Asymptotic behaviour of the Laguerre-Sobolev-Type Orthogonal Polynomials. A nondiagonal case.

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To professor Adhemar Bultheel with occasion of his 60^{th} ***birthday***

Abstract

In this paper we study the asymptotic behaviour of polynomials orthogonal with respect to a Sobolev-Type inner product

$$\langle p, q\rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A\mathbb{Q}(0),\ \alpha > -1,$$

where p and q are polynomials with real coefficients,

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},\ \mathbb{P}(0) = \begin{pmatrix} p(0) \\ p'(0) \end{pmatrix},\ \mathbb{Q}(0) = \begin{pmatrix} q(0) \\ q'(0) \end{pmatrix},$$

and A is a positive semidefinite matrix.

We will focus our attention on their outer relative asymptotics with respect to the standard Laguerre polynomials as well as on an analog of the Mehler-Heine formula for the rescaled polynomials.

Key words: Quasi-orthogonal polynomials, Laguerre Polynomials, Sobolev type inner products, outer relative asymptotics, Bessel functions, Mehler-Heine formula.
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1 Introduction.

Orthogonal polynomials with respect to a Sobolev-Type inner product

$$\langle p, q\rangle = \int_{\mathbb{R}} p(x)q(x)d\mu(x) + \mathbb{P}(c)^t A\mathbb{Q}(c), \tag{1}$$

where $d\mu$ is a nontrivial probability measure supported on the real line, $A \in \mathbb{R}^{(k,k)}$ is a positive semidefinite matrix, p, q are polynomials with real coefficients, and $\mathbb{Q}(c) = \left(q(c), q'(c), \ldots, q^{(k-1)}(c)\right)^t$ have been introduced in [1].

When $A = diag\,(M_0, M_1, \ldots, M_{k-1})$, the so-called diagonal Sobolev-Type case, many researchers were interested in the analytic properties of the polynomials orthogonal respect with (1). In particular, R. Koekoek [10] studied the second order linear differential equation satisfied by such orthogonal polynomials when $d\mu = x^\alpha e^{-x}dx$, $\alpha > -1$, and $c = 0$. They also satisfy a higher order recurrence relation as well as they can be represented as hypergeometric functions.

Later on, when $k = 2$ and $M_0, M_1 > 0$, in [11] the authors focus the attention in the location of the zeros of such orthogonal polynomials that are called Laguerre-Sobolev Type orthogonal polynomials. Finally, the analysis of their asymptotic properties was done in [3] as well in [13].

On the other hand, when $k \geq 2$ if $d\mu = x^\alpha e^{-x}dx$, $c = 0$, and $M_0 = M_1 = \cdots = M_{k-2} = 0$, $M_{k-1} > 0$ then the same analog problems were studied in [15] in the framework of the zero distribution. From an algebraic point of view and for more general measures, in [14] the authors deal with representations of Sobolev-Type orthogonal polynomials in terms of the polynomials orthogonal with respect to the measure μ assuming the same constraints for the inner product (1) as above.

The first situation of a non-diagonal Sobolev type inner product like (1) was considered in [2]. Here the authors deal with the measure $d\mu = e^{-x^2}dx$ supported on $\mathbb{R}$, $c = 0$, and $k = 2$. In particular, they analyze scaled asymptotics for the corresponding orthogonal polynomials (Mehler-Heine formulas) and, as a consequence, the asymptotic behaviour of their zeros follows.

Taking into account that generalized Hermite polynomials appear as a consequence of the symmetrization process for Laguerre orthogonal polynomials ([4],[6], and [16]) it seems to be very natural to analyze polynomial sequences orthogonal with respect to the inner product (1) when $d\mu = x^\alpha e^{-x}dx$, $A \in \mathbb{R}^{(k,k)}$ is a nondiagonal positive semi-definite matrix with $k \geq 2$, and $c = 0$.

In this contribution we focus our attention in the case $k = 2$. Thus we generalize some previous results from the diagonal case (see [3], [8], and [11]) as well as we give a nice interpretation of some results of [2] using a symmetrization process for our Laguerre-Sobolev type orthogonal polynomials.

The structure of the manuscript is the following. In section 2 we present the basic background about the properties of classical Laguerre polynomials which will be needed along the paper. Section 3 deals with the asymptotic properties of the Laguerre-Sobolev type polynomials, orthogonal with respect to the inner product

$$\langle p, q \rangle = \int_0^\infty p(x)q(x)x^\alpha e^{-x}dx + \mathbb{P}(0)^t A \mathbb{Q}(0), \ \alpha > -1,$$

where $A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix}$ is a positive semidefinite matrix and we denote $\mathbb{Q}(0) = (q(0), q'(0))^t$. We obtain the outer relative asymptotics of these polynomials in terms of Laguerre polynomials and a Mehler-Heine type formula as well as the behaviour of the Sobolev norm of the monic Laguerre-Sobolev type orthogonal polynomials.

2 Preliminaries.

Let $\{\mu_n\}_{n\geq 0}$ be a sequence of real numbers and let μ be the linear functional defined in the linear space $\mathbb{P}$ of the polynomials with real coefficients, such that

$$\langle \mu, x^n \rangle = \mu_n, \ n = 0, 1, 2, \ldots$$

μ is said to be a *moment functional* associated with $\{\mu_n\}_{n\geq 0}$. Furthermore μ_n is the $n-th$ *moment* of the functional μ.

Given a moment functional μ, a sequence of polynomials $\{P_n\}_{n\geq 0}$ is said to be a sequence of *orthogonal polynomials* with respect to μ if

(i) The degree of P_n is n.
(ii) $\langle \mu, P_n(x)P_m(x) \rangle = 0$, $m \neq n$.
(iii) $\langle \mu, P_n^2(x) \rangle \neq 0$, $n = 0, 1, 2, \ldots$

If every polynomial $P_n(x)$ has 1 as leading coefficient, then $\{P_n\}_{n\geq 0}$ is said to be a sequence of *monic orthogonal polynomials.*

The next theorem, whose proof appears in [6] , gives necessary and sufficient conditions for the existence of a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ with respect to a moment functional μ associated with $\{\mu_n\}_{n\geq 0}$.

Theorem 1 *([6]) Let μ be the moment functional associated with $\{\mu_n\}_{n\geq 0}$. There exists a sequence of monic orthogonal polynomials $\{P_n\}_{n\geq 0}$ associated with μ if and only if the leading principal submatrices of the Hankel matrix $[\mu_{i+j}]_{i,j\in\mathbb{N}}$ are non singular.*

A moment functional such that there exists the correspondient sequence of orthogonal polynomials is said to be *regular* or *quasi-definite* ([6]).

