NON-SIMULTANEOUS BLOW-UP FOR A QUASILINEAR PARABOLIC SYSTEM WITH REACTION AT THE BOUNDARY

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Abstract. We study a system of two porous medium type equations in a bounded interval, coupled at the boundary in a nonlinear way. Under certain conditions, one of its components becomes unbounded in finite time while the other remains bounded, a situation that is known in the literature as non-simultaneous blow-up. We characterize completely, in the case of nondecreasing in time solutions, the set of parameters appearing in the system for which non-simultaneous blow-up indeed occurs. Moreover, we obtain the blow-up rate and the blow-up set for the component which blows up. We also prove that in the range of exponents where each of the components may blow up on its own there are special initial data such that blow-up is simultaneous. Finally, we give conditions on the exponents which lead to non-simultaneous blow-up for every initial data.

1. Introduction and Main Results. We devote our attention to the formation of singularities in finite time of solutions \((u, v)\) of a parabolic system of two porous medium type equations,

\[
\begin{align*}
\begin{cases}
    u_t &= (u^m)_{xx}, \\ 
    v_t &= (v^n)_{xx}, 
\end{cases} 
  & (x, t) \in D_T = (0, L) \times (0, T), \\
\begin{cases}
    -(u^m)_x(0, t) &= u^{p_{11}} v^{p_{12}}(0, t), \\
    -(v^n)_x(0, t) &= u^{p_{21}} v^{p_{22}}(0, t), 
\end{cases} 
  & t \in (0, T), \\
\begin{cases}
    (u^m)_x(L, t) &= 0, \\
    (v^n)_x(L, t) &= 0, 
\end{cases} 
  & t \in (0, T).
\end{align*}
\]

with a nonlinear coupling at one of the ends of the interval

\[
\begin{align*}
\begin{cases}
    -(u^m)_x(0, t) &= u^{p_{11}} v^{p_{12}}(0, t), \\
    -(v^n)_x(0, t) &= u^{p_{21}} v^{p_{22}}(0, t), 
\end{cases} 
  & t \in (0, T), \\
\begin{cases}
    (u^m)_x(L, t) &= 0, \\
    (v^n)_x(L, t) &= 0, 
\end{cases} 
  & t \in (0, T).
\end{align*}
\]

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The initial data
\[
\begin{align*}
\begin{cases}
  u(x, 0) = u_0(x), \\
  v(x, 0) = v_0(x),
\end{cases}
\end{align*}
\] for \( x \in (0, L) \)

are assumed to be continuous and bounded. We consider all possible parameters satisfying \( m, n > 0 \), \( p_{ij} \geq 0 \). In this range the diffusivities may become degenerate or singular at level zero. Moreover, the reaction terms may not be Lipschitz, leading to non-uniqueness phenomena, [4]. To avoid the technicalities to which these difficulties may lead, we will assume that \( u_0, v_0 \geq \delta > 0 \). Since we are interested in the behaviour of the system for large values of the solutions, this is not a significant restriction. We will also assume that the initial data are compatible with the boundary conditions, so that solutions may be (and will be) understood in a classical sense.

If we extend solutions to (1.1)–(1.4) to the interval \((0, 2L)\) by symmetry, we get a solution to the same problem with the condition at \( x = L \), (1.3), substituted by a condition at \( x = 2L \),
\[
\begin{align*}
\begin{cases}
  (u^n)_x(2L, t) = u^{p_{11}}v^{p_{12}}(2L, t), \\
  (v^n)_x(2L, t) = u^{p_{21}}v^{p_{22}}(2L, t),
\end{cases}
\end{align*}
\quad t \in (0, T).
\]

Conversely, symmetric solutions to this latter problem are solutions to our original problem (1.1)–(1.4). This allows to translate many results of one of the problems to the other.

The time \( T \) denotes the maximal existence time for the solution \((u, v)\). If it is infinite we say that the solution is \textit{global}. If it is finite we say that the solution \textit{blows up}.

In the last years there has been an increasing interest in the study of blow-up due to reaction at the boundary, both for scalar problems and for systems, see for example the surveys [2], [7] and the references therein.

It is known that nontrivial solutions of (1.1)–(1.4) blow up if and only if the exponents \( p_{ij} \) verify any of the following conditions,
\[
\begin{align*}
  p_{11} &> \min \left\{ 1, \frac{m+1}{2} \right\}, \\
  p_{22} &> \min \left\{ 1, \frac{n+1}{2} \right\}, \\
  p_{12}p_{21} &> \left( \min \left\{ 1, \frac{m+1}{2} \right\} - p_{11} \right) \left( \min \left\{ 1, \frac{n+1}{2} \right\} - p_{22} \right),
\end{align*}
\]
see [18], [19]. In this case we have
\[
\limsup_{t \to T} \left\{ \|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \right\} = \infty.
\]
However, a priori there is no reason why both functions, \( u \) and \( v \), should go to infinity simultaneously at time \( T \). Indeed, as we will show, for certain choices of the parameters \( p_{ij} \) there are initial data for which one of the components of the system remains bounded while the other blows up. This phenomenon is known in the literature as \textit{non-simultaneous blow-up}. The possibility of non-simultaneous blow-up in nonlinear parabolic systems was first mentioned in [16], and has been studied more thoroughly later in [14], [15] and [17]. For problem (1.1)–(1.4) it was analyzed in [12] in the particular case \( m = n = 1 \).

The aim of this paper is to characterize the range of parameters for which non-simultaneous blow-up occurs under an extra condition on the monotonicity of the solution:
\[
(H) \quad (u, v) \text{ verifies } u_t, v_t \geq 0.
\]
This hypothesis is satisfied, for example, if \((u_m^n)'' \geq 0\) and \((v_n^0)'' \geq 0\). In the sequel we assume, without further mention, that \((H)\) holds.

Our first result gives a necessary and sufficient condition for the existence of non-simultaneous blow-up.

**Theorem 1.1.** (i) Let

\[ 2p_{21} < \max \{p_{11} - 1, 2p_{11} - (m + 1)\}. \tag{1.5} \]

Then, given \(v_0\), there exist initial data \(u_0\) such that \(u\) blows up while \(v\) remains bounded.

