their Applications*, C. Brezinski et al. Editors. Annals. Comput. Appl. Math. **9**. Baltzer, Basel. 1991, 95–130.

[31] A. F. Nikiforov and V. B. Uvarov, *Special Functions of Mathematical Physics: An unified approach*, Birkhauser Verlag, Basel. 1988.

[32] F. Peherstorfer, *Linear combination of orthogonal polynomials generating positive quadrature formulas,* Math. Comput. **55** (1990), 231—241 .

[33] F. R. Rafaeli, *Zeros de Polinômios Ortogonais na Reta Real*, Doctoral Dissertation, State University of Campinas - UNICAMP, 2010. In Portuguese.

[34] J. Shohat, *A differential equation for orthogonal polynomials*, Duke Math. J. **5** (1939), 401–417.

[35] H. Stahl and V. Totik, *General Orthogonal Polynomials*, Encyclopedia of Mathematics and its Applications Vol. **43**, Cambridge University Press, Cambridge UK. 1992.

[36] T. J. Stieltjes, *Sur certain polynômes que verifient une equation différentielle linéaire du second ordre et sur la théorie des fonctions de Lamé,* Acta Math. **6** (1885), 321—326 .

[37] T. J. Stieltjes, *Sur quelques théorèmes d'algèbre,* Comptes Rendus de l'Académie des Sciences, Paris **100** (1885), 439—440 .

[38] T. J. Stieltjes, *Sur les polynômes de Jacobi,* Comptes Rendus de l'Académie des Sciences, Paris 100, 620—622 (1885).

[39] T. J. Stieltjes, *Sur les racines de l'équation* $X_n = 0$, Acta Math. **9** (1886), 385—400.

[40] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Coll. Publ, Vol. **23**, 4th ed., Amer. Math. Soc., Providence, RI, 1975.

[41] V. B. Uvarov, *The connection between systems of polynomials that are orthogonal with respect to different distribution functions*, J. Comput. Math. and Math. Phys. **9** (1969), 25–36.

[42] A. Zhedanov, *Rational spectral transformations and orthogonal polynomials.* J. Comput. Appl. Math. **85** (1997), 67–83.

[43] G. J. Yoon, *Darboux transforms and orthogonal polynomials*, Bull. Korean Math. Soc. **39** (2002), 359–376