The proof of the next proposition can be founded in [4], [6], [9], [12], and [16].

Proposition 1 *(The Christoffel-Darboux formula). Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials. If we denote the n-th kernel polynomial by*

$$K_n(x,y) = \sum_{j=0}^{n} \frac{P_j(y)P_j(x)}{\left\langle \mu, P_j^2 \right\rangle},$$

then, for every $n \in \mathbb{N}$,

$$K_n(x,y) = \frac{1}{\langle \mu, P_n^2\rangle} \frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y}. \tag{2}$$

Using the following notation for the partial derivatives of the kernel $K_n(x,y)$

$$\frac{\partial^{j+k}\left(K_n(x,y)\right)}{\partial^j x \partial^k y} = K_n^{(j,k)}(x,y),$$

we present some properties about these derivatives. Let $\{P_n\}_{n\geq 0}$ be a sequence of monic orthogonal polynomials. From the Christoffel-Darboux Formula (2) , we have

$$K_{n-1}(x,y) = \frac{1}{\langle \mu, P_{n-1}^2\rangle} \frac{P_n(x)P_{n-1}(y) - P_{n-1}(x)P_n(y)}{x-y}.$$

The computation of the $j-th$ partial derivative with respect to y yields

$$K_{n-1}^{(0,j)}(x,y) = \frac{1}{\langle \mu, P_{n-1}^2\rangle} \left(P_n(x) \frac{\partial^j}{\partial y^j}\left(\frac{P_{n-1}(y)}{x-y}\right) - P_{n-1}(x)\frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x-y}\right)\right). \tag{3}$$

Using the Leibnitz rule

$$\frac{\partial^j}{\partial y^j}\left(\frac{P_n(y)}{x-y}\right)=\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}},$$

and replacing the last expression in (3), we get

$$\begin{aligned}K_{n-1}^{(0,j)}(x,y) &= \frac{1}{\langle\mu,P_{n-1}^2\rangle}\left(P_n(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_{n-1}^{(k)}(y)}{(x-y)^{j-k+1}}-P_{n-1}(x)\sum_{k=0}^{j}\frac{j!}{k!}\frac{P_n^{(k)}(y)}{(x-y)^{j-k+1}}\right)\\ &= \frac{j!}{\langle\mu,P_{n-1}^2\rangle\,(x-y)^{j+1}}\times\\ &\quad\left(P_n(x)\sum_{k=0}^{j}\frac{1}{k!}P_{n-1}^{(k)}(y)(x-y)^k-P_{n-1}(x)\sum_{k=0}^{j}\frac{1}{k!}P_n^{(k)}(y)(x-y)^k\right).\end{aligned}$$

As a consequence,

Proposition 2 *([1], [14]) For every $n\in\mathbb{N}$,*

$$K_{n-1}^{(0,j)}(x,0)=\frac{j!}{\langle\mu,P_{n-1}^2\rangle\,x^{j+1}}\left(P_n(x)Q_j(x,0;P_{n-1})-P_{n-1}(x)Q_j(x,0;P_n)\right)\quad(4)$$

where $Q_j(x,0;P_{n-1})$ and $Q_j(x,0;P_n)$ denote the Taylor Polynomials of degree j of the polynomials P_{n-1} and P_n around $x=0$, respectively.

The Laguerre orthogonal polynomials are defined as the polynomials orthogonal with respect to the inner product

$$\langle p,q\rangle=\int_0^\infty pqx^\alpha e^{-x}dx,\ \ \alpha>-1,\ \ p,q\in\mathbb{P}.\quad(5)$$

We will summarize some properties of the Laguerre monic orthogonal polynomials that we will use in the sequel. The details of the proof of Proposition 3 and the Theorem 2, can be founded in [4], [6], [9], [12], and [16].

Proposition 3 *Let $\{L_n^\alpha\}_{n\geq 0}$ be the sequence of Laguerre monic orthogonal polynomials.*

(1) For every $n\in\mathbb{N}$,

$$xL_n^\alpha(x)=L_{n+1}^\alpha(x)+(2n+1+\alpha)\,L_n^\alpha(x)+n\,(n+\alpha)\,L_{n-1}^\alpha(x),\quad(6)$$

with $L_0^\alpha(x)=1, L_1^\alpha(x)=x-\ (\alpha+1)$.

(2) For every $n \in \mathbb{N}$,

$$L_n^\alpha(x) = L_n^{\alpha+1}(x) + nL_{n-1}^{\alpha+1}(x). \tag{7}$$

(3) For every $n \in \mathbb{N}$,

$$\|L_n^\alpha\|_\alpha^2 = n!\Gamma(n+\alpha+1). \tag{8}$$

(4) For every $n \in \mathbb{N}$

$$L_n^\alpha(0) = (-1)^n \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}. \tag{9}$$

(5) For every $n \in \mathbb{N}$

$$(L_n^\alpha)'(x) = nL_{n-1}^{\alpha+1}(x). \tag{10}$$

(6) For every $n \in \mathbb{N}$,

$$x\left(L_n^\alpha(x)\right)' = nL_n^\alpha(x) + n(n+\alpha)L_{n-1}^\alpha(x). \tag{11}$$

In particular, for Laguerre polynomials we get

Proposition 4 *For every* $n \in \mathbb{N}$

$$K_{n-1}(x,0) = \frac{L_{n-1}^\alpha(0)L_{n-1}^{\alpha+1}(x)}{(n-1)!\Gamma(n+\alpha)}, \tag{12}$$

$$K_{n-1}^{(0,1)}(x,0) = \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)}L_{n-1}^{\alpha+2}(x) + \frac{(-1)^n n}{(n-2)!\Gamma(\alpha+2)}L_{n-2}^{\alpha+2}(x), \tag{13}$$

$$K_{n-1}^{(1,1)}(x,0) = \frac{(-1)^n(n-1)}{(n-2)!\Gamma(\alpha+2)}L_{n-2}^{\alpha+3}(x) + \frac{(-1)^n n}{(n-3)!\Gamma(\alpha+2)}L_{n-3}^{\alpha+3}(x). \tag{14}$$

The proof of (12) is given in [7]. For (13) see [8]. Finally, (14) is a consequence of (13) and (7).