(ii) If \(u\) blows up and \(v\) remains bounded, then \((1.5)\) holds.

Since \(p_{21} \geq 0\), in order to have non-simultaneous blow-up we need in particular that \(p_{11} > \min\{1, (m + 1)/2\}\), see \[8\]. Thus \(u\) can blow up by itself, without the help of \(v\). Condition \((1.5)\) says that \(p_{21}\) (which measures the influence of \(u\) in the equation for \(v\)) is small compared with \(p_{11}\) (which measures the capacity of \(u\) to blow up by itself); hence, when \(u\) blows up, it does not necessarily carry \(v\) along with it.

By interchanging the roles of \(u\) and \(v\), we get that the condition in order to have solutions such that \(v\) blows up while \(u\) remains bounded reads

\[ 2p_{12} < \max \{p_{22} - 1, 2p_{22} - (n + 1)\}. \tag{1.6} \]

It is remarkable that the non-simultaneous blow-up condition \((1.5)\) depends strongly on the diffusivity through the parameter \(m\). Indeed, a dramatic change occurs across the value \(p_{11} = m\). The main reason behind this fact is that in the case of non-simultaneous blow-up the blow-up rate of \(u\), that is, the speed at which it goes to infinity at time \(T\), also changes across this value.

**Theorem 1.2.** If \(u\) blows up at time \(T\) and \(v\) remains bounded, then there exist positive constants, \(C_1, C_2\), such that for \(0 < t < T\)

\[ C_1 \leq (T - t)^\gamma u(0, t) \leq C_2, \quad \gamma = 1/\max \{p_{11} - 1, 2p_{11} - (m + 1)\}. \tag{1.7} \]

In \[15\] the authors study conditions for non-simultaneous blow-up for the same pair of equations defined in \(\mathbb{R}_+ = (0, \infty)\) instead of an interval, with the same boundary conditions at the origin. In their case the non-simultaneous blow-up condition just reads \(2p_{21} < 2p_{11} - (m + 1)\), which is equal to the one we obtain when \(p_{11} > m\). The reason for this coincidence is the following: in this range the blow-up set of \(u\),

\[ B(u) = \{x : \text{there exist } x_n \to x, t_n \not\to T \text{ with } u(x_n, t_n) \to \infty\}, \]

when blow-up is non-simultaneous, contains only the point \(x = 0\). Hence the singularity does not see what happens away from the origin. Therefore, the boundary condition at \(x = L\) does not contribute to the blow-up behaviour of problem \((1.1)\)–\((1.4)\), when \(p_{11} > m\).

**Theorem 1.3.** If \(u\) blows up and \(v\) remains bounded, then the blow-up set of \(u\) is given by

\[ B(u) = \begin{cases} 
\{0\} & \text{if } p_{11} > m, \\
[0, L] \subseteq [0, L] & \text{if } p_{11} = m, \\
[0, L] & \text{if } p_{11} < m.
\end{cases} \]
The first case, where blow-up is localized at just one point, is known as single point blow-up. When the solution blows up at all the points where the problem is defined, we say that blow-up is global. If the blow-up set is an interval strictly contained in \([0, L]\), blow-up is regional.

When non-simultaneous blow-up is possible, the set of initial data for which it actually occurs is expected to be rather big. We will show that it is an open set in the \(L^\infty\)-topology. We conjecture that it should be a dense set, at least for some range of exponents.

**Theorem 1.4.** Let (1.5) hold. The set of initial data such that \(u\) blows up and \(v\) remains bounded is open in the \(L^\infty\)-topology.

However, we do not exclude the possibility of exceptional solutions with simultaneous blow-up. As an example, if (1.5) and (1.6) both hold at the same time, that is, if each of the components may blow up on its own, there exist initial data for which simultaneous blow-up indeed occurs.

**Theorem 1.5.** Let (1.5) and (1.6) both hold at the same time. Then, there is a solution \((u, v)\) that blows up simultaneously.

Nevertheless, sometimes blow-up is always non-simultaneous. This is the case if \(v\) cannot blow up without the help of \(u\).

**Theorem 1.6.** Let (1.5) hold. If \(p_{22} \leq \min\{1, (n + 1)/2\}\), then \(u\) blows up and \(v\) remains bounded for every initial data.

We think that blow-up should be also always non-simultaneous when \(p_{22} > \min\{1, (n + 1)/2\}\) and condition (1.5) holds, but (1.6) does not. This range of exponents allows non-simultaneous blow-up and also the possibility of both components to blow-up without the help of the other one. The fact that (1.6) does not hold implies that if \(v\) blows up then \(u\) has to blow up as well. We show two partial results that may be a guide for the general case.

**Theorem 1.7.** Let (1.5) hold. If \(2p_{12} \geq \max\{p_{22} - 1, 2p_{22} - (n + 1)\}\) and \(p_{22} > \min\{1, (n + 1)/2\}\), with \(p_{11} \geq m\) and \(p_{22} \geq n\), then \(u\) blows up and \(v\) remains bounded for every initial data.

When \(m, n \leq 1\), the hypotheses \(p_{11} \geq m\) and \(p_{22} \geq n\) in Theorem 1.7 always hold.

**Theorem 1.8.** Let (1.5) hold. If \(2p_{12} \geq \max\{p_{22} - 1, 2p_{22} - (n + 1)\}\) and \(p_{22} > \min\{1, (n + 1)/2\}\), with \(p_{21} = 0\) and \(m \geq 1\), then \(u\) blows up and \(v\) remains bounded for every initial data.

Consider again the problem studied in [15]. As we have seen, it behaves similarly as (1.1)–(1.4) when \(p_{11} > m\). As another example of this fact, the same proof of Theorem 1.7 can be used to show a new result for the problem posed in \(\mathbb{R}_+\).

**Theorem 1.9.** Let \(2p_{21} < 2p_{11} - (m + 1)\). If \(p_{22} > \min\{1, (n + 1)/2\}\) and \(2p_{12} \geq \max\{p_{22} - 1, 2p_{22} - (n + 1)\}\), then solutions of (1.1)–(1.4) in \(\mathbb{R}_+\) verify that \(u\) blows up and \(v\) remains bounded for every initial data.

