Viscosity solutions for quasilinear degenerate parabolic equations of porous medium type

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Abstract

We consider the Cauchy Problem for the class of nonlinear parabolic equations of the form
\[ u_t = a(u)\Delta u + |\nabla u|^2 , \]
with a function \( a(u) \) that vanishes at \( u = 0 \). Because of the degenerate character of the coefficient \( a \) the usual concept of viscosity solution in the sense of Crandall-Evans-Lions has to be modified to include the behaviour at the free boundary. We prove that the problem is well-posed in a suitable class of viscosity solutions. Agreement with the concept of weak solution is also shown.

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Introduction

The basis of the viscosity method for the study of existence and uniqueness of elliptic and parabolic PDE’s is comparison with classical super- and subsolutions. The theory was first stated in the 1980’s for first-order equations [CEL, CL]. Subsequently, a whole theory has been developed for fully nonlinear elliptic equations, [CIL, CC], and then extended to parabolic equations, see [Wa1, Wa2]. The book [Bl] contains some applications of the theory of viscosity solutions, like deterministic optimal control and singular perturbation problems for elliptic equations. See also [BCD] for Hamilton-Jacobi-Bellman equations and [LS] for stochastic PDE’s.

The primary virtues of this theory are that (i) it allows merely continuous functions to be solutions of fully nonlinear equations of first and second order; (ii) it provides very general existence and uniqueness theorems, and (iii) it yields great flexibility in passing to limits in various settings. The program of establishing the well-posedness of problems involving parabolic equations has been carried out for the most typical cases, linear or nonlinear, even in the fully nonlinear case to which the concept seems to be well-suited.

Typical examples in which this theory works are the case of the Hamilton-Jacobi equations of the form
\[ u_t = F(|\nabla u|), \]
and the diffusive approximation
\[ u_t = F(|\nabla u|) + \varepsilon \Delta u, \quad \varepsilon > 0, \]
where \( \nabla u \) denotes the spatial gradient of a function \( u = u(x,t) \) and \( \Delta u \) is the spatial Laplacian, cf. [CIL]. However, there is a problem in developing a similar theory for equations of degenerate type due to the absence of a sufficiently large family of classical super- and subsolutions to serve as test functions. In this paper we study the viscosity solutions for a quasilinear, parabolic, degenerate equation which is a good example where such difficulties are present. The equation is
\[ u_t = a(u) \Delta u + |\nabla u|^2. \]

There is no major problem in adapting the standard results when \( a(u) \) is smooth and bounded above and below in the form \( 0 < C_1 \leq a(u) \leq C_2 \), since we are dealing with a quasilinear uniformly parabolic equation. However, there is an interest in considering the case where function \( a(u) \) is continuous and positive for \( u > 0 \) but vanishes at \( u = 0 \). In that case we have a so-called degenerate parabolic equation, which serves as a nonlinear viscous approximation of the Hamilton-Jacobi equation and generalizes type (0.2).

A particular case of (0.3) in which \( a \) is linear, \( a(u) = \sigma u \) for a constant \( \sigma > 0 \), is well-known in the literature. The equation reads
\[ u_t = \sigma u \Delta u + |\nabla u|^2. \]
This equation appears in nonlinear diffusion to describe the evolution of the pressure of a gas in a porous medium under isothermal or adiabatic conditions. The concept and properties of viscosity solutions have been studied in [CV]. The main point of that paper is to show that a modification of the concept of viscosity solution given in [CIL] is needed in order to show that the Cauchy Problem is well-posed in the class of bounded and nonnegative viscosity solutions for this degenerate problem. It is to be noted that in this case equation (0.3) can be transformed into the well-known Porous Medium Equation (PME): \( \rho_t = \Delta \rho^m \), where \( m = \sigma + 1 \), and \( \rho \) stands for the density of the gas. Since the PME is a divergence-form equation, it has a weak theory, cf. [Ar, Pe, Va1], and this is used to complete the proof of the uniqueness theorem for viscosity solutions of (0.3) after the transformation.

Here we extend the analysis and results for equation (0.3) to the case where \( a(u) \) is a nonlinear function satisfying the following assumptions:

(H1) Degeneracy at \( u = 0 \): \( a : [0, \infty) \to \mathbb{R} \) is \( C^1 \)-smooth if \( u > 0 \), continuous at \( u = 0 \) and
\[
a(u) = 0 \quad \text{if} \quad u = 0, \quad a(u) > 0 \quad \text{if} \quad u > 0.
\]

(H2) Condition of linear growth of \( a \) near \( u = 0 \): for every \( M > 0 \) there exists a constant \( k = k(M) \) such that
\[
a(u) \leq k u \quad \text{for all} \quad 0 \leq u \leq M.
\]
In other words, \( a(u)/u \) is locally bounded.

Note that no control from below is imposed on the growth of \( a(u) \). The generality of these assumptions introduces a number of additional difficulties that we address. Thus, paper [CV] uses explicit free-boundary classical solutions (in the sense of Definition 1.3 below) as comparison functions. In our framework such solutions do not necessarily exist, and we work instead with two kinds of special functions, namely the Barenblatt functions and the spherical traveling waves, which are shown to be respectively sub- and supersolutions of (0.3). On the other hand, the correspondence with a divergence equation, in this case the Filtration Equation,
\[
\rho_t = \Delta \Phi(\rho),
\]
is not immediate and has to be carefully investigated. The present work not only extends, but also serves as a detailed exposition of the results and methods of [CV], a paper which proceeded at a rather quick pace. Modifications of the program of [CV] have been done at several places.

We consider continuous, bounded and nonnegative solutions defined in the space-time domain \( Q = \mathbb{R}^n \times (0, \infty) \), with continuous, bounded and nonnegative initial conditions
\[
u(x, 0) = u_0(x), \quad x \in \mathbb{R}^n.
\]

The aim of this article is to prove the following theorem.

**Theorem A** Under conditions (H1), (H2) the initial value problem (0.3), (0.7) is well-posed in the class of bounded, continuous and nonnegative viscosity solutions. Moreover, the viscosity solution coincides with the transform of the continuous weak solution of the associated Filtration Equation, \( \rho_t = \Delta \Phi(\rho) \). The Maximum Principle applies to viscosity solutions.
We point out that the solutions of our problem will not be classical when moving free boundaries appear, but the problem is well-posed in the framework of continuous functions. No requirement of uniform continuity has to be made.

The outline of the paper is as follows. We begin with four sections of preliminary material: a first section is devoted to the basic definitions, a second one to the relation of viscosity solutions with classical solutions, the third one deals with the transformation into the filtration equation, and another one examines some explicit examples of sub- and supersolutions that are necessary tools in the numerous comparisons on which the viscosity method relies.

In Section 5 we perform the construction of the Maximal Viscosity Solution for every given initial datum, which is an important though relatively easy part of the theory. Section 6 discusses the properties of support propagation and Section 7 introduces the sup and inf convolutions. This paves the way for the proof in Section 8 of the Comparison Theorem for separated solutions, which is a main result of the work. As a consequence, we derive in Section 9 the existence of the Minimal Viscosity Solution.

The uniqueness of viscosity solutions is then reduced to the uniqueness of ordered solutions, which is established in Section 10. This step uses the transformation into the filtration equation.

A number of typical results for parabolic equations also hold. Thus, comparison applies to super- and subsolutions if the initial data are correctly ordered, cf. Theorem 10.2. The solutions depend continuously on the initial data. Smooth viscosity solutions are solutions in the classical sense. There is a complete correspondence of viscosity solutions of equation (0.3) and bounded weak solutions of the associated filtration equation (0.6).

The proof of continuous dependence with respect to the initial data, Theorem 11.1, completes the proof of well-posedness of our problem. This is done in Section 11 where we also discuss variations of the definition of viscosity solution that lead to equivalent results. Section 12 shows that a free boundary test is needed in order to have a correct definition of viscosity solution (see (ii) in Definition 1.5 below).

We end the paper with an appendix on the Filtration Equation. A previous section is devoted to conclusions and comments. Let us recall here that there are a number of recent works concerned with the existence, uniqueness, continuity and other properties of viscosity solutions of nonlinear parabolic equations of the form

\[ u_t = F(x, t, u, Du, D^2u), \]

possibly of degenerate type, cf. e.g. [BRS, Bo, CGL, JK]. To our knowledge, all of them make assumptions like monotonicity of the function \( F \) with respect to the \( u \) variable that are not justified in our problem. Bénilan and Maliki [BeM, Ma] have considered existence of solutions of a general version of our problem,

\[ u_t = \alpha(u)\Delta u + |\nabla u|^2 + F(u), \]

and proved existence and different continuity estimates. The main problem where we focus our efforts is rather uniqueness and well-posedness.
1 Definitions

We gather here the main definitions we will be using. Let us first fix some notations for the sequel. \( C(D) \) is the space of continuous functions defined in a domain \( D \) of space \( \mathbb{R}^n \) or space-time \( \mathbb{R}^n \times \mathbb{R} \), \( C_b(D) \) the subspace of continuous and bounded functions, \( f \in C^{2,1}(Q) \) means that \( f \) is twice continuously differentiable in \( x \), and once in \( t \); i.e., \( f \in C^{2,1}_{x,t}(Q) \). \( B_r(x_0) \) denotes the open ball in \( \mathbb{R}^n \) with radius \( r \) and center \( x_0 \). The closed ball is denoted by \( \overline{B}_r(x_0) \). A parabolic neighborhood of a point \( P_0 = (x_0, t_0) \in Q \) is a cylinder of the form \( \mathcal{R} = B_r(x_0) \times (t_0 - \tau, t_0] \subset Q \) for some \( r, \tau > 0 \).

**Definition 1.1** By a classical solution of equation (0.3) we mean a function \( u(x, t) \) of the class \( C^{2,1} \) such that
\[
 u_t = a(u) \Delta u + |\nabla u|^2
\]
is satisfied everywhere in \( Q \). In a classical subsolution we ask that inequality holds
\[
 u_t \leq a(u) \Delta u + |\nabla u|^2.
\]
In a classical supersolution the inequality sign is reversed,
\[
 u_t \geq a(u) \Delta u + |\nabla u|^2.
\]

**Definition 1.2** A continuous and nonnegative function \( u \) defined in \( Q \) is a viscosity subsolution of equation (0.3) if and only if for every function \( \varphi \in C^{2,1}(Q) \), that touches \( u \) from above at a point \((x_0, t_0)\) the following inequality
\[
 \varphi_t \leq a(\varphi) \Delta \varphi + |\nabla \varphi|^2
\]
holds at \((x_0, t_0)\).

**Remark 1** We say that \( \varphi \) touches \( u \) from above at a point \( P_0 = (x_0, t_0) \) if \( \varphi - u \) reaches a local minimum zero in a parabolic neighborhood \( \mathcal{R} = B_r(x_0) \times (t_0 - \tau, t_0] \subset Q \) of \( P_0 \), i.e., \( u \leq \varphi \) in \( \mathcal{R} \) and \( u = \varphi \) at \( P_0 \).

An interesting technical trick is the following: after replacing \( \varphi \) by
\[
 \psi = \varphi + \delta((x - x_0)^4 + (t - t_0)^2), \quad \delta > 0,
\]
we may always assert that \( \psi - u \) has a strict local minimum zero in \( \mathcal{R} \) precisely at \( P_0 \) without changing the derivatives of \( \varphi \) at \( P_0 \) that appear in (1.1).

The definition of supersolution contains a novelty with respect to standard viscosity practice: we have to take into account the effect of the degeneracy of the equation by allowing for some free-boundary solutions as a possible test functions.

**Definition 1.3** We say that a continuous and nonnegative function \( u \) is a classical free-boundary solution of (0.3) if

(i) \( u \) is positive in an open set \( P(u) \subset Q \), where it is smooth and solves equation (0.3) in the classical sense.

(ii) The boundary of the positivity set, \( \Gamma = \partial P \cap Q \), called the free boundary, is a smooth hypersurface in space-time and \( u \in C^{2,1}_{x,t}(P \cup \Gamma) \).
(iii) On $\Gamma$ the following dynamic condition holds

$$v_n = |\nabla u|,$$

where we denote by $v_n$ the normal speed of advance of the boundary.

In case we impose the stronger condition: (iv) $\nabla u \neq 0$, on $\Gamma$, then we have a classical moving free-boundary solution.

A slight modification of the above definition allows us to define a classical free-boundary subsolution by replacing in condition (i) the equation by the inequality

$$u_t \leq a(u) \Delta u + |\nabla u|^2$$

and replacing (1.2) by $v_n \leq |\nabla u|$. In a similar way we can define the classical free-boundary supersolution replacing $\leq$ by $\geq$.

**Definition 1.4** We say that $u$ and $v$ are strictly separated, and we will write $u \prec v$, if

(i) the support of $u$, $\text{supp}(u)$, is a compact subset of $\mathbb{R}^n$ and

$$\text{supp}(u(x)) \subset \text{Int}(\text{supp}(v(x))).$$

(ii) Inside the support of $u$ the functions are strictly ordered,

$$u(x) < v(x).$$

**Definition 1.5** A continuous and nonnegative function $u$ defined in $Q$ is a viscosity supersolution of equation (0.3) if and only if the following two conditions are satisfied:

(i) For every function $\varphi \in C^{2,1}(Q)$, that touches $u$ from below at a point $(x_0, t_0)$ where $u(x_0, t_0) > 0$ the inequality

$$\varphi_t \geq a(\varphi) \Delta \varphi + |\nabla \varphi|^2$$

holds at $(x_0, t_0)$.

(ii) Every classical, moving free-boundary subsolution $v$ that is strictly separated from $u$ in the sense of Definition 1.4, at time $t = t_1 \geq 0$, $v(x, t_1) \prec u(x, t_1)$, cannot cross $u$ at a later time, i.e., $v(x, t) \leq u(x, t)$ for all $x \in \mathbb{R}^n$ and $t_2 > t_1$.

**Remark 2** The idea of comparison with free boundary subsolutions was introduced in [CV]. Note that the definition here is a small variation of the idea of [CV] (a slightly weaker test), which turns out quite convenient in presenting the theory. In that direction it must be stressed that only a particular family of classical moving free boundary subsolutions will be used in the proofs that follow, namely the Barenblatt functions introduced in Section 4, hence part (ii) of Definition 1.5 could be formulated in terms of comparison with that particular family. Several equivalent definitions are discussed in Proposition 11.2, but are not needed to develop the theory.

Other forms of comparison have been introduced in the literature for free boundary problems.
We get now to the goal of this section.

**Definition 1.6** A *viscosity solution* of (0.3) is a continuous and nonnegative function defined in $Q$ which is at the same time a viscosity subsolution and a viscosity supersolution.