Using (8) and (9) in (12), (13), and (14) we obtain

Proposition 5 *For every* $n \in \mathbb{N}$,

$$K_{n-1}(0,0) = \frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)\Gamma(\alpha+2)}, \tag{15}$$

$$K_{n-1}^{(1,0)}(0,0) = -\frac{\Gamma(n+\alpha+1)}{(n-2)!\Gamma(\alpha+1)\Gamma(\alpha+3)} = -\frac{n-1}{\alpha+2}K_{n-1}(0,0), \tag{16}$$

$$K_{n-1}^{(1,1)}(0,0) = \frac{\Gamma(n+\alpha+1)\left(n(\alpha+2)-(\alpha+1)\right)}{(n-2)!\Gamma(\alpha+2)\Gamma(\alpha+4)} = \frac{\left(n(\alpha+2)-(\alpha+1)\right)(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}K_{n-1}(0,0). \tag{17}$$

Theorem 2 *(The Mehler-Heine type formula) Let J_α be the Bessel function defined by*

$$J_\alpha(x) = \sum_{j=0}^{\infty} \frac{(-1)^j (x/2)^{2j+\alpha}}{j!\Gamma(j+\alpha+1)},$$

then

$$\lim_{n\to\infty} \frac{\widehat{L}_n^\alpha(x/(n+j))}{n^\alpha} = x^{-\alpha/2} J_\alpha(2\sqrt{x}) \tag{18}$$

uniformly on compact subsets $\mathbb{C}$ and uniformly in $j \in \mathbb{N} \cup \{0\}$. Here $\widehat{L}_n^\alpha(x) = (-1)^n/n! L_n^\alpha(x)$.

3 Asymptotic behaviour

If p is a polynomial with real coefficients, then we will denote

$$\mathbb{P}(x) = \begin{pmatrix} p(x) \\ p'(x) \end{pmatrix}.$$

Let p and q be polynomials with real coefficients. We define the following Sobolev type inner product

$$\langle p, q \rangle_S = \int_0^\infty p(x)q(x)x^\alpha e^{-x} dx + \mathbb{P}(0)^t A \mathbb{Q}(0), \;\; \alpha > -1, \tag{19}$$

where

$$A = \begin{pmatrix} M_0 & \lambda \\ \lambda & M_1 \end{pmatrix},$$

$M_0, M_1 \geq 0$, A is a positive semidefinite matrix, i.e $\det A = |A| \geq 0$. Notice that if $M_0 = 0$, $M_1 > 0$ or $M_1 = 0, M_0 > 0$ it implies that $\lambda = 0$. These situations have been considered in some previous papers by the authors (see [7] and [8]), as well as in [3] and [8].

From (19), $\langle p, q \rangle_S$ is an inner product in the linear space $\mathbb{P}$ of polynomials with real coefficients in the sense that

(1) $\langle \lambda p + \mu q, r \rangle = \lambda \langle p, r \rangle_S + \mu \langle q, r \rangle$, for $p, q, r \in \mathbb{P}$ and $\lambda, \mu \in \mathbb{R}$.
(2) $\langle p, q \rangle_S = \langle q, p \rangle_S$ for $p, q \in \mathbb{P}$.
(3) $\langle p, p \rangle_S > 0$, for every $p \in \mathbb{P} \backslash \{0\}$.

Let $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ be the sequence of monic polynomials orthogonal with respect to (19). Consider the Fourier expansion of $\widetilde{L}_n^\alpha$ in terms of the sequence of Laguerre monic orthogonal polynomials $\{L_n^\alpha\}_{n\geq 0}$

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) + \sum_{k=0}^{n-1} a_{n,k} L_k^\alpha(x),$$

where

$$a_{n,k} = \frac{\left\langle \widetilde{L}_n^\alpha(x), L_k^\alpha(x)\right\rangle_\alpha}{\|L_k^\alpha\|_\alpha^2},\ 0 \leq k \leq n-1.$$

From (19), we get

$$a_{n,k} = -\frac{\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \mathbb{L}_k^\alpha(0)}{\|L_k^\alpha\|_\alpha^2}.$$

As a consequence,

$$\begin{aligned}
\widetilde{L}_n^\alpha(x) &= L_n^\alpha(x) - \sum_{k=0}^{n-1} \frac{\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \mathbb{L}_k^\alpha(0)}{\|L_k^\alpha\|_\alpha^2} L_k^\alpha(x) \\
&= L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \sum_{k=0}^{n-1} \frac{\mathbb{L}_k^\alpha(0) L_k^\alpha(x)}{\|L_k^\alpha\|_\alpha^2},
\end{aligned}$$

i.e

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} K_{n-1}(x,0) \\ K_{n-1}^{(0,1)}(x,0) \end{pmatrix}. \tag{20}$$

From the above expression we obtain

$$\begin{aligned}
\widetilde{L}_n^\alpha(0) &= L_n^\alpha(0) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} K_{n-1}(0,0) \\ K_{n-1}^{(0,1)}(0,0) \end{pmatrix}, \\
\left(\widetilde{L}_n^\alpha\right)'(0) &= \left(L_n^\alpha\right)'(0) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} K_{n-1}^{(1,0)}(0,0) \\ K_{n-1}^{(1,1)}(0,0) \end{pmatrix}.
\end{aligned}$$

Thus

$$\left(\widetilde{\mathbb{L}}^\alpha(0)\right)^t = \left(\mathbb{L}^\alpha(0)\right)^t - \left(\widetilde{\mathbb{L}}^\alpha(0)\right)^t A \mathbb{K}_{n-1}(0,0), \tag{21}$$

where

$$\mathbb{K}_{n-1}(0,0) = \begin{pmatrix} K_{n-1}(0,0) & K_{n-1}^{(1,0)}(0,0) \\ K_{n-1}^{(0,1)}(0,0) & K_{n-1}^{(1,1)}(0,0) \end{pmatrix}.$$

As a consequence, from (21)

$$\left(\widetilde{\mathbb{L}}^{\alpha}(0)\right)^{t}(I + A\mathbb{K}_{n-1}(0,0)) = (\mathbb{L}^{\alpha}(0))^{t}, \tag{22}$$

where I is the 2×2 identity matrix. Notice that

$$I + A\mathbb{K}_{n-1}(0,0) =$$

$$K_{n-1}(0,0)\left[\begin{pmatrix} \frac{1}{K_{n-1}(0,0)} & 0 \\ 0 & \frac{1}{K_{n-1}(0,0)} \end{pmatrix} + A\begin{pmatrix} 1 & -\frac{n-1}{\alpha+2} \\ -\frac{n-1}{\alpha+2} & \frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)} \end{pmatrix}\right]$$
$$= K_{n-1}(0,0)\begin{pmatrix} G & H \\ J & K \end{pmatrix},$$

where

$$\begin{aligned}
G &= \frac{1}{K_{n-1}(0,0)} + \left(M_0 + \frac{\lambda}{\alpha+2}\right) - \frac{n\lambda}{\alpha+2} \\
H &= \frac{\lambda n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{M_0}{\alpha+2} + \frac{(2\alpha+3)\lambda}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)n + \\
&\quad \left(\frac{M_0}{\alpha+2} + \frac{\lambda}{(\alpha+2)(\alpha+3)}\right) \\
J &= -\frac{M_1}{\alpha+2}n + \left(\lambda + \frac{M_1}{\alpha+2}\right) \\
K &= \frac{M_1 n^2}{(\alpha+1)(\alpha+3)} - \left(\frac{\lambda}{\alpha+2} + \frac{(2\alpha+3)M_1}{(\alpha+1)(\alpha+2)(\alpha+3)}\right)n + \\
&\quad \left(\frac{\lambda}{\alpha+2} + \frac{M_1}{(\alpha+2)(\alpha+3)}\right) + \frac{1}{K_{n-1}(0,0)}.
\end{aligned}$$