Observe that the restrictions \(p_{11} \geq m\) and \(p_{22} \geq n\) follow directly from the conditions \(2p_{21} < 2p_{11} - (m + 1)\) and \(p_{22} > \min\{1, (n + 1)/2\}\).
ORGANIZATION OF THE PAPER. Section 2 deals with the blow-up rate in the case of non-simultaneous blow-up. Section 3 is devoted to the study of blow-up conditions for an auxiliary problem which will play an important role in the proof of our theorems. The non-simultaneous blow-up set is described in Section 4. In Section 5 we use the results of the previous sections to characterize the range of parameters for which non-simultaneous blow-up indeed occurs. Finally, in Section 6 we obtain results about the sets of initial data for which there is simultaneous or non-simultaneous blow-up.

Throughout the paper $C$ and $c$ denote positive constants that may change from one line to another, or even in the same line.

2. Non-simultaneous Blow-up Rates. A key point in the proof of Theorem 1.1 is the blow-up rate for $u$ when blow-up is non-simultaneous. In order to determine this rate, we consider $v^{p_{11}}(0, t)$ as a frozen coefficient. Hence we regard $u$ as a solution of

$$
\begin{align*}
 u_t - (u^m)_{xx}, & \quad x, t \in D_T, \\
 -(u^m)_x(0, t) = u^{p_{11}}(0, t)h(t), & \quad t \in (0, T), \\
 (u^m)_x(L, t) = 0, & \quad t \in (0, T), \\
 u(x, 0) = u_0(x), & \quad x \in (0, L),
\end{align*}
$$

(2.1)

where $h$ is a continuous, bounded, nondecreasing and strictly positive function. Hence by comparison, solutions of (2.1) blow up if and only if

$$p_{11} > \min\{1, (m + 1)/2\}.$$ 

Hence, by comparison, solutions of (2.1) blow up if and only if the same restriction on $p_{11}$ holds.

As a consequence of (H), $(u^m)_{xx} \geq 0$ and hence

$$\min_{x \in [0, L]} (u^m)_x(x, t) = (u^m)_x(0, t).$$

(2.2)

Moreover, $u_x \leq 0$ and therefore

$$\max_{x \in [0, L]} u(x, t) = u(0, t).$$

(2.3)

Hence, if $u$ blows up at a finite time $T$, $\{0\}$ belongs to the blow-up set of $u$.

First, we prove that when $u$ blows up and $p_{11} < m$ then $B(u) = [0, L]$. Moreover, $u$ blows up with the same speed at all points.

**Lemma 2.1.** Let $p_{11} > \min\{1, (m + 1)/2\}$ and let $u$ be a solution to (2.1). If $p_{11} < m$, then for every $c < 1$ there exists a value $t_0 = t_0(c) < T$ such that

$$c u(0, t) \leq u(x, t) \leq u(0, t), \quad x \in [0, L], \quad t_0 \leq t \leq T.$$

(2.4)

In particular $B(u) = [0, L]$.

**Proof.** We follow ideas from [6]. The second inequality is just (2.3). To prove the first one we apply (2.2) and the mean value theorem to get that there is a value $\xi \in (0, x)$ such that

$$u^m(x, t) - u^m(0, t) = x(u^m)_x(\xi, t) \geq x(u^m)_x(0, t) = -xu^{p_{11}}(0, t)h(t) \geq -CLu^{p_{11}}(0, t).$$

Thus, $u^m(x, t) \geq u^m(0, t)(1 - CLu^{p_{11} - m}(0, t))$. Since $p_{11} - m < 0$ and $u(0, t)$ blows up with $u_0(0, t) \geq 0$, we obtain that $1 - CLu^{p_{11} - m}(0, t) \geq c^m$ for $t$ close to $T$. \(\square\)

This lemma actually says more: the profile at the blow-up time is flat.
Corollary 2.1. Let \( p_{11} > \min\{1, (m+1)/2\} \) and let \( u \) be a solution to (2.1). If \( p_{11} < m \), then
\[
\lim_{\epsilon \to 0} u(x, \epsilon) = 1, \quad x \in [0, L].
\]

Remark Conditions \( p_{11} > \min\{1, (m+1)/2\} \) and \( p_{11} < m \) are compatible only if \( m > 1 \). In this case, \( \min\{1, (m+1)/2\} = 1 \).

We arrange the proof of Theorem 1.2 in two lemmas, according to the relation between \( p_{11} \) and \( m \). The first one uses Lemma 2.1 to obtain the blow-up rate for \( u \), when \( p_{11} < m \).

Lemma 2.2. Let \( p_{11} > \min\{1, (m+1)/2\} \) and let \( u \) be a solution of (2.1). If \( p_{11} < m \), then (1.7) holds for \( 0 < t < T \).

Proof. The mass \( F(t) = \int_0^L u(x, t) \, dx \) behaves as \( u(0, t) \). Indeed, from (2.3) we immediately obtain \( F(t) \leq Lu(0, t) \). On the other hand, given \( c < 1 \), we integrate (2.4) to obtain \( cL u(0, t) \leq F(t) \) for \( t_0(c) \leq t < T \). To extend this inequality to \( (0, t_0(c)) \) consider a positive constant \( C_1 \), small enough, and such that
\[
F(t) = \int_0^L u(x, t) \, dx \geq \int_0^L u(x, 0) \, dx \geq \frac{C_1}{cL} \int_0^L u(x, t_0(c)) \, dx.
\]
Using (2.4) we conclude that
\[
\frac{C_1}{cL} \int_0^L u(x, t_0(c)) \, dx \geq C_1 u(0, t).
\]
Therefore
\[
C_1 u(0, t) \leq F(t) \leq C_2 u(0, t), \quad 0 < t < T; \quad \text{(2.5)}
\]

From (2.5) we obtain that it is enough to prove an estimate analogous to (1.7) for \( F(t) \). For all \( t < T \) the mass satisfies the ordinary differential equation
\[
F'(t) = \int_0^L u_t(x, t) \, dx = \int_0^L (u^m)_t(x, t) \, dx = (u^m)_t(x, t) \, dx - (u^m)_t(0, t) = h(t)u^{p_{11}}(0, t).
\]
Since \( h \) is bounded from below, we get \( F'(t) \geq Cu^{p_{11}}(0, t) \geq C(F(t))^{p_{11}} \). Integrating in \((t, T)\) we get the upper estimate \( F(t) \leq C(T - t)^{-1/(p_{11} - 1)} \). The lower estimate is obtained in a similar way.