It is a standard practice of the viscosity theory to accept upper semicontinuous (resp. lower semicontinuous) functions as viscosity sub/solutions (resp. supersolutions). We will not need such extensions here.

## 2 Some basic preliminaries

Due to the standard theory for parabolic, quasilinear, nondegenerate equations, cf. e.g. [LSU], Chapter 6, we know that if $a(u)$ is smooth and positive for $u > 0$ and $u_0 \in C(\mathbb{R}^n)$, with $u_0 > 0$ and bounded, equation (0.3) admits a bounded classical solution $u > 0$.

We are interested in checking the consistency of the concepts introduced in the last section with the classical definitions.

**Lemma 2.1** A nonnegative function $u \in C^{2,1}(Q)$ is a viscosity subsolution of (0.3) if and only if $u$ is a classical subsolution.

**Proof.** We give the easy and rather standard arguments for the reader’s convenience.

$(\Rightarrow)$ Suppose $u$ to be a viscosity subsolution. Then for all $\phi \in C^{2,1}$ that touches $u$ from above at $(x_0, t_0)$ the inequality

$$\phi_t \leq a(\phi)\Delta \phi + |\nabla \phi|^2$$

holds at $(x_0, t_0)$. Since $u \in C^{2,1}$ we may take $u = \phi$, and therefore $u_t \leq a(u)\Delta u + |\nabla u|^2$ in a classical sense in $Q$.

$(\Leftarrow)$ If $u$ is a classical subsolution, $\phi \in C^{2,1}$, and $\phi - u$ has a local minimum zero at $(x_0, t_0)$, then at this point the following is true:

$$\phi = u, \quad \nabla \phi = \nabla u, \quad \phi_t \leq u_t, \quad \Delta \phi \geq \Delta u.$$

Hence,

$$\phi_t \leq u_t \leq a(u)\Delta u + |\nabla u|^2 \leq a(\phi)\Delta \phi + |\nabla \phi|^2$$

and $u$ is a viscosity subsolution. $\square$

**Lemma 2.2** If $u$ is a classical free boundary subsolution, then $u$ is in particular a viscosity subsolution.

**Proof.** Let $\phi \in C^{2,1}(Q)$ be a test function, that touches $u$ from above at $P_0 = (x_0, t_0)$. If $u(P_0) > 0$, since $u$ is a classical free boundary subsolution it satisfies $u_t \leq a(u)\Delta u + |\nabla u|^2$ at $P_0$ and we argue as before. Let now $P_0$ be a point of the free boundary of $u$. If $|\nabla u| = 0$ we
have $\varphi = |\nabla \varphi| = 0$ and $\varphi_t \leq 0$, so that $\varphi_t \leq a(\varphi) \Delta \varphi + |\nabla \varphi|^2$. Obviously, if $\nabla u \neq 0$ at the free boundary, the test function $\varphi$ cannot touch $u$ from above and no comparison is needed. □

For supersolutions we start with a weaker result.

**Lemma 2.3** If a nonnegative function $u \in C^{2,1}(Q)$ is a viscosity supersolution of (0.3) then $u$ is a classical supersolution.

**Proof.** If $u$ is a smooth viscosity supersolution, we argue as before at a point of positivity with $\varphi = u$ to show that $u_t \geq a(u) \Delta u + |\nabla u|^2$.

At a point where $u$ is zero, we attain a minimum, and we necessarily have $u = \nabla u = u_t = 0$, so the equation is satisfied. □

Showing that a classical (free boundary) supersolution is a viscosity supersolution is not as easy as the proof for subsolutions, since we cannot prove directly the comparison with classical moving free boundary subsolutions, without proving first a Maximum Principle for degenerate equations. We will settle such questions after establishing the main results. In the sequel we will only need a certain type of classical free boundary supersolutions, namely the spherical traveling waves, defined in Section 4. In fact, we will only need to prove that these functions are limits of viscosity supersolutions, see Lemma 4.4.

**Positive solutions.** The situation is better when the data and solutions are strictly positive since we can apply the quasilinear uniformly parabolic theory [LSU], either locally or globally. Thus, given $u_0 \in C(\mathbb{R}^n)$, $0 < \varepsilon \leq u_0 \leq M$, we can solve Problem (0.3), (0.7) by means of this theory and obtain a bounded classical solution $0 < \varepsilon < u(x, t) < M$. We can also pose the existence problem in a bounded cylinder in space-time with strictly positive and continuous data on the parabolic boundary and get in the same way a unique classical solution $u \geq \varepsilon$.

We have agreement of the concepts of solution for such solutions.

**Lemma 2.4** Let $u \in C(\overline{Q})$ be a positive and bounded viscosity solution of (0.3). If $u(x, 0) \geq \varepsilon > 0$ then the viscosity solution is classical in $Q$, is equal or larger than $\varepsilon$ and unique. If the viscosity solution is positive and smooth in a bounded cylinder $C$ in space-time, it is classical and is uniquely determined in $C$ by the data on the parabolic boundary of that cylinder.

**Proof.** If $u$ is a viscosity solution and $u_0 \geq \varepsilon > 0$ then we can put below $u_0$ a Barenblatt subsolution (to be constructed in the Section 4, see formula (4.1)) at different locations and with different parameters and compare according to point (ii) of the definition of supersolution to prove that $u \geq \varepsilon$. Being uniformly positive, $u$ is locally uniformly parabolic and then $C^{2+\alpha,1+\alpha/2}$, therefore the unique classical solution with these properties, cf. the parabolic theory of [Wa1, Wa2]. The argument on a bounded cylinder is almost the same and we omit it. □

To conclude this section let us proof a sort of Weak Maximum Principle for positive viscosity solutions.
Lemma 2.5 Let $u$ be a viscosity subsolution and $u_1$ a classical solution of (0.3) defined in a bounded and closed space-time cylinder, $\Omega$. If $u \leq u_1$ in $\Gamma = \partial \Omega$, then $u(x,t) \leq u_1(x,t)$ in $\Omega$.

Proof. By the standard quasilinear theory we may approximate $u_1$ from above by the classical solution $u_\varepsilon$ of equation

$$(u_\varepsilon)_t = a(u_\varepsilon)\Delta u_\varepsilon + |\nabla u_\varepsilon|^2 + \varepsilon,$$

taking the boundary data $u_\varepsilon = u_1 + \varepsilon$. It is known that as $\varepsilon \to 0$, $u_\varepsilon$ converges to $u_1$ in a monotonically decreasing, locally uniform way. Suppose now that $u_\varepsilon$ touches $u$ from above at a point $P_1$ of $\Omega$. By definition of viscosity subsolution we have

$$(u_\varepsilon)_t \leq a(u_\varepsilon)\Delta u_\varepsilon + |\nabla u_\varepsilon|^2,$$

which is a contradiction. Hence $u_\varepsilon$ and $u$ do not touch in $\Omega$, $u < u_\varepsilon$. In the limit $u \leq u_1$ in $\Omega$.

Lemma 2.6 Let $v$ be a positive viscosity supersolution and $v_1$ a classical solution of (0.3), defined in a bounded and closed space-time cylinder, $\Omega$. If $v_1 \leq v$ in $\Gamma = \partial \Omega$, then $v_1(x,t) \leq v(x,t)$ in $\Omega$.

Proof. This lemma is proved in the same way as Lemma 2.5. Here we approximate $v_1$ from below by solutions $v_\varepsilon$ of

$$(v_\varepsilon)_t = a(v_\varepsilon)\Delta v_\varepsilon + |\nabla v_\varepsilon|^2 - \varepsilon,$$

with boundary data $v_\varepsilon = v_1 - \varepsilon > 0$ and apply the definition of viscosity supersolutions to $v_\varepsilon$.

3 The pressure-to-density transformation

At several places in the development of the viscosity theory we need to use arguments based on the weak theory of a related variable, called the density, $\rho$, which is obtained from $u$ via a functional transformation. This process allows us to change the original equation in non-divergence form into a divergence equation, where the weak theory is well established.

More specifically, we transform solutions $u \geq 0$ of equation (0.3) into solutions $\rho = T(u)$ of the related equation

$$(3.1) \quad \rho_t = \Delta \Phi(\rho)$$

for some nondecreasing real function $\Phi$, which must obviously be defined in terms of $T$ and the function $a(u)$ that appears in (0.3). This direct problem is not immediate to solve. However, the transformation has been studied in some detail in the converse direction and this is the way we will attack the problem here. Indeed, in case $\Phi$ is a power, $\Phi(\rho) = \rho^m$, $m > 1$, the last equation is the well-known Porous Medium Equation, $\rho$ is thought of as the density of a gas and $u$, which stands for the pressure, is given by the formula $u = (m/(m-1))\rho^{m-1}$. We
get in this way equation (0.3) with \( a(u) = (m - 1)u \). The transformation was widely used by Aronson and others [Ar] in establishing typical properties of the solutions of the Porous Medium Equation, like finite speed of propagation, optimal regularity or interface behaviour.

In the general context of the filtration equation with function \( \Phi \), the relation between \( \rho \) and \( u \) is more involved. It is given by the formula

\[
(3.2) \quad u = P(\rho) \equiv \int_c^\rho \frac{\Phi'(s)}{s} \, ds.
\]

This is what we call density-to-pressure transformation. In order for the calculations to make sense we consider the class \( \mathcal{F} \) of nondecreasing continuous functions \( \Phi : [0, \infty) \to [0, \infty) \) which are \( \mathcal{C}^2 \) on \( (0, \infty) \) and such that \( \Phi(0) = 0 \) and \( \Phi'(\rho) > 0 \) for all \( \rho > 0 \). The constant \( c \geq 0 \) is an arbitrary constant that is set to 0 if the integral is convergent, otherwise we need to start from some \( c > 0 \).

Actually, in order to focus on the problem at hand we will assume that the integral defining \( P \) is convergent at \( \rho = 0 \), which selects a subclass \( \mathcal{F}_0 \) of functions \( \Phi \). Under these assumptions it is clear that \( P \) is an increasing bijection of \([0, \infty)\) onto an interval \( I \) of the real line. This interval has the form \([0, \infty)\) if \( P(\rho) \to \infty \) as \( \rho \to \infty \), or a finite interval \( I = [0, h) \) otherwise.

Since we are working with bounded solutions the problem as \( \rho \to \infty \) need not concern us here, and we conclude that there is an inverse transformation \( \rho = P^{-1}(u) \) defined for all \( u \in I \). \( P \) and \( P^{-1} \) are \( \mathcal{C}^1 \) functions for \( u, \rho > 0 \) and continuous down to the value zero.

The relation between the equations at the classical level is established in the next result, that is essentially taken from [Kn], see also [BeM].

**Lemma 3.1** Let \( \rho(x, t) \in \mathcal{C}^{2,1}(Q) \) be a nonnegative classical solution of equation (3.1) with \( \Phi \in \mathcal{F}_0 \). If \( u \) is defined through (3.2), \( u = P(\rho) \), then \( u \) is a nonnegative classical solution of

\[
(3.3) \quad u_t = a(u)\Delta u + |\nabla u|^2,
\]

with values in \( I \), where \( a(u) \) is given by

\[
(3.4) \quad a(u) = \Phi'(P^{-1}(u)) > 0.
\]

**Proof.** The conclusion is easy to reach under these regularity assumptions. Indeed, if \( \rho > 0 \) then

\[
 u_t = \frac{\Phi'(\rho)}{\rho} \rho_t \quad \text{and} \quad \nabla u = \frac{\Phi'(\rho)}{\rho} \nabla \rho,
\]

so that

\[
 u_t = \frac{\Phi'(\rho)}{\rho} \rho_t = \frac{\Phi'(\rho)}{\rho} \Delta \Phi(\rho) = \frac{\Phi'(\rho)}{\rho} \div(\Phi'(\rho) \nabla \rho) = \frac{\Phi'(\rho)}{\rho} \div(\rho \nabla u)
\]

\[
 = \frac{\Phi'(\rho)}{\rho} \rho \Delta u + \frac{\Phi'(\rho)}{\rho} \nabla \rho \nabla u = \Phi'(\rho) \Delta u + |\nabla u|^2 = a(u)\Delta u + |\nabla u|^2,
\]

with \( \Phi'(\rho) = a(u) \).

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We will see later that this relation can be extended into a full correspondence of weak solutions of the Filtration Equation with viscosity solutions of equation (0.3).

Our next step at this time is to make sure that the transformation can be inverted. Since equation (3.3) is characterized by the function \(a(u)\) and equation (3.1) by \(\Phi\), we only need to make sure that the combined map \(A: \Phi \mapsto P \mapsto a\), which we have defined for \(\Phi \in \mathcal{F}_0\), can be inverted in some context.

**Lemma 3.2** The image of the map \(P: \Phi \mapsto a\) defined for \(\Phi \in \mathcal{F}_0\) is the set \(\mathcal{A}\) of functions \(a: [0, h) \to \mathbb{R}_+\) such that \(a(u)\) is \(C^1\) and positive for \(u > 0\) and \(\int du/a(u)\) diverges as \(u \to 0\).

For every \(a \in \mathcal{A}\) the inverse image is a one-parameter family of functions of \(\mathcal{F}_0\) of the form \(\{\Phi_k(s): k > 0\}\), where

\[
\Phi_k(s) = k \Phi_1(s/k).
\]

Moreover, \(\Phi(s)\) is \(C^r\), \(r \geq 1\), for \(s > 0\) if and only if \(a(u) \in C^{r-1}\) for \(u > 0\).

**Proof.** Given a function \(a\) that has been obtained from \(\Phi\) and \(P\) (which is derived also from \(\Phi\)), we want to prove that \(a \in \mathcal{A}\) and that we recover the original \(\Phi\) up to a parameter. We do this in two steps. First, we observe that

\[
\Phi'(s) = a(P(s)), \quad s P'(s) = \int \frac{du}{a(u)}.
\]

for every \(s > 0\). Therefore, from (3.2) we get

\[
\frac{1}{s} = \frac{P'(s)}{a(P(s))},
\]

where we have used \(s\) instead of \(\rho\) to stress that we are interested in the functional dependence. Putting \(u = P(s)\) and integrating this expression we get

\[
\int \frac{ds}{s} = \int \frac{du}{a(u)},
\]

and thus,

\[
P^{-1}(u) = k \exp\{\int du/a(u)\}. \tag{3.5}
\]

This identifies the monotone function \(P^{-1}\) but for the integration constant \(k > 0\). Calling \(P_k\) the function with constant \(k\) we have \((P_k)^{-1}(u) = k (P_1)^{-1}(u)\). This means that \(P_k(s) = P_1(s/k)\).