On the other hand

$$|I + A\mathbb{K}_{n-1}(0)| =$$

$$(K_{n-1}(0,0))^2\left[\frac{1}{(K_{n-1}(0,0))^2}+\frac{1}{K_{n-1}(0,0)}traza\left[A\begin{pmatrix}1 & -\frac{n-1}{\alpha+2}\\ -\frac{n-1}{\alpha+2} & \frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}\end{pmatrix}\right]\right.$$

$$\left.+|A|\left(\frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{(n-1)^2}{(\alpha+2)^2}\right)\right]$$

$$=1+K_{n-1}(0,0)\left(M_1\frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{2\lambda}{\alpha+2}(n-1)+M_0\right)+$$

$$|A|\frac{n-1}{\alpha+2}\left(\frac{(n(\alpha+2)-(\alpha+1))}{(\alpha+1)(\alpha+3)}-\frac{n-1}{\alpha+2}\right)(K_{n-1}(0,0))^2$$

$$=1+K_{n-1}(0,0)\left(M_1\frac{(n(\alpha+2)-(\alpha+1))(n-1)}{(\alpha+1)(\alpha+2)(\alpha+3)}-\frac{2\lambda}{\alpha+2}(n-1)+M_0\right)+$$

$$(K_{n-1}(0,0))^2|A|\frac{n-1}{\alpha+2}\left(\frac{n}{(\alpha+1)(\alpha+2)(\alpha+3)}+\frac{1}{(\alpha+2)(\alpha+3)}\right).$$

Thus, if $|A|>0$ we get

$$|I+A\mathbb{K}_{n-1}(0,0)|\sim\frac{|A|\,n^{2\alpha+4}}{(\alpha+1)(\alpha+2)^2(\alpha+3)},\tag{23}$$

and, if $|A|=0, M_1>0$,

$$|I+A\mathbb{K}_{n-1}(0,0)|\sim\frac{n^{\alpha+3}M_1}{(\alpha+1)(\alpha+3)}.\tag{24}$$

As a consequence, from (20) and (22)

$$\widetilde{L}_n^{\alpha}(x)=$$

$$L_n^{\alpha}(x)-(\mathbb{L}_n^{\alpha}(0))^t(I+A\mathbb{K}_{n-1}(0))^{-1}A\begin{pmatrix}\frac{(-1)^{n-1}(\alpha+1)}{(n-1)!\Gamma(\alpha+2)} & 0\\ \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} & \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)}\end{pmatrix}\begin{pmatrix}L_{n-1}^{\alpha+1}(x)\\ L_{n-2}^{\alpha+2}(x)\end{pmatrix}$$

$$=L_n^{\alpha}(x)-\frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)K_{n-1}(0,0)}\begin{pmatrix}(-1)^n\frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\\ (-1)^{n-1}\frac{n\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)}\end{pmatrix}^t\begin{pmatrix}G & H\\ J & K\end{pmatrix}^{-1}A\times$$

$$\begin{pmatrix}-\frac{\alpha+1}{n-1} & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}L_{n-1}^{\alpha+1}(x)\\ L_{n-2}^{\alpha+2}(x)\end{pmatrix},$$

thus

$$\widetilde{L}_n^{\alpha}(x)=\tag{25}$$

$$= L_n^{\alpha}(x) + \begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix}^t \begin{pmatrix} G & H \\ J & K \end{pmatrix}^{-1} A \begin{pmatrix} -(\alpha+1) & 0 \\ n-1 & n-1 \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix}.$$

Furthermore, if we denote

$$M = \begin{pmatrix} G & H \\ J & K \end{pmatrix},$$

we get

$$M^{-1} = \frac{1}{|M|} \begin{pmatrix} K & -H \\ -J & G \end{pmatrix},$$

where

$$|M| = \frac{1}{(K_{n-1}(0,0))^2} \left| I + A\mathbb{K}_{n-1}(0,0) \right|.$$

Therefore, from (25), after some computations we get

$$\widetilde{L}_n^{\alpha}(x) = \tag{26}$$

$$= L_n^{\alpha}(x) + \frac{1}{|M|} \begin{pmatrix} \widetilde{A}_n n^2 + B_n n + C_n & \widetilde{A}'_n n^2 + B'_n n + C'_n \end{pmatrix} \begin{pmatrix} L_{n-1}^{\alpha+1}(x) \\ L_{n-2}^{\alpha+2}(x) \end{pmatrix},$$

with

$$\begin{aligned}
\widetilde{A}_n &= \frac{2\,|A|}{(\alpha+1)(\alpha+2)(\alpha+3)} + \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
B_n &= \frac{2\alpha\,|A|}{(\alpha+1)(\alpha+2)(\alpha+3)} - \frac{2\lambda}{K_{n-1}(0,0)} - \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
\widetilde{A}'_n &= \frac{|A|}{(\alpha+1)(\alpha+2)} + \frac{M_1}{(\alpha+1)K_{n-1}(0,0)} \\
B'_n &= \frac{\alpha\,|A|}{(\alpha+1)(\alpha+2)} - \frac{\lambda}{K_{n-1}(0,0)} - \frac{M_1}{(\alpha+1)K_{n-1}(0,0)},
\end{aligned}$$

and C_n and C'_n depend of M_0, M_1, λ, and α.

