First order structured perturbation theory for multiple zero eigenvalues of skew-adjoint matrices

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\textbf{Abstract}

Given a matrix belonging to some class of structured matrices, we consider the question of comparing the sensitivity of its eigenvalues under two different kinds of perturbations, either unstructured (i.e., arbitrary) or structured (i.e., those belonging to the same class of matrices as the unperturbed one). In a previous paper (Kressner et al., 2009 [13]), the authors compared the structured and unstructured condition numbers of (possibly multiple) eigenvalues for several different matrix and pencil structures. Only one case was left out of the analysis, namely the one where the asymptotic order of perturbed eigenvalues under structured perturbations is different from the asymptotic order under unstructured ones. This is precisely the case we consider in the present paper: given a matrix which is skew-adjoint with respect to a symmetric scalar product and has a zero eigenvalue with a certain Jordan structure, first order expansions are obtained for the perturbed eigenvalues under structured perturbation, as well as bounds on the structured condition number. Similar results are obtained for structured perturbations of symmetric/skew-symmetric and palindromic matrix pencils.

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1. Introduction

There is a growing interest in the structured perturbation analysis of eigenvalue problems, mainly due to the substantial development of structure-preserving algorithms for structured eigenproblems (we refer the reader to [2,4,15], and references therein). Since these algorithms preserve structure at every stage of the process, it is appropriate to consider structured condition numbers, which measure the sensitivity of eigenvalues under structured perturbations (see, e.g., [8,9,11,13,10] for definitions). The importance of these condition numbers derives from the well-known rule of thumb

\[ \text{(error)} \lesssim \text{(condition number)} \times \text{(backward error)}. \]  

More specifically, let \( \lambda \) be an eigenvalue of a complex \( n \times n \) matrix \( A \), and consider additive perturbations of the form \( A + \varepsilon E \), where \( \varepsilon \) is a small real parameter and \( E \) is any complex \( n \times n \) perturbation matrix. If \( \lambda \) has algebraic multiplicity \( m \), then \( A + \varepsilon E \) has generically \( m \) eigenvalues \( \hat{\lambda}_k(\varepsilon) \), each admitting a fractional expansion

\[ \hat{\lambda}_k = \lambda + \alpha_k \varepsilon^{\gamma_k} + o(\varepsilon^{\gamma_k}) \quad \text{as} \quad \varepsilon \to 0, \quad k = 1, \ldots, m, \]  

for some \( \alpha_k > 0, \gamma_k \leq 1 \) (see [14,23,17]). It is well known that, under generic conditions on \( E \), each \( n_j \times n_j \) Jordan block associated with \( \lambda \) gives rise to \( n_j \) perturbed eigenvalues satisfying the expansion (2) with \( \gamma_k = 1/n_j \). Motivated by this, the Hölder condition number for \( \lambda \) is defined in [17] as a pair

\[ \kappa(A, \lambda) = (n_1, \alpha_1), \]  

where \( n_1 \) is the largest size of Jordan blocks associated with \( \lambda \), so \( 1/n_1 \) is the smallest possible power \( \gamma_k \) of \( \varepsilon \) in the expansion (2) for any perturbation \( E \). The scalar \( \alpha_1^{1/n_1} > 0 \) is the largest possible magnitude of the coefficient of \( \varepsilon^{1/n_1} \) for all \( E \) with \( \|E\| \leq 1 \), where \( \| \cdot \| \) is any matrix norm. The bottom line is that for any perturbation \( E \) of size \( \|E\| \leq 1 \), the largest possible change in the eigenvalue \( \lambda \) is bounded roughly by \( \alpha_1^{1/n_1} \varepsilon^{1/n_1} \). Notice that the condition number \( \kappa(A, \lambda) \) depends on the matrix norm, i.e., different choices of the matrix norm usually produce different values for \( \alpha_1 \). The first component \( n_1 \), though, is independent of the matrix norm.