Once \(P^{-1}\) is known in terms of \(a\), we calculate \(\Phi\) from \(P\) thanks to the relation \(\Phi'(s) = s P'(s)\). Again, we need to check the integration constant, but \(\Phi(0) = P(0) = 0\). We get \(\Phi_k(s) = k \Phi_1(s/k)\). It is easy to see conversely that the family \(\{\Phi_k(s)\}\) produces the same \(a\).

Finally, formula (3.4) implies that \(a\) is \(C^1\) and positive function for \(u > 0\), and that

\[
\int_c^u \frac{du}{a(u)}
\]
must diverge as $c \to 0$. Under these conditions $P^{-1}$ is differentiable for $u > 0$ and $P^{-1}(0) = 0$. This means that $\Phi \in \mathcal{F}_0$. 

Next we point out sufficient conditions on $\Phi$ to ensure that the conditions on $a(u)$ stated in the Introduction hold. Note that since $a(P(s)) = \Phi'(s)$ we get $a'(P(s)) P'(s) = \Phi''(s)$. Using $\Phi'(s) = s P'(s)$ gives

$$a'(P(s)) = s \frac{\Phi''(s)}{\Phi'(s)}. \tag{3.6}$$

**Proposition 3.3**

(i) $\Phi$ is convex if and only if $a$ is monotone nondecreasing.

(ii) Suppose that $\Phi$ is positive and there exists $C > 0$ such that $s \Phi''(s) \leq C \Phi'(s)$. Then, $a'(u) \leq C$, so that $a(u) \leq C u$.

**Proof.** (i) This part is a direct consequence of the definition of $a$ and $\Phi$.

(ii) According to (3.6), $s \Phi''(s) \leq C \Phi'(s)$ implies $a'(P(s)) \leq C$. Putting $u = P(s)$ and integrating this means that $a(u) \leq C u$. 

Let us remark that a strengthened form condition (3.7) (involving a two-sided bound, $c \Phi'(s) \leq s \Phi''(s) \leq C \Phi'(s)$) has been chosen by Dahlberg and Kenig [DK1], [DK2] as a reasonable assumption in order to develop a theory of the Filtration Equation.

**Some examples.** It is well known that for the porous medium case, $\Phi(\rho) = c \rho^m$, we have

$$a(u) = (m - 1) u.$$ 

On the other hand, for an exponential function $\Phi(\rho) = e^{-1/\rho}$ we have

$$a'(u) = \frac{1}{\rho} - 2,$$

so that $a'(u) \to \infty$ as $u, \rho \to 0$, and $a$ is concave near $u = 0$. More precisely, since $u = P(\rho) \sim (1/\rho)e^{-1/\rho}$ as $\rho \to 0$, we get in first approximation $\log(1/u) = 1/\rho$, hence $a'(u) \sim \log(1/u)$, $a(u) \sim u \log(1/u)$. This example is not covered in our theory of viscosity solutions.

Next we go to the other limit, almost linear behaviour of $\Phi$. We set $\Phi(\rho) \sim \rho/(\ln^{\alpha} \rho)$, and take $\alpha > 1$ in order to have finite propagation. In this case, as $\rho \to 0$,

$$a(u) = \frac{1 - \alpha \ln^{-1} \rho}{\ln^{\alpha} \rho} \sim \frac{1}{\ln^{\alpha} \rho}.$$ 

Again, since $u = P(\rho) \sim (1 - \alpha)^{-1(1-\alpha)}(\rho)$, as $\rho \to 0$ we get $a(u) \sim cu^{1+\beta}$, with constants $c = (1 - \alpha)^{\alpha/(\alpha - 1)}$ and $\beta = 1/(\alpha - 1) > 0$.

Finally, we perform the computations in the reverse direction. We consider the case in which $a(u) = u^2$, then $\rho = P^{-1}(u) \sim k \exp\{-1/u\}$. If we take the constant $k = 1$ then $\Phi'(\rho) = 1/(\ln^2(\rho))$ and as $\rho \to 0$ we get $\Phi(\rho) \sim \rho/(\ln^2(\rho))$. 

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Remark 3 The pressure-density transformation can be applied for general monotone functions \( \Phi \) without the restriction that \( P(\rho) \) will be finite at \( \rho = 0 \), covering in this way the range of so-called fast diffusions. A detailed account of that generality will be given in [Va3]. The fast diffusion range is also studied in [ChV].

4 Examples of super- and subsolutions

In the sequel we will need two concrete examples of classical free boundary functions: the family of source-type subsolutions, also called Barenblatt functions; and the spherical traveling waves. We will also use a family of classical supersolutions of the so-called blow-up type. All three classes are modeled on explicit solutions of the the theory of the PME.

4.1 The main example of test functions we need are the Barenblatt functions.

Lemma 4.1 Let \( B(x, t; \tau, C) \) be the family of functions

\[
B(x, t; \tau, C) = \frac{(C(t + \tau)^{2\lambda} - K|x|^2)}{(t + \tau)},
\]

with constants \( \lambda, K, C, \tau > 0 \). If we assume \( a(u) \leq ku \) for some \( k > 0 \) in the range \( u \in [0, \text{max } B] \), and take \( \lambda = (kn + 2)^{-1}, 2K = \lambda \), then \( B(x, t; \tau, C) \) is a classical subsolution for equation (0.3).

Proof. If \( a(B) \leq kB \), with \( k > 0 \) we must prove that \( B_t \leq a(B)\Delta B + |\nabla B|^2 \) in the set of points where \( B > 0 \). First of all,

\[
B_t = (2\lambda - 1)C(t + \tau)^{2\lambda - 2} + K|x|^2(t + \tau)^{-2}.
\]

Since \( a(B) \leq kB \) and \( 2\lambda - 1 = -kn\lambda \), we also have

\[
a(B)\Delta B + |\nabla B|^2 = -2Kn(t + \tau)^{-1}a(B) + |2Kx(t + \tau)^{-1}|^2
\geq -\lambda nkC(t + \tau)^{2\lambda - 2} + \lambda nk(t + \tau)^{-2}K|x|^2 + 4K^2(t + \tau)^{-2}|x|^2
= (2\lambda - 1)C(t + \tau)^{2\lambda - 2} + 2K^2(nk + 2)|x|^2(t + \tau)^{-2},
\]

which implies that \( B \) is a classical subsolution for (0.3). \( \Box \)

Remark 4 (i) Since equation (0.3) is invariant under translations in space and time, these functions can be translated in an arbitrary way in space and time to form new subsolutions. 

(ii) Let us also point out that, as a consequence of Lemma 2.1, the Barenblatt functions are in particular viscosity subsolutions of (0.3).

(iii) Finally, the free boundary is given by the formula \( |x| = (C/K)^{1/2}(t + \tau)^\lambda \), hence there are no stationary points and it is a classical and moving free boundary subsolution.
The second important example are the traveling waves. It is convenient to consider first the case of one space dimension where we obtain plane traveling waves of the forms

\[ u_1(x, t) = A(x + ct - B)_+, \quad u_2(x, t) = A(-x + ct - B)_+, \]

with constants \( A, B, c > 0 \). A simple computation shows that both formulas (4.2) correspond to a classical moving free boundary supersolution defined in the whole region \( Q = \mathbb{R} \times (0, \infty) \) whenever \( 0 < A \leq c \). The first one moves to the left with speed \( c \), the second to the right with the same speed. Actually, when \( A = c \) they are exact classical free boundary solutions.

**Lemma 4.2** The plane traveling wave defined by formula (4.2) is a monotone limit of classical positive solutions.

*Proof.* We take one of the formulas, say the left one. First of all, we rewrite \( u(x, t) = A(x + ct - B)_+ \) as \( u = f(p \eta) \), with \( \eta = c(|x| + ct) \) and \( p = A/c \leq 1 \). When \( A = c \), \( u \) is a classical free boundary solution and \( f \) satisfies \( f' = a(f) f'' + (f')^2 \) whenever \( f > 0 \). Here primes indicate differentiation with respect to \( \eta \).

We now proceed in a rather standard way, approximating \( f \) by positive traveling waves \( f_{\varepsilon} \), such that \( f < f_{\varepsilon} \) and \( f_{\varepsilon}(-\infty) = \varepsilon, \quad f'_{\varepsilon}(-\infty) = 0 \).

Using the density-to-pressure transformation described in the Section 3, we may write the equation for \( f_{\varepsilon} \) as

\[ \rho_{\varepsilon}' = (\Phi(\rho_{\varepsilon}))''. \]

Integration of (4.4) yields \( \rho_{\varepsilon} = \Phi'(\rho_{\varepsilon}) + K \). Using the boundary conditions (4.3), we get \( (\rho_{\varepsilon} - \varepsilon_1) = \Psi'(\rho_{\varepsilon}) \), where \( \varepsilon_1 \) is the density corresponding to pressure \( \varepsilon \). Integrating this last expression gives

\[ \int_{\rho_{\varepsilon}}^{\rho_{\varepsilon} + K} \frac{\Phi'(\rho_{\varepsilon})}{\rho_{\varepsilon} - \varepsilon_1} d\rho_{\varepsilon} = \int_{0}^{\xi} d\xi. \]

The first integral diverges as \( \rho_{\varepsilon} \to \varepsilon_1 \), since \( \Phi'(\varepsilon_1) > 0 \) (see Section 3). In this way we obtain a classical solution \( \rho_{\varepsilon} \) of (4.4), which is always positive. We also see that \( \rho'_{\varepsilon} \) and \( (\Phi(\rho_{\varepsilon}))'' \) are positive and \( (\Phi(\rho_{\varepsilon}))'/\rho_{\varepsilon} \to 1 \) as \( \eta \to \infty \).

Undoing the transformation we get \( f_{\varepsilon} \), which is larger than \( \varepsilon \), increasing and such that \( f'_{\varepsilon}(\eta) \to 1 \) as \( \eta \to \infty \). It produces for \( A = c \) a classical positive exact solution \( u_{\varepsilon} \) of (0.3). If \( A < c \) it is easy to see that we get a classical supersolution.

Next step is passing to the limit as \( \varepsilon \) goes down to zero, to get that \( \rho_{\varepsilon} \to \rho \) uniformly over compact sets, and hence \( f_{\varepsilon} \to f \).

In several space dimensions we may consider the extension of these waves which is obtained by changing in (4.2) the argument \( x \) into \( x_1 \), or more generally into \( (x \cdot e) \), where \( e \) is a unit vector that denotes the direction of movement of the plane wave. But we also need to consider another type of supersolution, namely the spherical traveling waves, that have the form

\[ u(x, t) = A(|x| + ct - B)_+, \]

with \( A, B, c > 0 \). A simple computation shows that both formulas (4.5) correspond to a classical moving free boundary supersolution defined in the whole region \( Q = \mathbb{R} \times (0, \infty) \) whenever \( 0 < A \leq c \). The first one moves to the left with speed \( c \), the second to the right with the same speed. Actually, when \( A = c \) they are exact classical free boundary solutions.
with constants $A, B, c > 0$. We consider them as defined in a domain $\mathcal{R} = \{|x| < R, -T < t < 0\}$. They have radial symmetry in space with a hole in the support which is gradually filled as time passes. We observe that the support of $u$ above does not penetrate into the ball $B_{R/2}(0)$ if $2B > R$.

**Lemma 4.3** Assume that $a(u) \leq ku$, $k > 0$ constant, for $0 \leq u \leq M$, $R/2 \leq B < R$ and that

$$\frac{c}{A} \geq 1 + 2k(n - 1) \left(1 - \frac{B}{R}\right).$$

Then, the function given by (4.5) is a classical moving free-boundary supersolution for equation (0.3) in $\mathcal{R}$.

**Proof.** First of all, we have to prove that in the set $|x| > B - ct$, where $u > 0$,

$$u_t \geq a(u)\Delta u + |\nabla u|^2$$

holds. Therefore, we must show that

$$cA \geq a(u)\frac{A(n - 1)}{|x|} + A^2,$$

at all points of $\mathcal{R}$ where $u > 0$. Using the property that $a(u) \leq ku$, we need to show that

$$k(n - 1)u \leq (c - A)|x|.$$

Since $|x| < R$ and $t < 0$, it is true that

$$u = A(|x| + ct - B) \leq A(R - B).$$

Moreover, we have $2|x| \geq R$ at the points where $u > 0$, hence we get the sufficient condition

$$k(n - 1)A(R - B) \leq (c - A)R/2.$$

This condition is satisfied, therefore $u$ satisfies $u_t \geq a(u)\Delta u + |\nabla u|^2$ wherever $u > 0$ in $\mathcal{R}$. On the other hand, the moving free-boundary condition $v_n \geq |\nabla u| \neq 0$ is satisfied since $c > A$. \(

**Lemma 4.4** The spherical traveling wave defined by formula (4.5) is a monotone limit of classical positive supersolutions.

**Proof.** We retake the approximation of $f$ by $f_\varepsilon$ done in dimension one. When $n > 1$ we use the new condition on $A$ and $c$, given by (4.6), and observe that there is an extra first-order term, $(n - 1)a(f)f'(\eta)/|x|$, that has a positive sign, which is bad. However, it is uniformly similar to the term $(n - 1)a(u)A/|x|$ considered in Lemma 4.3 for $\varepsilon$ small enough, hence we still have that $f_\varepsilon$ is a classical supersolution if the assumptions of Lemma 4.3 are satisfied and $\varepsilon$ is small. \(\square\)
The next family of supersolutions does not play a big role, but we need them in order to control the maximal size of all solutions. We add an extra assumption: \( a(u) \leq k u \) for all \( u \geq 0 \). In that case we consider the family of functions

\[
U_{bu}(x, t) = M + b \frac{x^2}{T-t} + ct,
\]

where \( M \) and \( T \) are arbitrary positive constants, and \( b, c > 0 \) have to be chosen conveniently so that \( U_{bu} \) is a classical strict supersolution in the domain \( S = \mathbb{R}^n \times (0, T/2) \). Because of the term \( 1/(T-t) \) the functions are known in the PME theory as the blow-up solutions. Checking the strict supersolution condition in our context amounts to

\[
bx^2 + c(T-t)^2 > 2nbk(bx^2 + (M + ct)(T-t)) + 4b^2x^2.
\]

We will control separately the \( x^2 \)-terms and the time-depending terms. For the first we take \( 2b(nk + 2) < 1 \), which is fulfilled if \( b \) is small enough. For the latter, since \( 0 < t < T/2 \) it is enough that \( cT/2 > 2nbk(M + cT/2) \), hence

\[
2nbkM < \frac{cT}{2}(1 - 2nbk).
\]

In the application below we will fix \( M \) and \( T \) and will take \( b = \varepsilon > 0 \) small, and then the acceptable values of \( cT \) are also \( O(\varepsilon) \), uniformly in the rest of the data.

\section{Existence of the maximal viscosity solution}

We may now proceed with the construction of a viscosity solution for given data \( u_0 \). Indeed, the constructed solution will be shown to be maximal among all bounded and nonnegative viscosity solutions with the same data.