Let

$$\widehat{L}_n^\alpha(x) = \frac{(-1)^n}{n!} L_n^\alpha(x)$$
$$\widehat{\widetilde{L}}_n^\alpha(x) = \frac{(-1)^n}{n!} \widetilde{L}_n^\alpha(x),$$

then, from (26)

$$\widehat{\widetilde{L}}_n^\alpha(x) = \tag{27}$$

$$\widehat{L}_n^\alpha(x) + \frac{1}{|M|} \begin{pmatrix} \widetilde{A}_n n^2 + B_n n + C_n & \widetilde{A}'_n n^2 + B'_n n + C'_n \end{pmatrix} \begin{pmatrix} -\frac{1}{n} \widehat{L}_{n-1}^{\alpha+1}(x) \\ \frac{1}{n(n-1)} \widehat{L}_{n-2}^{\alpha+2}(x) \end{pmatrix}$$
$$= \widehat{L}_n^\alpha(x) + \frac{1}{|M|} \begin{pmatrix} \widetilde{A}_n n + B_n + \frac{C_n}{n} & \widetilde{A}'_n n + B'_n + \frac{C'_n}{n} \end{pmatrix} \begin{pmatrix} -\widehat{L}_{n-1}^{\alpha+1}(x) \\ \frac{1}{n-1} \widehat{L}_{n-2}^{\alpha+2}(x) \end{pmatrix}.$$

On the other hand, since

$$|M| = \tag{28}$$

$$\frac{1}{(K_{n-1}(0,0))^2} + \frac{1}{(K_{n-1}(0,0))} \left(\frac{M_1}{(\alpha+1)(\alpha+3)} n^2 + Rn + T \right)$$
$$+ |A| \left(\frac{n^2}{(\alpha+1)(\alpha+2)^2(\alpha+3)} + R'n + T' \right),$$

where R, T, R', and T'depend only of M_0, M_1, λ, and α; and, assuming that $|A| > 0$, we get

$$|M| \sim \frac{|A|}{(\alpha+1)(\alpha+2)^2(\alpha+3)} n^2.$$

Therefore

$$\widehat{\widetilde{L}}_n^\alpha(x) \tag{29}$$

$$\sim \widehat{L}_n^\alpha(x) + \frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{n^2 |A|} \begin{pmatrix} \widetilde{A}_n n + B_n + \frac{C_n}{n} & \widetilde{A}'_n n + B'_n + \frac{C'_n}{n} \end{pmatrix} \begin{pmatrix} -\widehat{L}_{n-1}^{\alpha+1}(x) \\ \frac{1}{n-1} \widehat{L}_{n-2}^{\alpha+2}(x) \end{pmatrix}.$$

As a consequence, for $x \in \mathbb{C} \backslash [0, \infty)$

$$\frac{\widehat{\widetilde{L}}_n^\alpha(x)}{\widehat{L}_n^\alpha(x)}$$

$$\sim 1+\frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{n^2\,|A|}\left(\widetilde{A}_n n+B_n+\tfrac{C_n}{n}\ \ \widetilde{A}'_n n+B'_n+\tfrac{C'_n}{n}\right)\begin{pmatrix}-\frac{\widehat{L}^{\alpha+1}_{n-1}(x)}{\widehat{L}^{\alpha}_n(x)}\\ \frac{1}{n-1}\frac{\widehat{L}^{\alpha+2}_{n-2}(x)}{\widehat{L}^{\alpha}_n(x)}\end{pmatrix}$$

$$=1+\frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{|A|}\left(\tfrac{\widetilde{A}_n}{n}+\tfrac{B_n}{n^2}+\tfrac{C_n}{n^3}\ \ \tfrac{\widetilde{A}'_n}{n}+\tfrac{B'_n}{n^2}+\tfrac{C'_n}{n^3}\right)\begin{pmatrix}-\frac{\widehat{L}^{\alpha+1}_{n-1}(x)}{\widehat{L}^{\alpha}_n(x)}\\ \frac{1}{n-1}\frac{\widehat{L}^{\alpha+2}_{n-2}(x)}{\widehat{L}^{\alpha}_n(x)}\end{pmatrix},$$

and taking into account

$$\lim_{n\to\infty}\frac{n^{(l-j)/2}\widehat{L}^{\alpha+j}_{n+k}(x)}{\widehat{L}^{\alpha+l}_{n+h}(x)}=(-x)^{-(j-l)/2}, \tag{30}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$, where $j,l\in\mathbb{R}$, $h,k\in\mathbb{Z}$, (see [3]) then

$$\frac{\widetilde{L}^{\alpha}_n(x)}{L^{\alpha}_n(x)}=1+\mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

On the other hand, if $|A|=0$, $M_1>0$, from (27) and (28)

$$\widehat{\widetilde{L}}^{\alpha}_n(x)$$

$$=\widehat{L}^{\alpha}_n(x)+\frac{1}{\frac{1}{(K_{n-1}(0,0))^2}+\frac{1}{(K_{n-1}(0,0))}\left(\frac{M_1}{(\alpha+1)(\alpha+3)}n^2+Rn+T\right)}\times$$
$$\left(\widetilde{A}_n n+B_n+\tfrac{C_n}{n}\ \ \widetilde{A}'_n n+B'_n+\tfrac{C'_n}{n}\right)\begin{pmatrix}-\widehat{L}^{\alpha+1}_{n-1}(x)\\ \frac{1}{n-1}\widehat{L}^{\alpha+2}_{n-2}(x)\end{pmatrix}.$$

thus, for $x\in\mathbb{C}\backslash[0,\infty)$

$$\frac{\widehat{\widetilde{L}}^{\alpha}_n(x)}{\widehat{L}^{\alpha}_n(x)}\sim 1+\frac{(\alpha+1)(\alpha+3)K_{n-1}(0,0)}{M_1}\left(\tfrac{\widetilde{A}_n}{n}+\tfrac{B_n}{n^2}+\tfrac{C_n}{n^3}\ \ \tfrac{\widetilde{A}'_n}{n}+\tfrac{B'_n}{n^2}+\tfrac{C'_n}{n^3}\right)\begin{pmatrix}-\frac{\widehat{L}^{\alpha+1}_{n-1}(x)}{\widehat{L}^{\alpha}_n(x)}\\ \frac{1}{n-1}\frac{\widehat{L}^{\alpha+2}_{n-2}(x)}{\widehat{L}^{\alpha}_n(x)}\end{pmatrix}.$$

Taking into account

$$\lim_{n\to\infty} K_{n-1}(0,0)\widetilde{A}_n = \frac{M_1}{\alpha+1}$$
$$\lim_{n\to\infty} K_{n-1}(0,0)\widetilde{A}'_n = \frac{M_1}{\alpha+1}$$
$$\lim_{n\to\infty} K_{n-1}(0,0)B_n = -2\lambda - \frac{M_1}{\alpha+1}$$
$$\lim_{n\to\infty} K_{n-1}(0,0)B'_n = -\lambda - \frac{M}{\alpha+1}$$
$$\lim_{n\to\infty} K_{n-1}(0,0)C_n = L_1$$
$$\lim_{n\to\infty} K_{n-1}(0,0)C'_n = L_2,$$

where L_1, L_2 are constants that do not depend of n, we obtain

$$\frac{\widehat{\widetilde{L}}{}_n^\alpha(x)}{\widehat{L}_n^\alpha(x)} = 1 + \mathcal{O}\left(\frac{1}{\sqrt{n}}\right),$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$. Thus

Theorem 3

$$\lim_{n\to\infty} \frac{\widetilde{L}_n^\alpha(x)}{L_n^\alpha(x)} = 1 \tag{31}$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,\infty)$.