Lemma 2.3. Let \( p_{11} > \min\{1, (m+1)/2\} \) and let \( u \) be a solution of (2.1). If \( p_{11} \geq m \), then (1.7) holds for \( 0 < t < T \).

Proof. We follow a technique from [10]. Define \( M(t) = \max u(\cdot, t) \) and
\[
\phi_M(y, s) = \frac{1}{M(t)} u(ay, bs + t),
\]
where \( a = M^{m-p_{11}}, b = M^{m+1-2p_{11}} \). As \( t \nearrow T, \ M \nearrow \infty \). Under the mentioned hypotheses, \( 2p_{11} > m + 1 \); hence \( b \searrow 0 \). Moreover, if \( p_{11} > m \) then \( a \searrow 0 \); on the other hand, \( a = 1 \) if \( p_{11} = m \).

We claim that there exist two positive constants \( c \) and \( C \) such that
\[
c \leq (\phi_M)_x(0, 0) \leq C. \quad \text{(2.6)}
\]
If we rewrite these inequalities in terms of \( M(t) \), we get
\[
c \leq M^{m-2p_{11}}(t) M'(t) \leq C.
\]
Integrating in \((t, T)\) and taking into account that \(M(t) = u(0, t)\), we obtain the desired result.

The proof of the claim (2.6) relies strongly on \(\{\phi_M\}\) being a family of uniformly bounded solutions of equations of porous medium type,

\[
\begin{cases}
(\phi_M)_y = (\phi_M^m)_{yy}, & (y, s) \in (0, \frac{L}{2}) \times (-\frac{T}{2}, 0), \\
-(\phi_M^m)_x(0, s) = \phi_M^{p_1 + 1}(0, s) h(bs + t), & s \in (-\frac{T}{2}, 0), \\
(\phi_M^m)_y(\frac{L}{2}, s) = 0, & s \in (-\frac{T}{2}, 0).
\end{cases}
\]

The uniform bound, \(0 \leq \phi_M \leq 1\), is a consequence of \(u_t \geq 0\). Uniformly bounded solutions to porous medium type equations turn out to be equicontinuous in compact subsets of their common domain, cf. \([5, 20]\). Observe that for any \(S < 0\), the domain of \(\phi_M\) contains the compact set \([0, L] \times [-S, 0]\), if \(M\) is large enough.

Given \(\{\phi_M\}\), there is a continuous function \(\Phi\) and a subsequence, which we denote again by \(\Phi\), such that \(\phi_M \to \Phi\) as \(M \to \infty\), uniformly on \([0, L] \times [-S, 0]\). Moreover, \(\Phi(0, 0) = 1\). Therefore, there exists a neighbourhood of \((0, 0)\), \(U\), such that \(\Phi \geq 1/2\) in \(U\). Since we have uniform convergence in \(\overline{U}\) (we can assume that \(\overline{U}\) is compact), for \(j\) large enough we have that \(1/4 \leq \phi_M \leq 1\) in \(\overline{U}\). Thus, the functions \(\phi_M\) are solutions of uniformly parabolic equations in \(U\). Since they are uniformly bounded we get, using well known Schauder estimates, \([11]\),

\[
\|\phi_M\|_{C^{1+\sigma}} \leq C \quad \text{in} \quad U.
\]

The upper bound in (2.6) follows immediately. To obtain the lower estimate assume, for contradiction, that there exists a sequence \(\{\phi_M\}\) such that \((\phi_M)_+(0, 0) \to 0\). Estimates (2.7) imply that \(w = \Phi_s\), satisfies \(w_{ss} = (m \Phi^{m-1} w)_{yy}\) in the positivity set of \(\Phi\). Moreover, \(- (m \Phi^{m-1} w)_y(0, s) = p_1 \Phi^{p_1 - 1} w(0, s) h(T)\). On the other hand, \((\phi_M)_+(0, 0) \to \Phi_+(0, 0)\), which implies \(\Phi_+(0, 0) = 0\). Since \(u_t \geq 0\), \(w = \Phi_+ \geq 0\); hence \(w\) has a minimum at \((0, 0)\) and, by Hopf’s Lemma, \(w \equiv 0\). We get that \(\Phi\) is a nonnegative stationary solution of the porous medium equation, \(\Phi^m(y) = c_1 y + c_2\). Besides, \(- (\Phi^m)'(0) \geq \Phi^{p_1 + 1}(0) h(T) = h(T)\); thus \(\Phi^m(y) \leq 1 - h(T)y\). Hence, if \(p_1 > m\), \(\Phi\) becomes negative, a contradiction. If \(p_1 = m\), either \(\Phi\) becomes negative or it does not verify \((\Phi^m)'(0) = 0\), again a contradiction.

3. Blow-up Results for an Auxiliary Scalar Problem. If \(u\) blows up while \(v\) remains bounded, then (3.1) holds. Thus \(v\) is a supersolution to

\[
\begin{cases}
v_t = (v^n)_{xx}, & (x, t) \in D_{T_0}, \\
-(v^n)_x(0, t) = c v^{p_{22}}(0, t)(T - t)^{-\gamma_{p_{21}}}, & t \in (0, T_0), \\
(v^n)_x(L, t) = 0, & t \in (0, T_0), \\
v(x, 0) = v_0(x), & x \in (0, L),
\end{cases}
\]

with \(c > 0\), and \(T_0\) the maximal existence time for \(v\). The key point is that, in order to have \(v\) bounded up to \(T\), \(\gamma_{p_{21}}\) has to be smaller than \(1/2\).

**Theorem 3.1.** (i) If \(\gamma_{p_{21}} \geq 1/2\), then every nonnegative, nontrivial solution of (3.1) blows up at a finite time \(T_0 \leq T\).