Given \( u_0 \in C(\mathbb{R}^n) \), \( u_0 \) nonnegative and bounded, say, \( 0 \leq u_0(x) \leq M \), we begin by solving Problem (0.3), (0.7) with bounded data \( u_{0,\varepsilon}(x) \geq u_0(x) + \varepsilon \). Applying the quasilinear uniformly parabolic theory [LSU] we obtain a classical solution \( u_{\varepsilon} \geq \varepsilon \). If the data are ordered, i.e., \( u_{0,\varepsilon'} < u_{0,\varepsilon} \) whenever \( 0 < \varepsilon' < \varepsilon \), by the Maximum Principle for classical solutions these solutions are ordered, \( u_{\varepsilon} \geq u_{\varepsilon'} \) if \( \varepsilon \geq \varepsilon' \). We then pass to the limit as \( \varepsilon \) goes down to zero to get

\[
\bar{u}(x, t) = \lim_{\varepsilon \to 0} u_{\varepsilon}(x, t),
\]

which is a nonnegative bounded function obtained as limit of a monotone sequence. In principle, we would be inclined to take \( u_{0,\varepsilon}(x) = u_0(x) + \varepsilon \) for simplicity, but it will be convenient in the development of the theory to take this definition in the ball of radius \( 1/\varepsilon \) while \( u_{0,\varepsilon}(x) = M + \varepsilon \) for \( |x| \geq 2/\varepsilon \) and define \( u_{0,\varepsilon}(x) \) in a continuous way for \( 1/\varepsilon < |x| < 2/\varepsilon \).

This behaviour at infinity will be needed in the proof of maximality. After the main result is established we will be able to show that the precise way of approximation is not important, so that the simpler definition also produces the maximal solution.

**Proposition 5.1** \( \bar{u} \) is a viscosity solution of Problem (0.3), (0.7).
Proof. We first observe that $\bar{\pi}$ is a continuous function. Instead of doing a study of regularity of our equation, we derive the uniform continuity result as a consequence of

(i) the pressure-to-density transformation, explained in the Section 3, that transforms solutions of equation (0.3) into solutions of the filtration equation $\rho_t = \Delta \Phi(\rho)$, and

(ii) the continuity of solutions to the filtration equation, which is described in Appendix A.

Let us point out that due to the continuity of $\bar{\pi}$ and $u_\varepsilon$ and the monotonicity of the sequence $\{u_\varepsilon\}$ we know that the convergence in (5.1) is locally uniform. But the continuity of the limit is also a consequence of the equicontinuity of bounded solutions of the filtration equation.

We continue with the proof by checking that $\bar{\pi}$ is a viscosity subsolution. Let $\varphi \in C^{2,1}(Q)$ be a test function such that $\bar{\pi} - \varphi$ has a local maximum zero at a point $P_0$, say $(0,0)$, in a parabolic neighborhood $\Omega = B_r(0) \times (-r^2,0]$, with $r$ small. As we have said, by replacing $\varphi$ by $\psi = \varphi + \delta(x^2 + t^2)$ we may assert that $v = \bar{\pi} - \psi$ has a strict maximum zero in $\Omega$ precisely at $P_0$ and $v_{\mid \Gamma} \leq -\delta r^4$, where $\Gamma$ is the parabolic boundary of $\Omega$.

Since $u_\varepsilon \to \bar{\pi}$ uniformly we can take $|u_\varepsilon - \bar{\pi}| \leq \delta r^4/4$ and then define $v_\varepsilon = u_\varepsilon - \psi_\varepsilon = u_\varepsilon - (\psi + c_\varepsilon)$, with a constant $c_\varepsilon$ which will be determined later. Under these assumptions we get

$$v_\varepsilon(P_0) \geq -\frac{\delta r^4}{4} - c_\varepsilon \text{ and } v_{\varepsilon|\Gamma} \leq -3\delta r^4/4 - c_\varepsilon = -\delta_1.$$

It follows that $v_\varepsilon$ must attain its maximum in $\Omega$ in an interior point $P_\varepsilon$. If $c_\varepsilon$ is chosen so that $\psi_\varepsilon(P_\varepsilon) = u_\varepsilon(P_\varepsilon)$, then $v_\varepsilon$ has a local maximum zero at $P_\varepsilon$. Now, since $u_\varepsilon$ are strictly positive classical solutions, they are, in particular viscosity solutions, and so we get that

$$(\psi_\varepsilon)_t \leq a(\psi_\varepsilon)\Delta \psi_\varepsilon + |\nabla \psi_\varepsilon|^2 \text{ at } P_\varepsilon.$$  

Finally, as $\varepsilon \to 0$ we can choose $r \to 0$ so that $P_\varepsilon \to P_0$ and $c_\varepsilon \to 0$. Therefore,

$$\varphi_t \leq a(\varphi)\Delta \varphi + |\nabla \varphi|^2 \text{ at } P_0.$$

We have shown that $\bar{\pi}$ is a viscosity subsolution.

To prove that $\bar{\pi}$ is a viscosity supersolution we choose a test function $\varphi$ such that $\bar{\pi} - \varphi$ has a local minimum zero. This leads to point (i) of the definition of supersolution, and we proceed analogously as it was done for the subsolution. But we also have to verify that every classical free-boundary subsolution that lies below $\bar{\pi}$ at a time $t_1$ cannot cross $\bar{\pi}$ at a later time $t_2 > t_1$. Let $v$ be a classical moving free-boundary subsolution such that $v(x,t_1) < \bar{\pi}(x,t_1)$. Since, by construction of $\bar{\pi}$ and by the strict monotonicity of $\{u_\varepsilon\}$ we have $u_\varepsilon(x,t) > \bar{\pi}(x,t)$, we get in particular $v(x,t_1) < \bar{\pi}(x,t_1) < u_\varepsilon(x,t_1)$. Now, since $u_\varepsilon$ are classical solutions, and they are also positive, and since $v$ is classical where it is positive, a standard application of the Maximum Principle says that $v$ cannot touch $u_\varepsilon$ from below, hence $u_\varepsilon(x,t_2) > v(x,t_2)$ (see the argument of the Proposition 5.3). Finally, passing to the limit as $\varepsilon$ tends to zero we get $\lim_{\varepsilon \to 0} u_\varepsilon(x,t_2) = \bar{\pi}(x,t_2) \geq v(x,t_2)$.

$\square$
Before we prove maximality we need a technical lemma.

**Lemma 5.2** Any bounded viscosity solution with data \( u_0(x) \leq M \) is bounded above by \( M \) in \( Q \).

**Proof.** If the solution is in principle less than a constant \( M_2 > M \) we take \( k \) as the supremum of \( a(u)/u \) for \( 0 \leq u < M_2 + 1 \). We fix some time \( T > 0 \) and for every small constant \( \varepsilon > 0 \) we consider the blow-up function

\[
U_1(x, t) = M' + b \frac{x^2}{T' - t} + ct,
\]

with parameters \( M' = M + \varepsilon, T' = 2T, b = \varepsilon \) and \( c = \mathcal{O}(\varepsilon) \) as prescribed in (4.8). Then \( U_1 \) is a strict classical supersolution of the equation satisfied by \( u \) in the strip \( S = \mathbb{R}^n \times (0, T) \) as long as \( U_1 \) is less than \( M_2 + 1 \). It is clear that \( U_1 \) is larger than \( u \) at the initial time and also that both functions are strictly separated for all large \( |x| \) uniformly in \( 0 < t < T \) in view of the assumed behaviour. Therefore, if \( U_1 \) is not strictly larger than \( u \) in \( S \), there must be a first point \( P_1 = (x_1, t_1), x_1 \in \mathbb{R}^n, 0 < t_1 < T \), at which \( U_1 \) touches \( u \) from above. At this point we apply the definition of subsolution to \( u \) with test function \( U_1 \) to arrive at the inequality

\[
(U_1)_t(P_1) \leq a(U_1(P_1)) \Delta U_1(P_1) + |\nabla U_1(P_1)|^2.
\]

This is a contradiction with the fact that \( U_1 \) is a strict supersolution in a small parabolic neighbourhood of \( P_1 \) (where \( U_1 \leq M_2 + 1 \)). We conclude that necessarily \( u(x, t) < U_1(x, t) \) in \( S \). Letting \( \varepsilon \to 0 \) we arrive at \( u(x, t) \leq M \) everywhere in \( S \).

**Proposition 5.3** The viscosity solution \( \bar{u} \) defined in (5.1) is the maximal viscosity solution of the Cauchy problem (0.3), (0.7).

**Proof.** Let \( \hat{u} \) be another viscosity solution for Problem (0.3) with data \( u_0 \). We compare \( \hat{u} \) with the family \( u_{\varepsilon} \) that produces \( \bar{u} \). Observe that at \( t = 0 \) the initial data satisfy \( u_{0,\varepsilon} > u_0 \). We have to prove that \( u_{\varepsilon} \) lies above \( \hat{u} \) at every time \( t > 0 \).

The possibility that \( \hat{u} \) jumps over \( u_{\varepsilon} \) and that the points of sign change move to infinity as \( t \to 0 \) is eliminated by the previous lemma and the construction of the \( u_{\varepsilon} \). Therefore, if \( u_{\varepsilon} \) is not larger than \( \hat{u} \) everywhere we may consider a point \( P_0 \) where \( \hat{u} \) touches \( u_\varepsilon \) from below, i.e., the difference reaches a local maximum zero. Since \( u_\varepsilon \) is positive, so is \( \hat{u} \) in a neighbourhood of \( P_0 \), it follows that there is a parabolic neighbourhood of \( P_0 \) where both functions are positive, hence smooth solutions, and they are ordered. By the Strong Maximum Principle, they cannot touch at \( P_0 \), a contradiction that eliminates the possibility of touching from below. We conclude that \( \hat{u} < u_{\varepsilon} \) in \( Q \).

Now we can pass to the limit as \( \varepsilon \to 0 \) and get \( \bar{u} = \lim u_{\varepsilon} \geq \hat{u} \) at every time and therefore \( \bar{u} \) is the maximal solution of (0.3), (0.7).
Proposition 5.4 (Comparison of maximal solutions) The maximal solutions are ordered in terms of the data: if \( \overline{u} \) and \( \hat{u} \) are two maximal solutions with ordered initial data, \( \overline{u}_0(x) \geq \hat{u}_0(x) \) in \( \mathbb{R}^n \), then \( \overline{u}(x,t) \geq \hat{u}(x,t) \) in \( Q \).

**Proof.** Let \( \overline{u}_0 \) and \( \hat{u}_0 \) be two initial data such that \( \overline{u}_0(x) \geq \hat{u}_0(x) \). We perform the above construction with \( \overline{u}_0,\varepsilon(x) \geq \hat{u}_0,\varepsilon(x) \). By the Maximum Principle it follows that \( \overline{u}_\varepsilon(x,t) \geq \hat{u}_\varepsilon(x,t) \). Passing to the limit, \( \overline{u}(x,t) \geq \hat{u}(x,t) \). \( \square \)

Proposition 5.5 If we apply the density-to-pressure transformation of Lemma 3.1 to the bounded weak solutions of equation (0.6), the image contains the maximal viscosity solutions of equation (0.3) obtained in (5.1).

**Proof.** Given \( \rho_0 \geq 0 \) continuous and bounded, we approximate \( \rho_0(x) \) by smooth and positive data \( \rho_\varepsilon(x) \) with the condition that the family \( \rho_\varepsilon \) approximates \( \rho_0 \) uniformly and monotonically from above as \( \varepsilon \to 0 \). The theory of quasilinear parabolic equations implies (see [LSU], Chapter 5) that the Cauchy problem can be solved in a unique way for all the approximate problems, that the corresponding solutions \( \rho_\varepsilon \) are indeed classical and that they form a monotone nonincreasing sequence. By the regularity results of [DbV] and [Sa] we know that \( \rho \) is a continuous solution, (see Appendix A for more details). Therefore the sequence \( \{\rho_\varepsilon\} \) converges locally uniformly to \( \rho \).

Due to Lemma 3.1 we know that \( u_\varepsilon = P(\rho_\varepsilon) \), is a classical solution of (0.3). Now, passing to the limit as \( \varepsilon \) goes down to zero we get

\[
\overline{u} = \lim_{\varepsilon \to 0} u_\varepsilon = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\rho_\varepsilon} \frac{\Phi'(s)}{s} ds = \int_{\rho}^{\rho} \frac{\Phi'(s)}{s} ds,
\]

therefore \( \overline{u} = P(\rho) \) and the limit is locally uniform. \( \square \)

6 Support properties

In this section we state some properties of the support of viscosity solutions.

**Proposition 6.1** The support of any viscosity supersolution is nondecreasing in time. In fact, it penetrates the whole space as \( t \to \infty \).

**Proof.** Let \( u \) be a viscosity supersolution and let \( u(x_0,t_0) > 0 \). We want to prove that \( u(x_0,t) > 0 \) for \( t > t_0 \). Let \( \mathcal{B} \) be a Barenblatt subsolution such that

\[
u(x_0,t_0) > \mathcal{B}(x-x_0;0;\tau,C).
\]

By part (ii) of Definition 1.5 \( \mathcal{B} \) stays below \( u \) at all later times, hence

\[
u(x_0,t) > \mathcal{B}(x-x_0,t-t_0;\tau,C) > 0 \quad \text{if} \quad t > t_0.
\]

\( \square \)

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Proposition 6.2 The support of any viscosity subsolution expands in a continuous way, i.e., the boundary of the support does not jump forward at any given time.

Proof. Let $u$ be a viscosity subsolution. We will prove that the possible expansion of $\text{supp}(u)$ must take place in a continuous way. We can reduce it to check the movement at $t = 0$. In technical terms we want to prove the following: given a small $\varepsilon > 0$ and a radius $R \gg 0$, we want to find $\tau > 0$ such that $\text{supp}(u(\cdot, t)) \cap B_R(0)$ is included in an $\varepsilon$-neighbourhood of $\text{supp}(u_0) \cap B_R(0)$ if $0 < t < \tau$.