We will find the corresponding Mehler-Heine formula for the Laguerre-Sobolev type orthogonal polynomials $\widetilde{L}_n^\alpha(x)$. As above, in the first case, we will assume that $|A| > 0$. From (29) we get

$$\begin{aligned}
&\frac{\widehat{\widetilde{L}}{}_n^\alpha(x/n)}{n^\alpha} \\
&\sim \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} + \frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{n^{\alpha+2}\,|A|}\begin{pmatrix} \widetilde{A}_n n + B_n + \frac{C_n}{n} & \widetilde{A}'_n n + B'_n + \frac{C'_n}{n}\end{pmatrix}\begin{pmatrix} -\widehat{L}_{n-1}^{\alpha+1}(x/n) \\ \frac{1}{n-1}\widehat{L}_{n-2}^{\alpha+2}(x/n)\end{pmatrix} \\
&= \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} + \frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{|A|}\begin{pmatrix} \widetilde{A}_n + \frac{B_n}{n} + \frac{C_n}{n^2} & \widetilde{A}'_n + \frac{B'_n}{n} + \frac{C'_n}{n^2}\end{pmatrix}\begin{pmatrix} -\frac{\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}} \\ \frac{n}{n-1}\frac{\widehat{L}_{n-2}^{\alpha+2}(x/n)}{n^{\alpha+2}},\end{pmatrix}
\end{aligned}$$

thus,

$$\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}{}_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}J_\alpha(2\sqrt{x}) +$$
$$\frac{(\alpha+1)(\alpha+2)^2(\alpha+3)}{|A|}\begin{pmatrix} \frac{2|A|}{(\alpha+1)(\alpha+2)(\alpha+3)} & \frac{|A|}{(\alpha+1)(\alpha+2)}\end{pmatrix}\begin{pmatrix} x^{-(\alpha+1)/2}J_{\alpha+1}(2\sqrt{x}) \\ x^{-(\alpha+2)/2}J_{\alpha+2}(2\sqrt{x})\end{pmatrix},$$

uniformly on compact subsets of $\mathbb{C}$. As a consequence, the second part of the previous expression is

$$x^{-\alpha/2}\left(J_{\alpha}(2\sqrt{x})-2(\alpha+2)x^{-1/2}J_{\alpha+1}(2\sqrt{x})+(\alpha+2)(\alpha+3)J_{\alpha+2}(2\sqrt{x})\right).$$

But, taking into account that

$$J_{\alpha}(2\sqrt{x})+J_{\alpha+2}(2\sqrt{x})=\frac{\alpha+1}{\sqrt{x}}J_{\alpha+1}(2\sqrt{x}),$$

then

$$\begin{aligned}
&x^{-\alpha/2}\left[J_{\alpha}(2\sqrt{x})-2(\alpha+2)x^{-1/2}J_{\alpha+1}(2\sqrt{x})+(\alpha+2)(\alpha+3)J_{\alpha+2}(2\sqrt{x})\right]\\
&=x^{-\alpha/2}\left[-\frac{(\alpha+3)}{\sqrt{x}}J_{\alpha+1}(2\sqrt{x})+\left(\frac{(\alpha+2)\,(\alpha+3)}{x}-1\right)J_{\alpha+2}(2\sqrt{x})\right]\\
&=x^{-\alpha/2}\left[-\frac{(\alpha+3)}{\sqrt{x}}\left(\frac{\alpha+2}{\sqrt{x}}J_{\alpha+2}(2\sqrt{x})-J_{\alpha+3}(2\sqrt{x})\right)+\right.\\
&\left.\left(\frac{(\alpha+2)\,(\alpha+3)}{x}-1\right)J_{\alpha+2}(2\sqrt{x})\right]\\
&=x^{-\alpha/2}\left[\frac{(\alpha+3)}{\sqrt{x}}J_{\alpha+3}(2\sqrt{x})-J_{\alpha+2}(2\sqrt{x})\right]\\
&=x^{-\alpha/2}J_{\alpha+4}(2\sqrt{x}).
\end{aligned}$$

Thus we get

Theorem 4 *Let* $\left\{\widehat{\widehat{L}}_n^{\alpha}\right\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to (19) and* $|A|>0$. *Then*

$$\lim_{n\to\infty}\frac{\widehat{\widehat{L}}_n^{\alpha}(x/n)}{n^{\alpha}}=x^{-\alpha/2}J_{\alpha+4}(2\sqrt{x}),\tag{32}$$

uniformly on compact subsets of $\mathbb{C}$.

Notice that the previous result coincides with [13] in the diagonal case, $M_0, M_1 > 0$.

Next, we will find the Mehler-Heine formula when $|A| = 0$, $M_1 > 0$. From (26),

$$\frac{\widehat{\widehat{L}}_n^{\alpha}(x/n)}{n^{\alpha}}$$

$$= \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} + \frac{1}{\frac{1}{(K_{n-1}(0,0))^2} + \frac{1}{(K_{n-1}(0,0))}\left(\frac{M_1}{(\alpha+1)(\alpha+3)}n^2 + Rn + T\right)} \times$$

$$\left(\widetilde{A}_n n + B_n + \frac{C_n}{n} \;\; \widetilde{A}'_n n + B'_n + \frac{C'_n}{n} \right) \begin{pmatrix} -n\frac{\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}} \\ \frac{n}{n-1}\frac{\widehat{L}_{n-2}^{\alpha+2}(x/n)}{n^{\alpha+2}} \end{pmatrix}$$

$$= \frac{\widehat{L}_n^\alpha(x/n)}{n^\alpha} + \frac{1}{\frac{1}{(K_{n-1}(0,0))^2} + \frac{1}{(K_{n-1}(0,0))}\left(\frac{M_1}{(\alpha+1)(\alpha+3)}n^2 + Rn + T\right)} \times$$