(ii) If \(\gamma_{p_{21}} < 1/2\), then given \(\varepsilon > 0\) and \(v_0\) there exists \(T\) small enough such that the solution of (3.1) verifies

\[
\sup_{0 < t < T} \|v(; t)\|_{\infty} \leq \|v_0\|_{\infty} + \varepsilon.
\]

In particular, \(v\) is bounded.
If $v$ remains bounded up to time $t = T$, we expect it to behave as a solution of
\begin{equation}
\begin{aligned}
v_t &= (v^n)_{xx}, & (x, t) \in D_T, \\
-(v^n)_x(0, t) &= c(T-t)^{-\gamma_{p21}}, & t \in (0, T), \\
(v^n)_x(L, t) &= 0, & t \in (0, T), \\
v(x, 0) &= v_0(x), & x \in (0, L).
\end{aligned}
\end{equation}

A result analogous to Theorem 3.1 holds for problem (3.2).

**Theorem 3.2.** (i) If $\gamma_{p21} \geq 1/2$, then every nonnegative, nontrivial solution of (3.2) blows up at time $T$.
(ii) If $\gamma_{p21} < 1/2$, then given $\varepsilon > 0$ and $v_0$ there exists $T$ small enough such that the solution of (3.2) verifies
\begin{equation}
\sup_{0 < t < T} \|v(\cdot, t)\|_\infty \leq \|v_0\|_\infty + \varepsilon.
\end{equation}
In particular, $v$ is bounded.

**Remark** The same result holds if there is a constant $C > 0$ in front of $v_1$.

Theorem 3.1 is a consequence of this latter result.

**Proof of Theorem 3.1.** (i) As we are considering strictly positive initial data, we have that $v$ is a supersolution of (3.2) with $\gamma_{p21} \geq 1/2$. Since solutions of this latter problem blow up, $v$ also blows up.
(ii) Given $\varepsilon > 0$ and $v_0$, let $z$ be a solution of (3.2) with initial datum $z(x, 0) = v_0(x)$ and $c$ big enough. If we choose $T$ small, $z$ has the property (3.3). Hence it is a supersolution of (3.1). We conclude that $v$ must satisfy (3.3).

We come now the proof of Theorem 3.2 We begin by showing that problem (3.2) has a contraction property in $L_\infty$.

**Lemma 3.1.** Let $v_1$ and $v_2$ be two solutions of (3.2), with ordered initial data, $v_1(x, 0) \geq v_2(x, 0)$. Then
\begin{equation}
\|v_1(\cdot, t_2) - v_2(\cdot, t_2)\|_\infty \leq \|v_1(\cdot, t_1) - v_2(\cdot, t_1)\|_\infty, \quad 0 \leq t_1 < t_2 < T.
\end{equation}

**Proof.** If $v_1(x, 0) \geq v_2(x, 0)$, comparison shows that $v_1 \geq v_2$. Defining
\begin{equation}
\Phi(x, t) = \begin{cases}
\frac{v_1^n(x, t) - v_2^n(x, t)}{v_1(x, t) - v_2(x, t)} & \text{if } v_1(x, t) \neq v_2(x, t), \\
\frac{v_1^{n-1}(x, t)}{v_1^n(x, t)} & \text{if } v_1(x, t) = v_2(x, t),
\end{cases}
\end{equation}
we get that $w = v_1 - v_2 \geq 0$ is a solution of the parabolic equation
\begin{equation}
\begin{aligned}
w_t &= (\Phi(x, t)w)_{xx}, & x \in (0, L), \quad t \in (t_1, t_2), \\
-(\Phi(x, t)w)_x(0, t) &= 0, & t \in (t_1, t_2), \\
(\Phi(x, t)w)_x(L, t) &= 0, & t \in (t_1, t_2), \\
w(x, t_1) &= v_1(x, t_1) - v_2(x, t_1), & x \in (0, L).
\end{aligned}
\end{equation}

Hence, the maximum of $w$ in $[t_1, t_2] \times [0, L]$ lies on the parabolic boundary. If the maximum is achieved at $t = t_1$, the result follows. If it is attained at $x = 0$ or $x = L$, Hopf’s Lemma (observe that the equation is uniformly parabolic in a neighbourhood of the maximum point, since $v_1 > 0$ there) implies that $w \equiv 0$, and the monotonicity of $\|w(\cdot, t)\|_\infty$ becomes trivial.

This contraction property implies that in order to prove Theorem 3.2 (and hence Theorem 3.1), it is enough to consider any initial data.
Corollary 3.1. Either every solution to (3.2) is bounded up to time $T$ or every solution blows up at time $T$.

Proof. Let $v_1$ be any solution of (3.2) and $v_2$ be the solution with initial data $v_2(x,0) = 0$. Then, by comparison, $v_1(x,t) \leq v_2(x,t)$. Since the difference $\|v_1(\cdot,t) - v_2(\cdot,t)\|_\infty$ does not increase with time, see (3.4), any solution $v_1$ blows up at time $T$ if and only if $v_2$ does so.

Lemma 3.2. Theorem 3.2 holds true in the linear case, $n = 1$.

Proof. (i) We use ideas from [15]. Thanks to the previous lemma, we can restrict to initial data $v_0 = 0$. Let $G$ be Green’s function for the problem. We have, $G(x,y,t,s) = \Gamma(x-y,t-s) + H(x,y,t,s)$, where $\Gamma$ is the fundamental solution for the heat equation,

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right),$$

and $H(x,y,t,s)$ is the solution of

$$\begin{align*}
H_x(x,y,t,s) + H_{yy}(x,y,t,s) &= 0, & (y,s) &\in D_T, \\
H_y(x,0,t,s) &= \Gamma_x(x,t-s), & s &\in (0,T), \\
H_y(x,L,t,s) &= \Gamma_x(x-L,t-s), & s &\in (0,T), \\
H(x,y,t,t) &= 0, & y &\in (0,L).
\end{align*}$$

Since $|H(0,0,t,s)|$ is bounded, we get from the representation formula using Green’s function, that

$$v(0,t) \geq C \int_0^t \frac{ds}{(t-s)^{1/2}(T-s)^{\gamma_{p21}}} - C. \quad (3.5)$$

Since $\gamma_{p21} \geq 1/2$, $v$ blows up at time $T$.