In order to prove this statement, we choose a point $x_0 \in B_R(0)$ that lies at distance $\varepsilon$ of $\text{supp}(u_0) \cap B_R(0)$ and show that $u(x_0, t)$ is zero for $0 < t < \tau$. This is done by comparison with the spherical traveling waves of Section 4, which have a finite speed of propagation. It is done as follows: Let $u_{A,c,R,B,T}$ the profile given in formula (4.5). We now put

$$W(x, t) = u_{A,c,\varepsilon,\varepsilon/2,\tau}(x - x_0, t - \tau)$$

and select the parameters $A$, $c$ and $\tau$ so that at $t = 0$ we have $u_0(x) \leq W(x, 0)$ in $B_{\varepsilon}(x_0)$, and that $W(x, t) \geq M = \|u\|_{\infty}$ on the parabolic boundary of the cylinder where $W$ is defined. We will have the conditions on the parameters of Lemma 4.3 plus $A(\varepsilon - c\tau - B) \geq M$. This leads for small $\varepsilon$ to high values of $c$ and $A$ and a very small values of $\tau$. Since $W$ is limit of smooth, positive supersolutions we necessarily have

$$u(x_0, t) \leq W(x_0, t) = 0$$

for all $0 \leq t \leq \tau$. The choice of the parameters can be done locally uniformly. We recall that the justification is done by comparing with the positive approximations to the spherical traveling waves and passing to the limit. Once the main comparison result (Theorem 10.2) is proved this trick will not be necessary.

Next, we prove that the support of a solution actually moves at a given time under certain conditions on the form of the solution near the free boundary. This result is not needed in the proof of well-posedness.

Lemma 6.3 Let $u$ be a viscosity solution such that $u_0 \in C^2$ on the closure of its positivity set $\Omega_0$ and $|\nabla u_0| \neq 0$ near a point $x_0 \in \Gamma_0 = \partial \Omega_0$. Then the free boundary of $u$ moves immediately in the outward direction at $t = 0$ near $x_0$.

Proof. (i) Let $x_0$ be a point of $\Gamma_0$. Since $|\nabla u_0| \neq 0$ at $x_0$ we may suppose without loss of generality that $\partial u_0 / \partial x_n(x) \neq 0$ for $x$ near $x_0$. Note that after translation and rotation of the axes we may assume that $x_0 = 0$ and that $\nabla u(x_0)$ points in the direction of the negative $x_n$ axis. Due to the Implicit Function Theorem, there exist a neighbourhood $U$ of the point $(x_0, \ldots, x_{n-1}, u) = (x'_0, u) = (0, 0) \in \mathbb{R}^n$ and a function $\varphi \in C^2(U)$ such that the following holds: if we write a point of $U$ as $(x', u)$ with $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}$, then the function

$$(6.1) \quad \Phi(x', u) = (x', \varphi(x', u)),$$
is a \( C^2 \) diffeomorphism from \( U \) onto its image \( V \), which is a neighbourhood of \( x_0 = 0 \), and \( \Phi(0, 0) = x_0 = 0 \). Moreover, \( \Gamma_0 \cap V \) is precisely the image by \( \Phi \) of \( U \cap \{ u = 0 \} \), with equation

\[(6.2) \quad x_n = \varphi(x', u), \]

and \( \Omega_0 \cap V = \Phi(U \cap \{ u < 0 \}) \), i.e. \( x_n < \varphi(x', u) \) for \( (x', u) \in U \).

(ii) By the \( C^1 \) regularity, the surface given by \( z = u_0(x) \) has a tangent plane at \( x_0 = 0 \). This plane is given by the equation \( \nabla u_0(0) \cdot (x) = z \) and since \( \nabla u_0(x_0) \neq 0 \) it is not horizontal. With our simplifying assumptions we may write it in the form

\[z = -\alpha x_n, \quad \alpha = -\frac{\partial u_0}{\partial x_n}(0).\]

On the other hand, since \( u_0 \in C^2 \) on its support and \( \nabla u_0 \neq 0 \) near \( x_0 = 0 \), the curvature of \( \Gamma_0 \) near \( x_0 \) is finite, \( 0 \leq k_0 < \infty \). Therefore \( \Omega_0 \) has the interior ball property with radius \( R = 1/k_0 \) at \( x_0 \).

(iii) Let \( B_1 \) be a small ball with center \( x_0 = 0 \) and radius \( \delta \) and let \( B \) be a Barenblatt subsolution, as in (4.1), chosen such that \( B \) almost touches \( x_0 \). We also choose the constants in \( B \) in such a way that the support of \( B(x, 0) \) is a small ball \( B' \) contained inside \( \Omega_0 \cap B_1 \).

(iv) We want to prove that we may place the surface \( z = B(x, 0) \) below \( z = u_0(x) \); i.e \( u_0(x) > B(x, 0) \) in \( B' \). If we take \( |\nabla B(x_0, 0)| \) small enough, for instance \( |\nabla B(x_0, 0)| < \alpha/2 \), then the tangent plane to \( B \) at \( x_0 \) is below the tangent plane to \( u_0 \) at \( x_0 \) for all \( x \in B_1 \) and hence \( B(x, 0) \) is below \( u_0(x) \) in \( B_1 \cap B' \). Indeed, let \( x_1 = (x'_n, x') \) be the projection of \( x = (x_n, x') \) on \( \partial B' \) parallel to the \( x_n \)-axis and \( x_2 = (x''_n, x') \) the projection of \( x \) on \( \Gamma_0 \) also parallel to the \( x_n \)-axis, with \( x_n < x'_n < x''_n < 0 \). Due to the Taylor Theorem, if \( x \in B_1 \cap B' \) we may write

\[u_0(x) \sim u_0(x_2) + \frac{\partial u_0}{\partial x_n}(x_2)(x_n - x''_n) \sim \alpha(x''_n - x_n),\]

\[B(x, 0) \sim B(x_1, 0) + \frac{\partial B}{\partial x_n}(x_1)(x_n - x'_n) \leq \frac{\alpha}{2}(x'_n - x_n).\]

Since \( x'_n - x_n < x''_n - x_n \) we conclude that \( B(x, 0) < u_0(x) \) for all \( x \in B' \), (see Figure 1).

(v) Finally, since the free boundary of \( B \) moves immediately and \( B \) and \( u \) are strictly separated at \( t = 0 \), \( B \) cannot cross \( u \) at a later time \( t > 0 \), and therefore the free boundary of \( u \) also moves immediately at \( t = 0 \), at a linear rate.

\[\square\]

\section{Supremum and infimum convolutions}

Given a viscosity subsolution \( u \) and a radius \( r > 0 \) we define the function \( \overline{v}_r \) as

\[(7.1) \quad \overline{v}_r(x, t) = \sup_{\overline{B}_r(x, t)} u(y, \tau),\]

where \( \overline{B}_r(x, t) = \{(y, \tau) : |y - x|^2 + (\tau - t)^2 \leq r^2\} \) is a closed ball in space-time.
In the same way, given a viscosity supersolution \( u \) we define

\[
\nu_r(x, t) = \inf_{B_r(x, t)} u(y, \tau).
\]

The interest of these constructions, taken from [CV], lies on the following properties.

**Lemma 7.1** Function \( \nu_r \) is a subsolution and function \( \nu_r \) is a supersolution of (0.3) for \( t \geq r \).

**Proof.** First of all let us point out that we need \( t \geq r \) to have the balls \( B_r(x, t) \) defined for positive times.

(i) Let us show that \( \nu_r \) is a viscosity subsolution. Given a point \( P_0 = (x_0, t_0) \) in the domain, let \( P_1 = (x_1, t_1) \) be the point in \( B_r(P_0) \) where the supremum in the definition of \( \nu_r(P_0) \) is attained. We define the translate

\[
u_h(x, t) = u(x + h_1, t + h_2)
\]

where \( h_1 = x_1 - x_0, \ h_2 = t_1 - t_0. \) It is immediate that \( \nu_h \) is a viscosity subsolution in the translated domain, that \( \nu_h(P_0) = u(P_1) = \nu_r(P_0). \) By the definition of \( \nu_r \) as a supremum we have \( \nu_h \leq \nu_r \) in a small parabolic neighborhood of \( P_0. \) We conclude that whenever a test function \( \varphi \) touches \( \nu_r \) from above at \( P_0, \) then it also touches \( \nu_h \) from above at the same point, hence the condition of subsolution is satisfied by \( \varphi \) at \( P_0. \) We conclude that \( \nu_r(x, t) \) is a viscosity subsolution.

(ii) In the same way it can be proved that \( \nu_r \) is a viscosity supersolution. Note that comparison with classical free boundary subsolutions is also necessary. Indeed, let \( v \) be a classical free boundary subsolution, such that \( v(x, t_0) < \nu_r(x, t_0). \) First of all, observe that
(7.2) is equivalent to
\[ \varphi_r(x, t) = \inf_{|h| \leq r} u_h(x, t), \]
with \( u_h(x, t) = u(x + h, t + h) \), for \( |h| \leq r \). We have \( v(x, t_0) - \varphi_r(x, t_0) \leq u_h(x, t_0) \). Since the functions \( u_h \) are viscosity supersolutions we have \( v(x, t) \leq u_h(x, t) \) for \( t \geq t_0 \) and finally taking the infimum over the values \( |h| \leq r \) we get the desired result. \( \square \)

These constructions are important for us because of the special boundary regularity they enjoy that is reflected in the following result.

**Lemma 7.2** (i) The positivity set of \( \varphi_r \) has the interior ball property with radius \( r \) at every point of its boundary and the support of \( \varphi_r \) has the exterior ball property with radius \( r \).

(ii) At the points of the boundary of the support of \( u \) where these balls are centered we have the complementary statements: an exterior ball in the first case and an interior ball in the second.

**Proof.** (i) Let \( \text{Pos}(\varphi_r) \) be the positivity set of \( \varphi_r \) and let \( P_0 = (x_0, t_0) \) a point of its boundary, \( \Gamma_r = \partial(\text{Pos}(\varphi_r)) \). We necessarily have \( u(P_0) = 0 \). By the definition of \( \varphi_r \) as a supremum, there must exist a point \( P_1 \) at distance \( r \) from \( P_0 \) lying at the boundary of the positivity set of \( u \). Clearly, \( \varphi_r(P_1) > 0 \).

Now, we will prove that the ball of radius \( r \) and center \( P_1 \) is contained in \( \text{Pos}(\varphi_r) \), hence it is the interior ball to the boundary of \( \varphi_r \) mentioned in the statement.

Since \( P_1 \in \partial \text{Pos}(u) \), there exists a sequence of points \( P_{1n} \to P_1 \), such that \( u(P_{1n}) > 0 \). Let now \( P' \) be such that \( d(P', P_1) < r \). Observe that for large \( n \) we also have \( d(P_{1n}, P') < r \), and that by definition \( \varphi_r(P') = \sup_{B_r(P')} u \). Consequently, \( \varphi_r(P') \geq u(P_{1n}) > 0 \). Therefore \( P' \) belongs to the positivity set of \( \varphi_r \).

(ii) Conversely, at every point \( P_1 \) of the boundary of \( u \) where we can locate the center of an inner ball of radius \( r \) to the boundary of \( \text{Pos}(\varphi_r) \), we have an exterior ball of radius \( r \) to the support of \( u \). Indeed, if \( P_0 \) is the point of the previous step we have \( d(P_1, P_0) = r \) and \( \text{supp} u \cap B_r(P_0) = \emptyset \) (otherwise \( \varphi_r(P_0) > 0 \)).

Arguing analogously we can prove the other statements. \( \square \)

**Inf convolution with decreasing radius.** In the sequel we will need a slight modification of the construction for supersolutions that shrinks the balls as time proceeds. Thus we will take as definition for \( \varphi_{r, \tau} \),
\[ \varphi_{r, \delta}(x, t) = \inf_{B_{r-\delta t}(x, t)} u(y, \tau). \]
If \( \delta > 0 \), \( \varphi_{r, \delta} \) will still be a supersolution and Lemma 7.1 still applies as long as \( \delta t < r \). Indeed, when we try to repeat the proof at a point \( P_0 \) we still find a point \( P_1 \) at a distance (in space-time) equal or less than \( r_0 = r - \delta t_0 \) at which \( \varphi_{r, \delta}(P_0) = u(P_1) \). Defining again the translate \( u_h(P) = u(P + h) \) with \( h = P_0 - P_1 \) we have \( \varphi_{r, \delta}(P_0) = u_h(P_0) \). Let us now
consider $\Omega$ a parabolic neighborhood of $P_0$. For every $P = (x, t) \in \Omega$ we have $t \leq t_0$, so that $d(P + h, P) \leq r_0 \leq r$, hence

$$u_h(P) = u(P + h) \geq v_{r, \delta}(P).$$

There is however a difference with respect to the result stated in Lemma 7.2: given a point $P_0 = (x_0, t_0)$ of the free boundary of $v_{r, \delta}$ we still conclude that the ball of radius $r_0 = r - \delta t_0$ centered at $P_0$ is an interior ball to $\text{Pos}(u)$ with tangency at a certain point $P_1 \in \Gamma(u)$. But now, the property of the exterior ball for $\text{supp}(v_{r, \delta})$ has to be re-examined since the radius of the ball where the minimum is taken changes with the point. Instead, we have

**Lemma 7.3** Given points $P_0 = (x_0, t_0)$ and $P_1 = (x_1, t_1)$ as above, the set of points where we can assert that $v_{r, \delta} = 0$ includes the region limited by the ellipsoid

$$S = \{(x, t) : (x - x_1)^2 + (t - t_1)^2 = (r - \delta t)^2\},$$

which passes trough $P_0$. An exterior ball to $\text{supp}(v_{r, \delta})$ exists therefore at $P_0$ but it has a smaller radius than $r_0$.

An important observation for the next section is the following: for given $P_0$ and $P_1$ the tangent hyperplane $H$ to the surface $S$ at $P_0$ depends on $\delta$. In fact, it always has the same trace on the plane $t = t_0$, which is an $n$-dimensional sphere of radius $|x_0 - x_1| = r_0 \cos \beta$ (here $\beta$ denotes the angle formed by the vector $P_1 P_0$ with the space hyperplanes $t =$constant). But the time component of the unit normal to $H$ in $\mathbb{R}^{n+1}$ is given by $n_{n+1}(P_0) = \sin \theta$, where

$$\tan(\theta) = \frac{\sin \beta + \delta}{\cos \beta},$$

which increases as $\delta$ grows. Note that $\tan(\theta)$ is precisely the velocity with which the space projection of this tangent hyperplane advances in time.

**8 Comparison of strictly separated viscosity solutions**

We are now ready to establish a main result of the paper, the monotone ordering among viscosity solutions. It is here that the main ideas of the theory of viscosity solutions will be used, together with a detailed geometric analysis of possible touching points which was first described in [CV].

**Theorem 8.1** Suppose that $u$ is a viscosity subsolution and $v$ is a viscosity supersolution of equation (0.3) in $Q$, and they have strictly separated initial data, $u_0 \prec v_0$. Then the solutions remain ordered for all time, $u(x, t) \leq v(x, t)$ for every $(x, t) \in Q = \mathbb{R}^n \times (0, \infty)$. 

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Proof. Step i: Sup and inf functions.