$$\left(-\widetilde{A}_n n^2 - B_n n - C_n \;\; \frac{n}{n-1}\left(\widetilde{A}'_n n^2 + B'_n n + C'_n\right) \right) \begin{pmatrix} -n\frac{\widehat{L}_{n-1}^{\alpha+1}(x/n)}{n^{\alpha+1}} \\ \frac{\widehat{L}_{n-2}^{\alpha+2}(x/n)}{n^{\alpha+2}} \end{pmatrix}$$

$$\to x^{-\alpha/2} J_\alpha(2\sqrt{x}) + \left(-(\alpha+3) \;\; \alpha+3 \right) \begin{pmatrix} x^{-(\alpha+1)/2} J_{\alpha+1}\left(2\sqrt{x}\right) \\ x^{-(\alpha+2)/2} J_{\alpha+2}\left(2\sqrt{x}\right) \end{pmatrix}$$

uniformly on compact subsets of $\mathbb{C}$. Then we get

Theorem 5 *Let* $\left\{\widehat{\widetilde{L}}_n^\alpha\right\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to (19) and assume* $|A| = 0, M_1 > 0$. *Then*

$$\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}_n^\alpha(x/n)}{n^\alpha} = x^{-\alpha/2}\left(J_\alpha(2\sqrt{x}) - \frac{\alpha+3}{\sqrt{x}} J_{\alpha+1}\left(2\sqrt{x}\right) + \frac{\alpha+3}{x} J_{\alpha+2}\left(2\sqrt{x}\right)\right),$$

uniformly on compact subsets of $\mathbb{C}$.

Notice that the previous result coincides with [3] and [8], where the case $M_0 = 0$ and $\lambda = 0$ is studied.

In order to find a scaled strong asymptotic formula, we will write the Laguerre-Sobolev type orthogonal polynomials $\widetilde{L}_n^\alpha(x)$ as a combination of the Laguerre monic orthogonal polynomials $L_n^{\alpha+2}(x), L_{n-1}^{\alpha+2}(x)$, and $L_{n-2}^{\alpha+2}(x)$. Replacing (12) and (13) in (20)

$$\begin{aligned}
\widetilde{L}_n^\alpha(x) &= L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} K_{n-1}(x,0) \\ K_{n-1}^{(0,1)}(x,0) \end{pmatrix} \\
&= L_n^\alpha(x) - \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} \frac{(-1)^{n-1}}{(n-1)!\Gamma(\alpha+1)} L_{n-1}^{\alpha+1}(x) \\ \frac{(-1)^n}{(n-2)!\Gamma(\alpha+2)} L_{n-1}^{\alpha+2}(x) + \frac{(-1)^n n}{(n-2)!\Gamma(\alpha+2)} L_{n-2}^{\alpha+2}(x) \end{pmatrix} \\
&= L_n^\alpha(x) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)}\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} -\frac{1}{n-1} L_{n-1}^{\alpha+1}(x) \\ \frac{1}{(\alpha+1)} L_{n-1}^{\alpha+2}(x) + \frac{n}{\alpha+1} L_{n-2}^{\alpha+2}(x) \end{pmatrix}.
\end{aligned}$$

From (7) we get

$$\widetilde{L}_n^\alpha(x) = L_n^\alpha(x) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} -\frac{1}{n-1}\left(L_{n-1}^{\alpha+2}(x) + (n-1)L_{n-2}^{\alpha+2}(x)\right) \\ \frac{1}{(\alpha+1)}L_{n-1}^{\alpha+2}(x) + \frac{n}{\alpha+1}L_{n-2}^{\alpha+2}(x) \end{pmatrix}$$
$$= L_n^\alpha(x) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \left[\begin{pmatrix} -\frac{1}{n-1} \\ \frac{1}{(\alpha+1)} \end{pmatrix} L_{n-1}^{\alpha+2}(x) + \begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix} L_{n-2}^{\alpha+2}(x) \right],$$

where

$$\left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t = \left(I + A\mathbb{K}_{n-1}(0,0)\right)^{-1}\left(\mathbb{L}_n^\alpha(0)\right)^t.$$

But from (7)

$$L_n^\alpha(x) = L_n^{\alpha+2}(x) + 2nL_{n-1}^{\alpha+2}(x) + n(n-1)L_{n-2}^{\alpha+2}(x).$$

As a consequence, we have the following

Theorem 6 *For every $n \in \mathbb{N}$*

$$\widetilde{L}_n^\alpha(x) = L_n^{\alpha+2}(x) + A_{n,\alpha}L_{n-1}^{\alpha+2}(x) + B_{n,\alpha}L_{n-2}^{\alpha+2}(x) \tag{33}$$

where

$$A_{n,\alpha} = 2n - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} -\frac{1}{n-1} \\ \frac{1}{(\alpha+1)} \end{pmatrix} \sim 2n - (\alpha+1)(\alpha+2)$$
$$B_{n,\alpha} = n(n-1) - \frac{(-1)^n}{(n-2)!\Gamma(\alpha+1)} \left(\widetilde{\mathbb{L}}_n^\alpha(0)\right)^t A \begin{pmatrix} -1 \\ \frac{n}{\alpha+1} \end{pmatrix} \sim n(n-1) - (\alpha+1)(\alpha+2)(n-1).$$

This means that the sequence $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ is quasi-orthogonal with respect to the Laguerre weight $d\mu_{\alpha+2} = x^{\alpha+2}e^{-x}dx$. See [5] for more information about quasi-orthogonal families, in particular, the analysis of the zero distribution.

Introducing the change of variable nx in (33), we get

$$\widehat{\widetilde{L}}_n^\alpha(nx) = \widehat{L}_n^{\alpha+2}(nx) - \frac{A_{n,\alpha}}{n}\widehat{L}_{n-1}^{\alpha+2}(nx) + \frac{B_{n,\alpha}}{n(n-1)}\widehat{L}_{n-2}^{\alpha+2}(nx).$$

From the definition of $A_{n,\alpha}$ and $B_{n,\alpha}$,

$$
\begin{aligned}
\frac{A_{n,\alpha}}{n} &= 2 - \frac{(\alpha+1)(\alpha+2)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\\
\frac{B_{n,\alpha}}{n(n-1)} &= 1 - \frac{(\alpha+1)(\alpha+2)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\widehat{\widehat{L}}{}_n^{\alpha}(nx) = {} & \widehat{L}_n^{\alpha+2}(nx) - 2\widehat{L}_{n-1}^{\alpha+2}(nx) + \widehat{L}_{n-2}^{\alpha+2}(nx)\\
& + \frac{(\alpha+1)(\alpha+2)}{n}\widehat{L}_{n-1}^{\alpha+2}(nx) - \frac{(\alpha+1)(\alpha+2)}{n}\widehat{L}_{n-2}^{\alpha+2}(nx) +\\
& - \widehat{L}_{n-1}^{\alpha+2}(nx)\mathcal{O}\left(\frac{1}{n^2}\right) + \widehat{L}_{n-2}^{\alpha+2}(nx)\mathcal{O}\left(\frac{1}{n^2}\right).
\end{aligned}
$$