(ii) Consider any initial data $v_0$. Then $v$ can be written as $v = v_1 + v_2$, with $v_1$ a solution of the heat equation with zero flux and $v_1(x,0) = v_0(x)$ and $v_2$ a solution of (3.2) (with $n = 1$) with initial datum $v_0(x) = 0$. From the general theory for the heat equation we have that $v_1(x,t) \leq v_0(x)$. On the other hand, from (3.5) we have that $v_2(x,t) \leq CT^{1/2-\gamma_{p21}}$. Hence $v(x,t) \leq v_0(x) + CT^{1/2-\gamma_{p21}}$, from where (3.3) follows, if $T$ is small.

Proof of Theorem 3.3. In order to extend the proof to a general $n$ we follow again ideas from [15]. The idea is to use that

$$0 < c \leq ne^{n-1} \leq C \quad (3.6)$$

to obtain supersolutions of the heat equation.

(i) Assume that there is a solution of (3.2), $v$, bounded up to time $T$. Let $z = v^n$. We get

$$\begin{align*}
\frac{1}{n} z^{(1-n)/n}_t &= z_{xx}, & (x,t) &\in D_T, \\
-z_{x}(0,t) &= c(T-t)^{-\gamma_{p21}}, & t &\in (0,T), \\
z_{x}(L,t) &= 0, & t &\in (0,T), \\
z(x,0) &= v_0^n(x), & x &\in (0,L).
\end{align*}$$

As either all solutions blow up or none of them does, we can take any initial data without loss of generality. We take $z(x,0) \geq \delta > 0$ such that $z_{xx}(x,0) \geq 0$. Thus,
\[ z_t \geq 0 \text{ and } z(x, t) \geq \delta \text{ for all } t. \] Hence, condition (3.6) holds, and we have that

\[
\begin{align*}
Cz_t &\geq z_{xx}, & (x, t) &\in D_T, \\
-z_x(0, t) = c(T - t)^{-\gamma p_2}, & t &\in (0, T), \\
z_x(L, t) = 0, & t &\in (0, T), \\
z(x, 0) &\geq \delta, & x &\in (0, L).
\end{align*}
\]

Therefore, \( z \) is a bounded supersolution of a heat equation with a flux at the boundary given by \( c(T - t)^{-\gamma p_2} \), a contradiction.

(ii) Given \( v_0 \), let \( z \) be a solution of (3.2) with \( n = 1 \), \( z_0(x) = v_0^n(x) \), and a constant \( k(z_0) = n(z_0 + \varepsilon)^{(n-1)/n} \) in front of \( z_t \). Let \( \tilde{v} = z^{1/n} \), then

\[
\begin{align*}
\nu_t^n - \nu_{xx} = c(T - t)^{-\gamma p_2}, & (x, t) \in D_T, \\
-(\nu^n)_x(0, t) = c(T - t)^{-\gamma p_2}, & t \in (0, T), \\
(\nu^n)_x(L, t) = 0, & t \in (0, T), \\
\nu(x, 0) = z_0^{1/n}(x), & x \in (0, L).
\end{align*}
\]

As \( \gamma p_2 < 1/2 \), we can find \( T \) small enough such that \( z(x, t) \leq \|z_0\|_{\infty} + \varepsilon \). Since \( z \) is also bounded away from zero and \( z_t > 0 \), we have, no matter the value of \( n > 0 \), taking \( \delta \) appropriately,

\[ n\nu^n - \nu_t \leq k(z_0). \]

Therefore \( \tilde{v} \) is a supersolution of problem (3.2). Hence,

\[ \sup_{0 < t < T} \|v(\cdot, t)\|_\infty \leq \sup_{0 < t < T} \|\tilde{v}(\cdot, t)\|_\infty \leq (\|v_0\|_{\infty} + \varepsilon)^{1/n}, \]

from where (3.3) follows. \( \square \)

**Remark** The mass \( \int_0^L v \) satisfies

\[ \int_0^L v(x, t) \, dx - \int_0^L v_0(x) \, dx = \int_0^T c(T - \tau)^{-\gamma p_2} \, d\tau. \]

Hence, a solution of (3.2) remains bounded in the \( L^1 \)-norm up to time \( T \) if and only if \( \gamma p_2 < 1/2 \). This condition differs from the condition for blow-up in the \( \infty \)-norm, \( \gamma p_2 < 1/2 \).

4. **Non-simultaneous Blow-up Set.** The results of sections 2 and 3 allow us to obtain the blow-up set when blow-up is non-simultaneous.

**Proof of Theorem 4.1.** If \( p_1 \leq m \), since (1.7) holds, \( u \) is a supersolution of

\[
\begin{align*}
u_t = (u^m)_{xx}, & \quad (x, t) \in D_T, \\
u(0, t) = c(T - t)^{-\gamma}, & \quad t \in (0, T), \\
(u^m)_x(L, t) = 0, & \quad t \in (0, T), \\
u(x, 0) = u_0(x), & \quad x \in (0, L).
\end{align*}
\]

Hence if \( \tilde{u} \) is a subsolution of (4.1) \( B(\tilde{u}) \subseteq B(u) \). In our case, we take as subsolution, \( \tilde{u} \), the solution of problem (4.1) defined for \( x \in \mathbb{R}_+ \). The blow-up set for such solutions has been described in [3] and [9]. The authors show that if \( p_1 = m \) then \( B(\tilde{u}) = [0, L] \) and if \( p_1 < m \) then \( B(\tilde{u}) = \mathbb{R}_+ \).