We take $\delta > 0$ and $r > 0$ small with $r \gg \delta$ and introduce the function

\begin{equation}
W(x,t) = \inf_{B_{r-\delta t}(x,t)} v(y,\tau),
\end{equation}

which is well defined in $\mathbb{R}^n \times (t',T)$, with $T = r/\delta$, $t' = r/(1 + \delta)$ and

\begin{equation}
Z(x,t) = \sup_{B_r(x,t)} u(y,t'),
\end{equation}

well defined in $\mathbb{R}^n \times (r,T)$. Two facts are then true if $\delta$ and $r$ are small enough and $T \leq r/\delta$:

(i) $W$ is a supersolution and $Z$ is a subsolution of equation (0.3) in $Q_T = \mathbb{R}^n \times (r,T)$,

(ii) $Z(\cdot, r) \prec W(\cdot, r)$.

The first statement comes from Section 7. The second from the fact that $u_0 \prec v_0$ and from the Proposition 6.2 which ensures that the support of $u$ cannot expand in a discontinuous way at $t = 0$. Indeed, if the support of $u$ does not expand in a discontinuous way at $t = 0$, at a small time $t = r > 0$, we still have $u(\cdot, r) \prec v(\cdot, t)$, for $0 \leq t \leq r$. By the continuity of the solutions and the construction of $Z$ and $W$, we have that if $r$ is chosen small enough, then $Z(\cdot, r) \prec W(\cdot, r)$.

Now, if we can prove that $W$ stays above $Z$ for every $x \in \mathbb{R}^n$, and $r < t < T$ and also for every choice of $r$ and $\delta$ small enough, we first let $\delta \to 0$ and then $r \to 0$, to conclude that $v \geq u$ and the theorem is proved. Otherwise, we would have for some of those choices a first point where $W$ touches $Z$ from above.

Step ii: No interior contact.

Let us first prove that the interior contact between $Z$ and $W$ is not possible.

Assume that the touching point occurs for a positive value of $Z$, which means a positive value of $W$, $Z(P_0) = W(P_0) > 0$. Then, in a parabolic neighbourhood of $P_0$, $\Omega$, both functions are positive and they are separated on the boundary of $\Omega$. Then, we can find two positive classical solutions, $z$ and $w$, defined in $\partial \Omega$ as $z = Z + \varepsilon_1$ and $w = W + \varepsilon_2$, with $\varepsilon_1$ and $\varepsilon_2$ chosen such that $z < w$ in $\partial \Omega$. We may apply the Strong Maximum Principle for classical solutions to $z$ and $w$ and get $z(P_0) < w(P_0)$. On the other hand, due to the Lemma 2.5 and since $Z < z$ on $\partial \Omega$ we have $Z(x,t) \leq z(x,t)$ in $\Omega$. Analogously, thanks to Lemma 2.6 we get that $w(x,t) \leq W(x,t)$ in $\Omega$. Summing up,

\[ Z(P_0) \leq z(P_0) < w(P_0) \leq W(P_0), \]

which implies that $W$ cannot touch $Z$ from above in a point of positivity of the functions.

Step iii: Free-boundary contact.

We are therefore confronted with the analysis of a first contact point of $W$ and $Z$, $P_0 = (A,t_0)$, located at the free boundary (i.e., the boundary of the positivity set) of both functions. We will prove that no such point exists for $r < t_0 < T$. 

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At the free boundary contact point we have $Z(P_0) = W(P_0) = 0$ and the support of $Z$ has interior ball of radius $r$, while the support of $W$ has an exterior ball of radius $r' < r - \delta t_0$. On the other hand, since the first contact point between the supports occurs at $t_0$, for all $t < t_0$ the support of $Z$ is included in the support of $W$. This means that the two balls are disjoint for all $t < t_0$. At $P_0$ they may be tangent or not, i.e., the tangent hyperplanes at $P_0$ may or not be the same, depending on the tangency of the balls, but in any case the tangent hyperplanes have a common space projection and must be ordered in time so that the supports are included as explained.

Moreover, there is a point $P_1 = (B, t_1)$ on the free boundary of $u$ located at distance $r$ from $P_0$; in fact, by the definition of $Z$ as a supremum, there exists a ball of radius $r$ and center $P_0$ where $u = 0$. At distance $r + \varepsilon$ from $P_0$ and near the point $P_1$, $u$ has to be positive somewhere, otherwise $P_0$ would not be at the boundary of $Z$.

There exists another point $P_2 = (C, t_2)$ on the free boundary of $v$ and at distance $r_0 = r - \delta t_0$ from $P_0$ by the definition of $W$. We have said that for $\delta > 0$ the hyperplane that bounds the exterior ball to the free boundary of $W$ at $P_0$ is not perpendicular to the line that joins $P_0$ and $P_2$, hence $P_1, P_0$ and $P_2$ are not aligned. But the space projections of those hyperplanes at $t_0$ must coincide, hence the space projections of $P_1, P_0$ and $P_2$ are aligned.

**Lemma 8.2** The tangent hyperplane $H$ to the free boundary of $Z$ at $P_0$ in space-time is neither horizontal nor vertical.

**Proof.** (i) First, we will show that $H$ is not vertical. Due to the definition of $W$ as the infimum of $v$ taken over the balls $B_{r-\delta t}(x, t)$, there is a $\delta$ difference between the speed of propagation of the supports of $W$ and $v$. Indeed, the balls $B_{r-\delta t}(x, t)$ shrink as time proceeds, at time $t$ these balls have radius $r - \delta t$ but at a later unit of time their radius is $r - \delta(t + 1)$. By subtracting both expressions we obtain the difference in the speed of propagation of the supports. Due to Proposition 6.2 the support of $v$ does not contract, hence neither do the supports of $Z$ nor $W$, and we may assert that $Z$ and $W$ move forward with speed $\geq \delta$ at $P_0$. We can thus eliminate the possibility of a tangent hyperplane to the supports of $W$ and $Z$ at $P_0$.

(ii) Eliminating the possibility of a horizontal hyperplane is a bit more involved and relies on a scaling argument and on the comparison with traveling waves.

If there is a horizontal hyperplane, after translating the axes we may suppose that $P_0 = (0, -r)$ and $P_1 = (0, 0)$. Then the support of $Z$ has a horizontal tangent hyperplane $t = -r$ at $P_0$ and the support of $u$ has an exterior ball at $P_1$ with horizontal tangency. By continuity for every small $\delta > 0$ we may take a cylinder $C_\lambda = \{(x, t) : |x| \leq \lambda, -\lambda^2 \leq t \leq 0\}$ where $u \leq \delta$.

The next step is based on the scaling argument of the function $u$, $$u_\lambda(x, t) = u(\lambda x, \lambda^2 t),$$ which is a new subsolution of (0.3) defined in the unit cylinder $C_1$. This function takes the value zero on the bottom of $C_1$, i.e., for $t = -1$, and its values are smaller than $\delta$ on the lateral boundary. Due to the finite propagation property, we arrive at a contradiction, then for a very small $\delta$ the free boundary of $u$ cannot reach $x = 0$ at $t = 0$. This can be proved by
constructing a radially symmetric supersolution that moves from the boundary to the interior at very small speed,

\[ u_1(r, t) = A(r + ct - 1 + \varepsilon)_+, \]

with \( A \geq \varepsilon/\delta \) and \( c \) and \( \varepsilon \) small. Comparison of \( u \) with \( u_1 \) in \( C_1 \) implies that \((0, 0)\) is not a free-boundary point of \( u \) against the definition.

**Step IV: Linear boundary behavior.**

Now we establish the linear behavior of the free boundary of \( Z \). Before we do that let us review the situation and fix some notations.

We may displace for convenience the origin of coordinates so that \( P_0 \) becomes \((0, t_0)\) and then choose space coordinates so that the positive direction of the \( x_1 \)-axis points along the vector \( \overline{AB} \), space projection of \( P_0P_1 \). Let us also write \( x = (x_1, x') \), with \( x' \in \mathbb{R}^{n-1} \).

Let \( H \) be the tangent hyperplane to the internal ball to \( Z \) at \( P_0 \). We will write the internal normal vector to \( H \) at \( P_0 \) as \((e_1, m)\), with \( m = \tan \alpha \), for some \( 0 < \alpha < \pi/2 \); \( m \) is therefore the advance speed of the free boundary of \( Z \) and \( P_1 = (r \cos \alpha, 0, t_0 + r \sin \alpha) = (d_1, 0, t_0 + r \sin \alpha) \) is a point of the free boundary of \( u \).

The tangent hyperplane \( H_1 \) to \( W \) at \( P_0 \) has advance speed equal or less than \( m \). That speed is related to \( \beta \), the angle formed by \( P_0P_2 \) with the space-like hyperplanes, by relation (7.5). In any case \( \beta < \alpha \).

**Lemma 8.3** Near the point \( P_0 \) we have the nontangential estimate

\[ Z(x, t_0) \geq m(1 - \varepsilon)(x_1)_+, \]  

and as \( x_1 \to 0, \varepsilon \to 0 \) in any nontangential cone.

**Proof.** A nontangential space cone at 0 is a cone \( K \) in \( \mathbb{R}^n \), with vertex 0, axis \( e_1 \) and aperture less than \( \pi/2 \).

Assume for contradiction that (8.3) does not hold and that for a sequence of points \( A_n = (x_{1n}, x'_{nn}) \), converging to \( 0 \in \mathbb{R}^n \) and lying in a nontangential cone \( K \) at 0 we have for some fixed \( \varepsilon > 0 \),

\[ Z(Q_n) \leq m(1 - \varepsilon)(x_{1n})_+, \]

with \( Q_n = (A_n, t_0) \).
By the definition of $Z$ as a supremum we conclude that the function $u$ has value equal or less than $\mu_0 = m(1-\varepsilon)x_1$ in a ball $B_r(Q_0)$ centered at $Q_0$. Besides, we know that $u$ is actually 0 in a circle of the same radius centered at $P_0 = (0, t_0)$, see Figure 3.

Let us extend the vanishing set of $u$. Since $P_2$ belongs to the free boundary of $v$, $v$ vanishes at this point, situated at distance $r - \delta t_0$ from $P_0$ in the direction of a vector $-(e_1, \tan \beta)$. By the definition of $W$ as an infimum it follows that for $t = t_0$, $W$ vanishes in the space ball $B'$ of radius $0 < d_2 < r - \delta t_0$, centered at the point $Q' = -d_2 e_1$; in fact, any ball of radius $r - \delta t_0$ centered at a point $P'$ includes the point $P_2$, and therefore the infimum of $v$ in that ball has to be 0 and consequently $W$ vanishes at $P'$. Since $Z \leq W$ at $t = t_0$, $Z$ also vanishes at $P'$, and by the definition of $Z$ as a supremum we conclude that $u$ vanishes in the set $\Sigma$ formed as the union of the space-time balls of radius $r$ centered at the points of $B'$.

We come to a crucial step. For $x_1 = \lambda > 0$ small we consider a certain space-time set $L = L_\lambda$ where $u \leq \mu_\lambda = m(1-\varepsilon)\lambda$. We describe the set $L_\lambda$ through its boundary formed of several pieces. First, we have the ‘covers’, which are parts of the hyperplanes $t = t_1 - \tau$ and $t = t_1$. $L_\lambda$ will be contained in that time interval. Then there is a boundary piece closer to the origin, formed by a part of the boundary of $\Sigma$ containing the point $P_1$. It is a concave surface and there we have $u = 0$. There is also an opposite boundary part of $L_\lambda$ formed by a piece of the sphere $S_r(Q_\lambda)$, boundary of $B_r(Q_\lambda)$, where we know that $u \leq \mu_\lambda$. As $\lambda \to 0$ the set $L_\lambda$ takes the form of a thin flat sliver around $P_1$, with depth $\lambda$ and width $O(\sqrt{\lambda})$. Asymptotically around $P_1$, it looks like a region between two parallel planes advancing with speed $m > 0$, and hence the depth $\lambda$. These planes are curved in time and towards the space origin, which complicates the analysis, but finally we get the width estimate $O(\sqrt{\lambda})$.

We will analyze $u$ as a solution of equation (0.3) in $L_\lambda$ and prove that whenever (8.4) holds for $x_1$ small enough, then $u$ has to vanish in a neighborhood of $P_1$, thus arriving at a contradiction. The proof is done by comparison of $u$ with a curved supersolution of the form

$$\tilde{u}(x,t) = A(|x| + ct - B)_+,$$

with suitable parameters $A, B$ and $c$. It will be convenient to displace the $t$ axis so that $t_1 = t_0 + r \sin \alpha$ is set to 0 and the point $P_1$ becomes $(d_1, 0, 0)$. The speed $c$ and the slope $A$
of the spherical traveling wave $\tilde{u}$ will be taken close to $m$

(8.5) \quad c = m \left(1 - \frac{\varepsilon}{3}\right), \quad A = m \left(1 - \frac{2\varepsilon}{3}\right),

so that $c/A > 1$. We will also take $B = d_1 + \lambda\varepsilon > d_1$ which ensures that for $t = 0$, $\tilde{u}$ vanishes in $P_1$:

$$\tilde{u}(P_1) = \tilde{u}(d_1, 0, 0) = A(d_1 - B)_+ = m \left(1 - \frac{2\varepsilon}{3}\right)(d_1 - d_1 - \lambda\varepsilon)_+ = 0,$$

and also in a neighborhood of $P_1$.

Besides, according to Lemma 4.3, $\tilde{u}$ is a supersolution of (0.3) in a domain $R = \{|x| < R, -\tau < t < 0\}$, with $R > d_1$ and $\tau$ small, if $B$ is chosen close to $R$ so that

$$B < R, \quad \frac{R - B}{R} \ll \varepsilon.$$

And so, choosing $c$ and $A$ as in (8.5) we obtain the condition (4.6).

We compare $\tilde{u}$ with $u$ in $L_{\lambda, \tau} = L_\lambda \cap \{-\tau \leq t \leq 0\}$ for $\lambda$ and $\tau$ much smaller than $\varepsilon$. Actually we take $\varepsilon$ fixed small, $\tau$ much smaller and $6\lambda = m\varepsilon \tau$.