From (7) we get that $\widehat{L}_n^{\alpha}(x) = \widehat{L}_n^{\alpha+2}(x) - 2\widehat{L}_{n-1}^{\alpha+2}(x) + \widehat{L}_{n-2}^{\alpha+2}(x)$, thus

$$
\frac{\widehat{\widehat{L}}{}_n^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = 1 + \frac{(\alpha+1)(\alpha+2)}{n}\frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} - \frac{(\alpha+1)(\alpha+2)}{n}\frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)} +
$$

$$
-\frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)}\mathcal{O}\left(\frac{1}{n^2}\right) + \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_n^{\alpha}(nx)}\mathcal{O}\left(\frac{1}{n^2}\right). \tag{34}
$$

We want to find the limit when n tends to ∞ in the left hand side of the previous identity. Using that (see [3] and [16])

$$
\lim_{n\to\infty} \frac{\widehat{L}_{n-1}^{\alpha}(nx)}{\widehat{L}_n^{\alpha}(nx)} = -\frac{1}{\varphi\left((x-2)/2\right)} \tag{35}
$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$, where φ is the mapping of $\mathbb{C}\backslash[-1,1]$ onto the exterior of the unit circle given by

$$
\varphi(x) = x + \sqrt{x^2-1},
$$

R.Alvarez-Nodarse and J. J. Moreno-Balcázar proved in [3] that

$$
\lim_{n\to\infty} \frac{\widehat{L}_n^{\alpha}(nx)}{\widehat{L}_{n-1}^{\alpha+2}(nx)} = -\frac{\left(\varphi\left(\left(x-2\right)/2\right)+1\right)^2}{\varphi\left(x-2\right)/2}. \tag{36}
$$

Then, using (35) and (36) we conclude that

$$
\begin{aligned}
\lim_{n\to\infty} \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_{n}^{\alpha}(nx)} &= \lim_{n\to\infty} \frac{\widehat{L}_{n-2}^{\alpha+2}(nx)}{\widehat{L}_{n-1}^{\alpha+2}(nx)} \frac{\widehat{L}_{n-1}^{\alpha+2}(nx)}{\widehat{L}_{n}^{\alpha}(nx)} \\
&= \left(-\frac{1}{\varphi\left((x-2)/2\right)}\right)\left(-\frac{\varphi\left(\left(x-2\right)/2\right)}{\left(\varphi\left(\left(x-2\right)/2\right)+1\right)^{2}}\right) \\
&= \frac{1}{\left(\varphi\left(\left(x-2\right)/2\right)+1\right)^{2}}
\end{aligned}
$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$. As a conclusion, from (34) we get the relative asymptotics for the scaled Laguerre-Sobolev-type orthogonal polynomials

Proposition 6 *For* $n \in \mathbb{N}$,

$$
\lim_{n\to\infty} \frac{\widehat{\widetilde{L}}_{n}^{\alpha}(nx)}{\widehat{L}_{n}^{\alpha}(nx)} = \lim_{n\to\infty} \frac{\widetilde{L}_{n}^{\alpha}(nx)}{L_{n}^{\alpha}(nx)} = 1 \tag{37}
$$

uniformly on compact subsets of $\mathbb{C}\backslash[0,4]$.

On the other hand, using (19) we get

$$
\left\|\widetilde{L}_{n}^{\alpha}\right\|_{S}^{2} = \|L_{n}^{\alpha}\|_{\alpha}^{2} + \mathbb{L}^{\alpha}(0)^{t}(I + A\mathbb{K}_{n-1}(0,0))^{-1}A\mathbb{L}^{\alpha}(0).
$$

If B is a nonsingular matrix, it is straightforward to prove that

$$
\begin{vmatrix} 0 & u^{t} \\ v & B \end{vmatrix} = -\left|B\right| u^{t}B^{-1}v
$$

where

$$
B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.
$$

Thus

$$
\begin{aligned}
\left\|\widetilde{L}_{n}^{\alpha}\right\|_{S}^{2} &= \|L_{n}^{\alpha}\|_{\alpha}^{2} - \frac{1}{|I + A\mathbb{K}_{n-1}(0,0)|} \begin{vmatrix} 0 & \mathbb{L}_{n}^{\alpha}(0)^{t} \\ A\mathbb{L}_{n}^{\alpha}(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \\
&= \frac{\|L_{n}^{\alpha}\|_{\alpha}^{2}}{|I + A\mathbb{K}_{n-1}(0,0)|} \left(|I + A\mathbb{K}_{n-1}(0,0)| + \begin{vmatrix} 0 & \mathbb{L}_{n}^{\alpha}(0)^{t}/\|L_{n}^{\alpha}\|_{\alpha}^{2} \\ -A\mathbb{L}_{n}^{\alpha}(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix} \right) \\
&= \frac{\|L_{n}^{\alpha}\|_{\alpha}^{2}}{|I + A\mathbb{K}_{n-1}(0,0)|} \begin{vmatrix} 1 & \mathbb{L}_{n}^{\alpha}(0)^{t}/\|L_{n}^{\alpha}\|_{\alpha}^{2} \\ -A\mathbb{L}_{n}^{\alpha}(0) & I + A\mathbb{K}_{n-1}(0,0) \end{vmatrix}.
\end{aligned}
$$

Finally, using the fact that

$$I + A\mathbb{K}_n(0,0) = I + A\mathbb{K}_{n-1}(0,0) + \frac{A}{\|L_n^\alpha\|_\alpha^2}\mathbb{L}_n^\alpha(0)\mathbb{L}_n^\alpha(0)^t,$$

then

$$\frac{\left\|\widetilde{L}_n^\alpha\right\|_S^2}{\|L_n^\alpha\|_\alpha^2} = \frac{|I + A\mathbb{K}_n(0,0)|}{|I + A\mathbb{K}_{n-1}(0,0)|}. \tag{38}$$

Therefore using (38), (23), and (24) we get

Proposition 7 *Let* $\left\{\widetilde{L}_n^\alpha\right\}_{n\geq 0}$ *be the sequence of polynomials orthogonal with respect to (19). Then*

$$\lim_{n\to\infty}\frac{\left\|\widetilde{L}_n^\alpha\right\|_S^2}{\|L_n^\alpha\|_\alpha^2} = 1.$$

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