If \( p_1 > m \) we use that \( u \) is a subsolution of (4.1) that blows up. Hence if \( \tilde{u} \) is a solution of that problem, \( B(\tilde{u}) \subseteq B(\tilde{u}) \). We are therefore confronted with the study of the blow-up set of \( \tilde{u} \). This was studied for \( m \geq 1 \) in [8], where it was proved that \( B(\tilde{u}) = \{0\} \). When \( m < 1 \) we can argue as follows: consider a special solution to \( \tilde{u}_t = (\tilde{u}^m)_{xx} \) for \( (x, t) \in (0, 2L) \times (0, \infty) \) with initial datum \( u_0 \) (extended to \( (0, 2L) \))
1.1 Let ( allows to conclude that \( \hat{u}(x,t) \to \infty \) as \( x \to 0,2L \) for every \( t \) (this kind of solution was constructed in [1]). This solution provides us with a maximal solution for our problem, i.e. \( \hat{u}(x,t) \geq u(x,t) \). As \( \hat{u} \) is bounded in the interior of the interval \((0,2L)\), we conclude that \( B(\hat{u}) = \{0\} \). \( \square \)

5. Conditions for Non-simultaneous Blow-up. We arrive to the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) Let \( N = \|v_0\|_{\infty} \). Since \( p_{11} > \min\{1,(m+1)/2\} \), we can choose the initial data \( u_0 \) large enough in order to obtain a blow-up solution of (1.1) - (1.4) with \( T \) small. Let \( t_0 \) be the first time where \( v \) touches the height \( 2N \). If such \( t_0 \) does not exist, \( \|v(\cdot,t)\|_{\infty} \leq 2N \), and the result follows. Hence we assume that \( t_0 < T \). For any \( t \in [0,t_0) \) we have that \( u \) is a solution of (2.1) with \( \delta \leq h(t) \leq 2N \). Using the non-simultaneous blow-up rate from above, which is valid up to time \( t_0 \), we get that \( v \) is a subsolution of (3.1) for \( t \leq t_0 \). Let \( \bar{v} \) be the solution of (3.1). Since \( \gamma_{p_{21}} < 1/2 \), \( \bar{v} \) does not blow up. Moreover, as \( T \) is small, \( \bar{v}(0,t_0) < 3N/2 \). But then, \( 2N = v(0,t_0) \leq \bar{v}(0,t_0) < 3N/2 \), a contradiction. We conclude that \( v \) remains bounded up to time \( T \).

(ii) Since \( v(x,t) \leq C \) for all \( 0 < t < T \), \( u \) is a subsolution of (2.1). Hence we need \( p_{11} > \min\{1,(m+1)/2\} \) for \( u \) to blow up. \( \square \). Now, if we use the blow-up rate for \( u \) from below, and plug it into the equation for \( v \), which is assumed to be bounded, we get that \( v \) is a supersolution of (3.1). By Theorem 3.1 we have \( \gamma_{p_{21}} < 1/2 \). \( \square \)

6. Simultaneous / Non-simultaneous Blow-up. Our next aim is to study the set of initial data for which blow-up is non-simultaneous. We show that it is an open set in the \( L^\infty \)-topology.

Proof of Theorem 1.3. Let \((u,v)\) be a solution such that \( u \) blows up at time \( T \) and \( v \) remains bounded up to that time, say \( v \leq C \). As \( u \) blows up at time \( T \), it becomes large at time \( T - \varepsilon \). We can find a neighbourhood of \((u(x,0),v(x,0))\) in \( L^\infty \) such that, if \( (\hat{u},\hat{v}) \) has initial data in such neighbourhood and \( \hat{u}_t \geq 0 \), then \( \hat{u} \) becomes large at time \( T - \varepsilon \) and \( \hat{v} \) is bounded up to \( T - \varepsilon \) (this follows by continuity with respect to the initial conditions since up to \( T - \varepsilon \) we are dealing with bounded solutions). The argument in the proof of Theorem 1.1 allows to conclude that \( \hat{u} \) blows up and \( \hat{v} \) remains bounded. \( \square \)

This result suggests that the set of initial data leading to non-simultaneous blow-up is large. However, exceptional solutions with simultaneous blow-up may exist. We show that this happens indeed if (1.5) and (1.6) both hold at the same time. Observe that in this range both components of the solution may blow-up by themselves, without the help of the other component.

Proof of Theorem 1.5. Let \((u_0,v_0)\) be such that the solution of (1.1) - (1.4) blows up. For any \( \lambda \in (0,1) \) we denote by \((u_\lambda,v_\lambda)\) the solution of (1.1) - (1.4) with initial data \((u_0/\lambda,v_0/(1-\lambda))\). Define

\[
A_\lambda = \{ \lambda \in (0,1) : u_\lambda \text{ blows up and } v_\lambda \text{ remains bounded} \},
\]

\[
B_\lambda = \{ \lambda \in (0,1) : v_\lambda \text{ blows up and } u_\lambda \text{ remains bounded} \}.
\]

Thanks to the previous results, we know that if \( \lambda \) is small enough \( u \) blows up and \( v \) remains bounded. Hence, \( A_\lambda \neq \emptyset \). Analogously, if \( \lambda \) is close to 1, then \( \lambda \in B_\lambda \). On the other hand, we know from Theorem 1.3 that \( A_\lambda \) and \( B_\lambda \) are open. A
connectedness argument shows that there exist \( \lambda_0 \in (0, 1) \) such that \( \lambda_0 \notin A_1 \cup B_1 \). Since \((u_{\lambda_0}, v_{\lambda_0})\) blows up, we have constructed a solution with simultaneous blow up.

Let us show now that, under certain conditions on \( p_{12} \) and \( p_{22} \), blow-up is always non-simultaneous.

**Proof of Theorem 1.6.** Under our assumptions \( u \) must blow up. Then, there are two possibilities: either \( u \) blows up alone, or \( u \) and \( v \) blow up simultaneously. Let us exclude the second one.

By considering the solution at time \( T - \varepsilon \) as initial data, we may assume that the blow-up time is as small as pleased. Let \( \|v_0\|_\infty = N \) and let \( t_0 \) be the first time where \( v(0, t_0) = 2N \). If such \( t_0 \) does not exist, \( \|v(\cdot, t)\|_\infty \leq 2N \), and \( v \) is bounded.

Assume that such \( t_0 \) exists. As \( v \) is bounded at least up to \( t_0 \), using the same argument as in the proof of Theorem 1.4 we conclude that \( v \) does not exceed \( 3N/2 \), a contradiction. This argument is valid for any \( N \) as long as \( v_0 \geq \delta > 0 \).

**Proof of Theorem 1.7.** Since \( p_{11} \geq m \), then \( 2p_{11} > m + 1 \) and condition (1.5) becomes \( 2p_{21} < 2p_{11} - (m + 1) \). Analogously, \( 2p_{22} > n + 1 \). Trivially we have that \( 2p_{12} \geq 2p_{22} - (n + 1) \).