We begin with the comparison at the bottom side, where $t = -\tau$. We have chosen the parameters $A, B$ and $c$, so that $\tilde{u} \geq m(1 - \varepsilon)\lambda$ whenever $u$ is positive, which automatically implies that $\tilde{u} \geq u$. Indeed, we have $u = 0$ in a space ball $B_{d(\tau)}(0)$ of radius

$$d(\tau) = d_1 + m\tau + O(\tau^2).$$

Hence, $\tilde{u} \geq u$ at $t = -\tau$, if

$$\tilde{u} = m \left(1 - \frac{2\varepsilon}{3}\right) \left(|x| - m \left(1 - \frac{\varepsilon}{3}\right) \tau - B\right)_+ \geq m(1 - \varepsilon)\lambda,$$

which is true under our assumptions. To prove it we replace $B$ by its value $B = d_1 + \lambda\varepsilon$ and have into account that $6\lambda = m\varepsilon \tau$. We get

$$\tilde{u} \geq m \left(1 - \frac{2\varepsilon}{3}\right) \left(d_1 + m\tau + O(\tau^2) - m\tau + \frac{m\varepsilon \tau}{3} - d_1 - \lambda\varepsilon\right)_+$$

$$\geq m \left(1 - \frac{2\varepsilon}{3}\right) (2\lambda - \lambda\varepsilon)_+ \geq m(1 - \varepsilon)\lambda = u.$$

In the inner boundary of $L_{\lambda, \tau}$ we have the easiest comparison, $u = 0 \leq \tilde{u}$.

The outer boundary of $L_{\lambda, \tau}$ consists of points $(x_1, x', t)$ of the sphere $S_r(Q_0)$, which are given by the expression

$$|x_1 - \lambda|^2 + |x' - k\lambda|^2 + (t + r \sin \alpha)^2 = r^2,$$

with $k$ bounded independently of $\lambda$, since we are working in a nontangential cone. Then

$$|x|^2 = x_1^2 + (x')^2$$

(8.6) \quad = 2x_1\lambda + 2x'k\lambda - \lambda^2(1 + k^2) - t^2 - 2tr \sin \alpha + r^2(1 - \sin^2 \alpha)$$

$$= 2x_1\lambda + 2x'k\lambda - \lambda^2(1 + k^2) - t^2 - 2tr \sin \alpha + d_1^2.$$
In a first approximation, on this boundary we have $x_1 \approx d_1 = r \cos \alpha$ plus the bound $x' = O(\sqrt{\lambda})$. If $t \in (-\tau, 0)$ and up to terms of higher order in $\lambda$, and therefore up to terms of higher order in $t$, we have

$$|x| \approx d_1 + \lambda + m(-t).$$

This easily proved comparing this last expression with (8.6), suppressing the terms of higher order and remembering that $r \sin \alpha = d_1 m$. Finally $\tilde{u}$ is larger than $u$ if

$$\tilde{u} \approx m \left(1 - \frac{2 \varepsilon}{3}\right) \left(d_1 + \lambda + m(-t) + \left(1 - \frac{\varepsilon}{3}\right) mt - d_1 - \lambda \varepsilon\right)_+$$

$$= m \left(1 - \frac{2 \varepsilon}{3}\right) \left(\lambda - \frac{\varepsilon}{3} mt - \lambda \varepsilon\right)_+ = m \left(1 - \frac{2 \varepsilon}{3}\right) \left(\lambda(1 - \varepsilon) + \frac{\varepsilon}{3} m|t|\right)_+$$

$$\geq m(1 - \varepsilon) \lambda.$$

It follows that $u \leq \tilde{u}$ in the parabolic boundary of $L_{\lambda, \tau}$, hence $u \leq \tilde{u}$ in $L_{\lambda, \tau}$. This means that $u$ vanishes identically around $P_1$, the desired contradiction.

**STEP v: Conclusion.**

We use the behavior (8.3) to derive consequences on the movement of the free boundaries. Thus we may place a small Barenblatt subsolution, $B$, below $Z$ at $t_0$ so that it almost touches $P_0$ and has speed $m$ like the hyperplane $H$. We choose $B$ so that it crosses the same hyperplane advancing with any smaller speed $m' < m$ after a very short time. More precisely, we can make sure that for a small time interval $t_0 + \tau_1 \leq t \leq t_0 + \tau_2$ the support of $B$ covers a space ball centered at $P_0$ with radius $m'(t - t_0)$. By moving $B$ closer to $P_1$ we can choose $\tau_1$ and $\tau_2$ small enough.

Next we transport this information to $v$. Since $Z$ lies below $W$ at $t_0$, then $B$ lies also below $W$ at $t_0$. By the definition of $W$ as an infimum, we can place a copy of $B$ below the graph of $v$ after displacing it at all distances equal or less than $r_0 = r - \delta t_0$ in space-time. The property of comparison with classical free boundary subsolutions says that $B$ will stay below the graph of $v$ at later times, and since the support of $B$ moves with speed almost $m$ for a short time in the sense explained above, we get the conclusion that the support of $v$ has grown to at least a distance of $m'$ times the increment of time at all those points located at distances equal or less than $r_0$ from $P_0$.

Going back to $W$, this implies that the support of $W$ has expanded at least at speed $m' + \delta$ in the already mentioned time interval. If $m' + \delta > m$ such a result is in contradiction with the fact that there exists a tangent hyperplane $H_1$ for $W$ at $P_0$ with advance speed equal or less than $m$.

The contradiction implies that no free boundary contact point may exist, which ends the proof of the Comparison Result.

**9 Construction of the minimal viscosity solution**

Due to the Comparison Theorem 8.1 and the construction of the maximal viscosity solution, we are now able to build up another kind of solution, namely the minimal solution. This
solution is constructed as a limit of maximal solutions as follows. Let \( u_0 \) be a general initial datum for (0.3), (0.7), with \( \bar{u} \) the maximal viscosity solution of that problem. Let also \( \{u_{0,n}\} \) be a sequence of initial data for the same problem, such that

\[
u_{0,1} \prec u_{0,2} \prec \ldots \prec u_{0,n} \prec \ldots \prec u_0.
\]

Each of these problems has an associated maximal solution \( \bar{u}_n \). Observe that, due to the Comparison Theorem 8.1, \( \bar{u}_1 \leq \bar{u}_2 \leq \ldots \leq \bar{u}_n \leq \ldots \leq \bar{u} \). If we now pass to the monotone limit as \( n \to \infty \), we get

\[
\bar{u}(x, t) = \lim_{n \to \infty} \bar{u}_n \leq \bar{u}.
\]

**Proposition 9.1** If \( u_{0,n} \not\to u_0 \) as \( n \to \infty \) then \( \bar{u} \) is the minimal viscosity solution of Problem (0.3), (0.7).

*Proof.* (i) The continuity of the limit is obtained by the same regularity argument as in Proposition 5.1. Observe that due to the continuity of the solutions and the monotonicity of the sequence \( \{u_{0,n}\} \), the convergence from \( u_{0,n} \) to \( u_0 \) is locally uniform. Let us also point out that the supports of \( u_{0,n} \) converge; i.e. \( \text{supp}(u_{0,n}) \to \text{supp} u_0 \), in the sense of the local set distance.

(ii) We have to prove that \( \bar{u} \) is a viscosity solution. The argument is similar to the one used to prove that the limit of classical solutions is a viscosity solution. We only have to take care when proving that the second part of the definition of supersolution is also true for \( \bar{u} \). Indeed, let \( v \) be a classical moving free boundary subsolution, such that \( v(x, 0) \prec u_0(x) \). Due to the convergence of the supports and the convergence of the initial data, for \( n \) big enough we can find \( u_{0,n} \) such that \( v(x, 0) \prec u_{0,n}(x) \prec u_0(x) \). Since \( u_{0,n} \) is the initial data to the problem with the maximal viscosity solution \( \bar{u}_n \) we get that \( v(x, t) \leq \bar{u}_n(x, t) \). Finally, by the construction of \( \bar{u} \), we have that \( v(x, t) \leq \bar{u}(x, t) \).

(iii) Let us now show that \( \bar{u} \) is minimal. Let \( \tilde{u} \) be another viscosity solution for the problem with initial datum \( u_0 \). Since \( u_{0,n} \not\prec u_0 \) we may apply the Comparison Theorem 8.1 and conclude that \( \bar{u}_n \leq \tilde{u} \). Passing to the limit \( \bar{u} \leq \tilde{u} \), so that \( \bar{u} \) is a minimal solution of (0.3), (0.7). \( \square \)

**Remark 5** Existence of the maximal and minimal viscosity solution has been obtained for general bounded and continuous data. It is a main reduction of the problem of uniqueness: all possible solutions are sandwiched between them, and besides they have special properties. Thus, both can be obtained as (locally uniform) limits of smooth solutions. Besides, the argument used in Proposition 5.5 applies also here to show that the minimal solution is the transform of a weak solution of the associated filtration equation.

We will prove in the next section a general uniqueness result, Theorem 10.1. Nevertheless, we will give here a less general but direct proof, that uses the previous Comparison Result for separated solutions and the properties of the support. This proof avoids having to pass to the filtration equation.
Proposition 9.2 Let $u_0$ be as in Lemma 6.3, so that the free boundary moves immediately. Then, the viscosity solution of Problem (0.3) is unique and it coincides with a weak solution of the associated filtration problem (0.6).

Proof. Let $\overline{u}$ and $\underline{u}$ be the maximal and minimal viscosity solutions of (0.3). We want to prove that $\underline{u} \equiv \overline{u}$, which implies uniqueness.

Under the conditions of Lemma 6.3 we use the natural scaling invariance of the equation to define

$$u_\varepsilon(x, t) = (1 + \varepsilon)u(x, (1 + \varepsilon)t),$$

which is another viscosity solution of (0.3) with initial data

(9.2) $$u_\varepsilon(x, 0) = (1 + \varepsilon)u_0(x) > u_0(x).$$

With such an initial data the free boundary of $u_{\varepsilon, 0}$ moves in the same way. We conclude that for some small $\tau(\varepsilon) > 0$ we will have strict separation of $u_\varepsilon$ and $\underline{u}$ in the form

$$u_\varepsilon(x, \tau) > \underline{u}(x, 0), \quad \text{supp}(\underline{u}(\cdot, 0)) \subset \text{Int}(\text{supp}(u_\varepsilon(\cdot, \tau))).$$

Indeed, we can choose $\varepsilon > 0$ such that $\text{supp}(\underline{u}(\cdot, 0)) \subset \text{Int}(\text{supp}(u_\varepsilon(\cdot, \tau')))$, for some $\tau' > 0$ small. If we define $u_\varepsilon(x, \tau) = (1 + \varepsilon)u(x, \tau') = (1 + \varepsilon)u(x, (1 + \varepsilon)\tau)$ then the assertion is proved. It follows from the monotonicity result, Theorem 8.1, that $\overline{u}$ will lie below $u_\varepsilon$ for all $t \geq \tau$ and all $\varepsilon > 0$. In the limit, as $\varepsilon \to 0$ we get $\overline{u} \leq \underline{u}$. \hfill \square

10 General uniqueness and comparison

As it was pointed out in the Introduction, the uniqueness of solutions of (0.3), (0.7) stated in Theorem A can only be proved (to our knowledge) using the transformation $T$ of Section 3 and some uniqueness result for the associated filtration problem.

Let $\overline{u}$ and $\underline{u}$ be the maximal and minimal solutions to (0.3) with continuous, bounded and nonnegative initial data $u_0$. We want to prove that $\overline{u} = \underline{u}$. We use the transformation into the filtration equation. Indeed, each of these solutions has its corresponding weak solution of (0.6), $\overline{\rho}$ and $\underline{\rho}$, constructed due to the pressure-to-density transformation studied in Section 3. We use now the following uniqueness result.

Theorem 10.1 Two continuous, nonnegative, ordered weak solutions $\rho$ of the filtration equation (0.6) with the same data are the same if $\Phi$ is Lipschitz continuous on the range of the solutions.

Proof. Let $\rho(x, t) = \overline{\rho}(x, t) - \underline{\rho}(x, t) \geq 0$. Then, $\rho(x, 0) = 0$. Let us define

$$M(t) = \int_{\mathbb{R}^n} \rho(x, t)\varphi(x) \, dx$$

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for some smooth $\varphi \in L^1(\mathbb{R}^n)$ to be chosen below. Integrating by parts we get

$$M'(t) = \frac{d}{dt} \int_{\mathbb{R}^n} (\bar{\rho} - \rho) \varphi \, dx = \int_{\mathbb{R}^n} \Delta (\Phi(\bar{\rho}) - \Phi(\rho)) \varphi \, dx$$

$$= \int_{\mathbb{R}^n} \Delta \varphi (\Phi(\bar{\rho}) - \Phi(\rho)) \, dx.$$ 

We now choose $\varphi$ as the solution of the equation $-\Delta \varphi + \varphi = \psi$, for some smooth $\psi \in L^1(\mathbb{R}^n)$, $\psi \geq 0$. It is well known that a unique $\varphi$ exists, $\varphi > 0$ and $\int \varphi \, dx = \int \psi \, dx$. Using also the fact that $\Phi$ is monotone and Lipschitz continuous we get

$$M'(t) = \int_{\mathbb{R}^n} \varphi (\Phi(\bar{\rho}) - \Phi(\rho)) \, dx - \int_{\mathbb{R}^n} \psi (\Phi(\bar{\rho}) - \Phi(\rho)) \, dx \leq K \int_{\mathbb{R}^n} \rho \varphi \, dx = KM(t),$$

where $K$ is an upper bound of $\Phi'$ in the range of $\bar{\rho}$ and $\rho$. Note that the last term before the inequality is negative and can be dropped. Since $M(0) = 0$, we conclude from the differential inequality $M'(t) \leq KM(t)$ that $M \equiv 0$, hence uniqueness. \hfill \Box

**Theorem 10.2 (Comparison of viscosity solutions)** Let $u_1$ and $u_2$ be respectively a viscosity subsolution and a viscosity supersolution of (0.3) with initial data $u_{0,1}$ and $u_{0,2}$, such that $u_{0,1}(x) \leq u_{0,2}(x)$. Then $u_1(x,t) \leq u_2(x,t)$ for all $(x,t) \in Q$.

**Proof.** We consider a viscosity solution $u$ of (0.3) with initial data $u_0$ between $u_{0,1}$ and $u_{0,2}$, i.e., $u_{0,1}(x) \leq u_0(x) \leq u_{0,2}$. The proof of the theorem is done in two steps. First we prove that the subsolution lies below the viscosity solution and then that the supersolution is above the solution.

**Step I:** We first prove that $u_1(x,t) \leq u(x,t)$ whenever $u_{0,1}(x) \leq u_0(x)$. In the construction of the maximal solution we have defined a sequence of classical solutions to (0.3), $\{u_\varepsilon\}$, with initial data $u_{0,\varepsilon}(x) \geq u_0(x) + \varepsilon > u_0(x)$. As we have seen in Section 5 and here above, the limit $u$ of this sequence is the maximal viscosity solution of the problem and hence the unique solution. On the other hand, once we fix an $\varepsilon > 0$ we may approximate $u_\varepsilon$ from above by standard quasilinear theory with a smooth solution $\varphi = \varphi_{\varepsilon,\delta}$ of the equation

$$u_t = a(u) \Delta u + |\nabla u|^2 + \delta,$$

taking the same initial data $u_{0,\varepsilon}$. As $\delta \to 0$ we have that $\varphi_{\varepsilon,\delta}$ converges to $u_\varepsilon$ in a monotonically decreasing, locally uniform way. Suppose now that $\varphi$ touches $u_1$ from above at a point $P_1 = (x_1,t_1)$. By the definition of viscosity subsolution we have

$$\varphi_t \leq a(\varphi) \Delta \varphi + |\nabla \varphi|^2$$

at $P_1$, and this contradicts (10.1). Hence, they do not touch and $u_1 < \varphi_{\varepsilon,\delta}$. Letting first $\delta \to 0$ and then $\varepsilon \to 0$ we get $u_1 \leq u$ in $Q$ as desired.

**Step II:** Let $u_0 \leq u_{0,2}$. Following the construction of the minimal solutions, Section 9, we can approximate $u_0$ by a sequence of initial data to (0.3), such that

$$u_{0,1} \prec v_{0,2} \prec \ldots \prec v_{0,n} \prec \ldots \prec u_0 \leq u_{0,2}.$$
Due to the Comparison Theorem for solutions with separated initial data, Theorem 8.1, we know that,
\begin{equation}
  v_1 \leq v_2 \leq \ldots \leq v_n \leq \ldots \leq u_2.
\end{equation}
On the other hand we have that \( \lim v_n = u \) as \( n \to \infty \), \( u \) is the minimal solution of Problem (0.3), and hence the unique solution. Passing to the limit in (10.2) we get \( u(x,t) \leq u_2(x,t) \) in \( Q \).

11 Continuous dependence. Equivalent definitions

A standard property of the theory is continuous dependence on the initial data, that we prove next.

**Theorem 11.1** Let \( u \) be a viscosity solution and let \( \{u_n\} \) be a sequence of viscosity solutions, such that the sequence of their corresponding initial data, \( u_{0,n} \) is uniformly bounded and converges locally uniformly to the initial data \( u_0 \). Then \( u_n \) converges to \( u \) in the same way.

**Proof.** After the Comparison Theorem 10.2 we can reduce the proof to the case of monotone sequences. In one of the situations we consider a sequence of positive and smooth initial data \( u_{0,n} \geq u_0 + 1/n \) that converge locally to \( u_0 \) in a decreasing way. The proof that the smooth solutions \( u_n(x,t) \) converge to a viscosity solution repeats the ideas of Proposition 5.1. The solution equals \( u \) by uniqueness.

On the other side, we have to consider the situation of data \( u_{0,n} \) with compact support that converge monotonically to \( u_0 \) from below. We may also assume that these solution are separated. We have a monotone limit that is a viscosity solutions and it is not difficult to see that the initial data is \( u_0 \).

The next question we want to settle is the possibility of weakening the concept of supersolution. We define a weak viscosity supersolution as a continuous function \( u(x,t) \) such that Definition 1.5 holds with condition (ii) replaced by

(ii*) Every Barenblatt subsolution \( v \) that is strictly separated from \( u \) at a time \( t = t_1 \geq 0 \) cannot cross \( u \) at a later time, i.e., \( v(x,t) \leq u(x,t) \) for all \( x \in \mathbb{R}^n \) and \( t_2 > t_1 \).

In the opposite direction, in the definition of a strict viscosity supersolution we replace the Barenblatt subsolution by any classical free boundary subsolution. We have

**Proposition 11.2** The concepts of viscosity supersolution, weak viscosity supersolution and strict viscosity supersolution are equivalent in our problem.

**Proof.** Clearly, a strict viscosity supersolution is a viscosity supersolution and the latter is a weak viscosity supersolution. On the contrary, let \( u \) be a weak viscosity supersolution and let \( v \) be any classical free boundary subsolution that is strictly separated from \( u \) at \( t = 0 \).
First, we may replace \( v \) by the solution \( V \) with same initial data. Since a classical free boundary subsolution is in particular a viscosity subsolution we conclude from the Comparison Theorem 10.2 that \( v \leq V \) in \( Q \).

We now repeat the proof of the Comparison Theorem 8.1 between \( u \) and \( V \) that only uses property (ii*). Therefore, \( u \geq V \).

We now complete the study of agreement of the classical and viscosity concepts for smooth functions that we have started discussing in Section 2. After that section the result that we want to prove is

**Proposition 11.3** A smooth classical supersolution \( u \in C^{2,1}(Q) \) is also a viscosity supersolution. The same is true for a classical free boundary supersolution.

**Proof.** We only need to prove condition (ii*). Now, let \( v = B(x,t) \) be a Barenblatt function that is strictly separated from a smooth classical supersolution \( u \) at \( t = 0 \). Then no contact may happen because a classical supersolution must have zero gradients at all points of the free boundary, hence the free boundary contact of the Comparison Theorem 8.1 may not happen.

In the case of a classical free boundary supersolution \( u \) we compare it with a Barenblatt function that has been modified so that \( B_t \leq a(B)\Delta B + \varepsilon|\nabla B|^2 \) with \( \varepsilon < 1 \). This means that the speed of propagation of \( B \) is given by \( v_n = \varepsilon|\nabla B| \). We may assume that \( B \) and \( u \) are strictly separated at \( t = 0 \). We contend that there can be no contact at a positivity set (standard argument) and no contact at the free boundary. Indeed, at such a point we would have \(|\nabla B| \leq |\nabla u|\), while the speed of the interface of \( B \) must be larger or equal than the speed of \( u \). This cannot be according to the speed laws. Therefore, \( B \) stays below \( u \). \( \Box \)

12 Counterexample without boundary test

In this section we want to emphasize the need in the definition of viscosity solution for a condition to control the behaviour of the solutions near the free boundary. If Condition (ii) of Definition 1.5 is eliminated and no other condition is included, then the concept is too vague and uniqueness of solutions may fail. We exhibit here an example based on transformation of a solution of the Porous Medium Equation defined in a bounded space domain with zero boundary data and extended to the whole space by zero.

We consider the Porous Medium Equation (PME) in a bounded domain \( \Omega \),

\[
\rho_t = \Delta \rho^m, \quad m > 1.
\]  

We take as initial data a nontrivial function

\[
\rho(\cdot,0) = \rho_0 \in L^1(\Omega), \quad \rho_0 \geq 0,
\]

with zero boundary data

\[
\rho = 0 \quad \text{on } \Sigma = \partial \Omega \times (0,\infty).
\]
This problem admits a unique nonnegative weak solution, and it depends continuously on the data in the $L^1(\Omega)$-norm. In [AP] it is shown that there exists a unique self-similar solution of the PME, which describes the asymptotic behavior of all solutions of the Cauchy-Dirichlet problem. This solution has the form

\begin{equation}
\tilde{\rho}(x,t) = t^{-\alpha} f(x), \quad \alpha = \frac{1}{m-1},
\end{equation}

where $f$ is solution of a stationary equation

\begin{equation}
\Delta f^m + \alpha f = 0.
\end{equation}

They also show that $f^m \in C^\infty(\Omega)$, if it is assumed that $f^m$ is a bounded solution of (12.5). The Maximum Principle implies that $f^m$ is strictly positive inside $\Omega$ unless it is identically zero. The solution is continuous up to the boundary. See also [Va2] for details. We now define the extension of this classical solution to the whole space as follows

\[ \tilde{\rho}(x,t) = 0 \quad \text{for } x \neq \Omega, \quad t > 0. \]

We let $\tilde{u}$ be the function obtained from $\tilde{\rho}$ via the density-to-pressure transformation defined in Section 3, i.e.

\[ \tilde{u}(x,t) \begin{cases} 
  u(x,t) = T^{-1}(\rho) & x \in \Omega, \\
  0 & x \notin \Omega.
\end{cases} \]

Due to Lemma 3.1 we know that $\tilde{u}$ is a classical solution of

\[ \tilde{u}_t = a(\tilde{u}) \Delta \tilde{u} + |\nabla \tilde{u}|^2, \]

whenever $\tilde{\rho}$ is a classical solution of (12.1), i.e. in $\Omega \times (0, \infty)$. On the other hand, $u$ is also a classical solution of the equation in the exterior of $\Omega$, where it vanishes. The only pending points are those of the set $\Sigma$ where we have not checked the conditions for viscosity supersolution. Indeed, they do not hold since every classical free boundary subsolution (e.g., one of the Barenblatt functions) that is strictly separated from $\tilde{u}$ at a certain time $t$, has to cross $\tilde{u}$ at a later time because of its expanding support, against the definition. Hence, $\tilde{u}$ is not a viscosity solution.

The problem is well understood in the weak theory, and it is known that the weak solution is different. By transformation we get the real viscosity solution of the problem.

13 Conclusions and Comments

Let us recall that the equivalence of viscosity and distribution solutions for linear, second-order, elliptic and parabolic equations has been established in the past by P. L. Lions and H. Ishii, [IL]. The authors of [JLM] prove the equivalence of weak solutions and viscosity solutions of the $p$-Laplace equation

\[ -\text{div}(|Du|^{p-2} Du) = 0 \]
for $1 < p < \infty$ and its parabolic version. The presence of the degeneracy at the level of $u$-dependence of the second order operator is a special difficulty that makes the machinery of the present work different. A more related study is performed by C. I. Kim [Ki] for viscosity solutions of the Hele-Shaw and Stefan Problems. Solutions are obtained in the limit of the solutions of the PME case treated in [CV].

We want to point out again that the uniqueness proof is indirect, because it is done via a transformation into another equation. It would be desirable to obtain a proof, whose arguments rely only on viscosity techniques. Even in the case of the PME treated in [CV] no direct proof of well-posedness for viscosity solutions independent of this connection, is available to our knowledge. The approach of the present paper decreases the dependence on such a method.

There are examples of related equations where there is non-uniqueness for the weak solutions and viscosity solutions. In particular, consider the limit case where the quadratic gradient term in (0.3) disappears and $a(u) = u$. We get the equation

$$u_t = u\Delta u,$$

which is proposed in [CV] as one that presents particular difficulties in order to develop the theory: the uniqueness of viscosity solutions in the sense proposed in this paper is false in the class of all nonnegative and continuous initial data. However, the theory produces a maximal and a minimal solution for each initial data as in our main result.

More generally, it would be interesting to study the problems with data $u_0$ not necessarily bounded. An interesting open problem is to consider equations of the general form

$$u_t = a(x, u)\Delta u + b(x, u)F(|\nabla u|^2),$$

under suitable conditions on the functions $a$, $b$, and $F$. The methods of the present paper do not allow to produce a theory similar to the one presented here.

### A Appendix. Continuity of solutions of $\rho_t = \Delta \Phi(\rho)$

The continuity of weak solutions to the filtration equation

(A.1) \[ \rho_t = \Delta \Phi(\rho), \]

under different conditions over $\Phi$, has been widely studied, cf. [Db], [DbV], [Sa], [Zi]. In our case $\Phi$ is given by (3.4) which introduces some restrictions that we have to take into account when studying the continuity of $\rho$. In fact, the standard theory of the Filtration Equation is done under the assumption that $\Phi'$ is locally bounded from above and below, cf. [Db] and [Zi]. Now, in view of the definition of $\Phi'$ and condition (0.5) on the function $a(u)$, we cannot assume a lower bound for $\Phi'$. Fortunately, such a situation is covered in [DbV] and [Sa].

To reduce (A.1) to the problem studied in [DbV] put $v = \Phi(\rho)$, $\beta = \Phi^{-1}$, and rewrite it as

(A.2) \[ \beta(v)_t - \Delta v = 0. \]
Let \( v \in L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(\Omega)) \). Here \( \Omega \) is a domain in \( \mathbb{R}^n \), and for \( T > 0 \) we denote by \( \Omega_T \equiv \Omega \times (0,T) \). By a weak solution of equation (A.2) we mean that\n
\[
\begin{align*}
(A.3) & \quad t \to \beta(v(\cdot,t)) \text{ is weakly continuous in } L^2_{\text{loc}}(\Omega) \\
(A.4) & \quad \int_{\Omega} \beta(v(\varphi(x,s)) \, dx \bigg|_{s=t_2}^{s=t_1} + \int_{t_1}^{t_2} \int_{\Omega} -\beta(v)\varphi_t + \nabla v \cdot \nabla \varphi \, dx \, ds = 0,
\end{align*}
\]

for every test function \( \varphi \in W^{1,2}_{\text{loc}}(0,T;L^2_{\text{loc}}(\Omega)) \cap L^2_{\text{loc}}(0,T;W^{1,2}_{\text{loc}}(\Omega)). \)

Since \( \beta = \Phi^{-1} \), it inherits the properties of \( \Phi \); i.e. \( \beta : [0,\infty) \to [0,\infty) \) is \( C^2(0,\infty) \), we have \( \beta' > 0 \) for all \( v > 0 \), \( \beta(0) = 0 \) and \( \beta \) is bounded over compacts. Moreover, since \( \Phi'(\rho) = a(u) \leq C \), \( \beta'(v) \geq C \) as required by \([DbV]\) in order to establish that bounded weak solutions \( v \) are continuous, and moreover they have a uniform modulus of continuity. We state here precise theorem in \([DbV]\).

**Theorem A.1** Let \( v \) be a weak solution of (A.2) in the sense stated before. If the assumptions over \( \beta \) hold and \( \|v\|_{\infty,\Omega_T} \leq M \), then \( v \) is continuous in \( \Omega_T \). Moreover, for every compact subset \( K \subset \Omega_T \), there exists a uniform modulus of continuity, i.e., a continuous, nonnegative and increasing function \( s \to \omega_{\text{data},K} \), \( \omega_{\text{data},K}(0) = 0 \), that can be determined a priori in terms of only the data and the distance from \( K \) to the parabolic boundary of \( \Omega_T \), such that

\[
|u(x_1,t_1) - u(x_2,t_2)| \leq \omega_{\text{data},K}(|x_1 - x_2| + |t_1 - t_2|^{1/2}),
\]

for every pair of points \( (x_i,t_i) \in K \), \( i = 1,2 \).

Finally we have to verify that \( v \), a weak solution of (A.2), is such that \( \|v\|_{\infty,\Omega_T} \leq M \), with \( M > 0 \) constant. But this is easily done by taking into account that \( v = \Phi(\rho) \) and again, by the definition of \( \Phi' \) we get \( \|\Phi(\rho)\|_{\infty,\Omega_T} \leq M \).

The continuity of \( v \) translates into the continuity of \( \rho \) automatically. Thus, \( \rho(x,t) = \beta(v(x,t)) \) and \( \beta \) is a continuous function. We obtain a uniform modulus of continuity for bounded solutions \( \rho \) of the filtration equation.

**References**


[Bo] M. Bourgoing. $C^{1,\beta}$-regularity of viscosity solutions via continuous dependence result. To appear