Assume that there is an initial data \((u_0, v_0)\), such that \( u \) and \( v \) both blow up at time \( T \). Following [13], define

\[
M(t) = \max u(\cdot, t) \quad \text{and} \quad N(t) = \max v(\cdot, t),
\]

and set, for \( t < T \),

\[
\varphi_M(y, s) = \frac{1}{M(t)}u(ay, bs + t), \quad y > 0, \quad -\frac{1}{b} < s < 0,
\]

\[
\psi_N(y, s) = \frac{1}{N(t)}v(cy, ds + t), \quad y > 0, \quad -\frac{1}{a} < s < 0,
\]

with

\[
a = \frac{M^{m-p_{11}}}{N^{p_{12}}}, \quad b = \frac{M^{-2p_{11}+m+1}}{N^{2p_{12}}}, \quad c = \frac{N^{n-p_{22}}}{M^{p_{21}}}, \quad d = \frac{N^{-2p_{22}+n+1}}{M^{2p_{21}}}.
\]

With the same ideas of the proof of Lemma 2.3 (see also [13]) it is easy to show that

\[
c \leq (\varphi_M)_\ast(0, 0) \leq C, \quad c \leq (\psi_N)_\ast(0, 0) \leq C. \tag{6.1}
\]

Writing (6.1) in terms of \( M \) and \( N \), we have

\[
M^{m-2p_{11}} M' \geq cN^{2p_{12}}, \quad N^{n-2p_{22}} N' \leq CM^{2p_{21}}.
\]

Assume first that \( 2p_{12} > 2p_{22} - (n + 1) \). After a straightforward computation we obtain

\[
CM(t)^{2p_{21} - 2p_{11} + m + 1} + C \geq CN(t)^{2p_{12} - 2p_{22} + n + 1} - C.
\]

As \( 2p_{21} < 2p_{11} - (m + 1) \) and \( 2p_{12} > 2p_{22} - (n + 1) \), we obtain a contradiction with the assumption of simultaneous blow-up.

If \( 2p_{12} = 2p_{22} - (n + 1) \), we get

\[
CM(t)^{2p_{21} - 2p_{11} + m + 1} + C \geq C \ln N(t) - C,
\]

which is again a contradiction.
Proof of Theorem 1.8. Let us first assume that \( m = 1 \). Since \( p_{21} = 0 \), we have that \( v \) is a solution of

\[
\begin{align*}
  v_t &= (v^n)_{xx}, & (x, t) &\in D_{T_v}, \\
  -(v^n)_x(0, t) &= v^{p_{22}}(0, t), & t &\in (0, T_v), \\
  (v^n)_x(L, t) &= 0, & t &\in (0, T_v), \\
  v(x, 0) &= v_0(x), & x &\in (0, L).
\end{align*}
\]

As \( p_{22} > \min\{1, (n + 1)/2\} \), \( v \) blows up at a time \( T_v \). We can apply the results of Section 2 with \( h(t) = 1 \) (see also [8]) and get that \( v \) verifies the blow-up rate,

\[
c(T_v - t)^{-\eta} \leq v(0, t) \leq C(T_v - t)^{-\eta}, \quad \eta = 1/\max\{p_{22} - 1, 2p_{22} - (n + 1)\}.
\]

This implies that \( u \) is a supersolution of

\[
\begin{align*}
  u_t &= u_{xx}, & (x, t) &\in D_{T_u}, \\
  -u_x(0, t) &= C' u^{p_{11}}(T_v - t)^{-\eta p_{12}}, & t &\in (0, T_u), \\
  u_x(L, t) &= 0, & t &\in (0, T_u), \\
  u(x, 0) &= u_0(x), & x &\in (0, L),
\end{align*}
\]

with \( \eta p_{12} \geq 1/2 \). From Theorem 3.1 we know that every nonnegative solution \( \tilde{u} \) of (6.2) blows up in time \( T_{\tilde{u}} \leq T_v \). Our purpose is to show that we have a strict inequality. Hence \( u \) blows up at time \( T_u \leq T_{\tilde{u}} < T_v \), which means that blow-up is non-simultaneous.

Since \( m = 1 \), from the representation formula, we get

\[
\tilde{u}(0, t) \geq C \int_0^t \frac{\tilde{u}(0, s)^{p_{11}}}{(T_v - s)^{\eta p_{12}}(t - s)^{1/2}} ds \geq C \int_0^t \frac{\tilde{u}(0, s)^{p_{11}}}{(T_v - s)^{\eta p_{12} + 1/2}} ds = C u(t).
\]

Therefore \( \tilde{u}(0, t)^{p_{11}}(T_v - t)^{-\eta p_{12}} = w'(t) \) and hence

\[
\frac{w'(t)}{w(t)^{p_{11}}} \geq \frac{C}{(T_v - t)^{\eta p_{12} + 1/2}}.
\]

Integrating this expression in \( (t, T_v) \) we get

\[
\frac{c_1}{w(T_v)^{p_{11}} - 1} \leq c_2 - \int_t^{T_v} \frac{C}{(T_v - s)^{\eta p_{12} + 1/2}} ds = c_2 - I(t).
\]

Thus,

\[
c w(T_v) \geq (c_2 - I(t))^{-1/(p_{11} - 1)}.
\]

Since \( I(t) \) diverges as \( t \nearrow T_v \), there is a time \( T_0 < T_v \), such that \( c_2 = I(T_0) \) and therefore \( w(T_0) \geq \infty \). Now, since \( \tilde{u}(0, t) \geq w(t) \) we get the desired result, \( T_{\tilde{u}} \leq T_0 < T_v \).

To extend the proof to \( m > 1 \) we follow the ideas of the proof of Theorem 3.2. Observe that

\[
0 < c \leq m u^{m-1}.
\]

If we define \( z = u^m \), using (6.3) we get that \( z \) is a supersolution of the heat equation. Hence \( z \) blows up at \( T_0 < T_v \). So, \( u \) blows up at \( T_u \leq T_0 < T_v \) and blow-up is non-simultaneous.

Let us point out that this idea cannot be used when \( m < 1 \), since in this case (6.3) is false. \( \square \)
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