In many practical situations the matrix \( A \) has a certain particular structure, and it makes sense to restrict the perturbation matrices \( E \) to belong to that same class of structured matrices. This is the case, for instance, when the perturbation analysis is motivated by the error analysis for a structure-preserving eigenvalue algorithm. Therefore, if we denote by \( S \) the class of structured matrices under scrutiny, and \( \lambda \) is an eigenvalue of a matrix \( A \in \mathbb{S} \), the goal in a structured perturbation analysis becomes to assess the sensitivity of the eigenvalue \( \lambda \) under structured perturbations \( A + \varepsilon E \), with \( E \in \mathbb{S} \). The crucial quantity in this respect is now the structured condition number of the eigenvalue. In the case of a multiple eigenvalue \( \lambda \), arguing as above via the expansion (2), one arrives at a definition of the structured Hölder condition number as

\[ \kappa_S(A, \lambda) = (n_S, \alpha_S), \]  

where \( 1/n_S \) is the smallest possible power \( \gamma_k \) of \( \varepsilon \) in (2) for any structured perturbation \( E \in \mathbb{S} \), and the scalar \( \alpha_S^{1/n_S} > 0 \) is the largest possible magnitude of the coefficient of \( \varepsilon^{1/n_S} \) for all structured perturbations \( E \in \mathbb{S} \) with \( \|E\| \leq 1 \). Both definitions (3) and (4) boil down to the usual definitions for simple eigenvalues if we take \( n_1 = 1 \). Again, the value of \( \kappa_S(A, \lambda) \) depends on the matrix norm we choose to measure the size of the perturbations \( E \in \mathbb{S} \).

Notice that from the definitions above it is clear that in general \( n_1 \geq n_S \), and that whenever \( n_1 \) and \( n_S \) are equal, \( \alpha_1 \) is never exceeded by \( \alpha_S \). The comparison between the two condition numbers, structured and unstructured, has been the subject of several papers [5,7,9–11,13,18–21] for both simple and multiple eigenvalues. It has been shown [21] that for several matrix structures (including symmetric, Toeplitz, Hankel, circulant) the unstructured condition number (3) and the structured one (4) are exactly the same for both the Frobenius and the spectral norm. In many other cases under study, the leading exponents \( 1/n_1 \) and \( 1/n_S \) coincide, and the coefficients \( \alpha_1 \) and \( \alpha_S \) have been shown to be...
equal up to a moderate factor (see, for instance, [5] for $S = \mathbb{R}^{n \times n}$, or [21,22] for many other structures including real skew-symmetric, Hamiltonian, persymmetric, orthogonal and unitary).

For some other families, however, there can be a significant difference between the structured and the unstructured condition number, either because $n_1 = n_S$ but the constants $\alpha_1$ and $\alpha_S$ can be arbitrarily wide apart in size, or because the leading exponents $1/n_1$ and $1/n_S$ in the expansion (2) are different. Examples of the former have been found for certain zero-structured [19] or symplectic matrices [11], while one simple example of the latter case is the following 3 by 3 example, taken from [21]: consider the complex skew-symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & i \\ 0 & -i & 0 \end{bmatrix}.$$ (5)

which has a unique eigenvalue $\lambda = 0$ with one single $3 \times 3$ Jordan block. Therefore, $n_1 = 3$. However, one can prove that no structured perturbation $A + \varepsilon E$ with complex skew-symmetric $E$ produces perturbed eigenvalues of order $O(\varepsilon^{1/3})$. Instead, most complex skew-symmetric perturbations give rise to perturbed eigenvalues of order $O(\varepsilon^{1/2})$, and one can prove that $n_S = 2$ for the class $S$ of 3 by 3 complex symmetric matrices. We will see later that this kind of behavior is quite special for this particular class of matrices, and requires very strong restrictions on the eigenvalue and its multiplicities.

Notice that the case when $n_S < n_1$ is the most promising one from the point of view of error analysis, in the sense that a structure-preserving algorithm has a better chance of being significantly more accurate than conventional methods, since the eigenvalue is much less sensitive to structured perturbations than to unstructured ones. However, no explicit formulas are available in the literature, either for $n_S$ or for $\alpha_S$ in this degenerate situation. This is precisely why we shall focus on this kind of highly nongeneric perturbations:

**Definition 1.1.** Let $\lambda$ be a (possibly multiple) eigenvalue of a matrix $A \in \mathbb{C}^{n \times n}$, let $S$ be a family of structured $n$ by $n$ matrices, and let

$$\kappa(A, \lambda) = (n_1, \alpha_1), \quad \kappa_S(A, \lambda) = (n_S, \alpha_S)$$

be, respectively, the unstructured and the structured condition numbers of $\lambda$, as defined in (3) and (4). We say that the class $S$ of matrices is **fully nongeneric** for $A$ and $\lambda$ if $n_S < n_1$.

If a structure $S$ is not fully nongeneric for $A$ and $\lambda$, then there exists some $E \in S$ such that the perturbed eigenvalues of $A + \varepsilon E$ closest to $\lambda$ behave generically. Saying that a family $S$ of matrices is fully nongeneric amounts to precluding the generic behavior for every single perturbation in the family $S$.

Notice that in Definition 1.1 we are not imposing the unperturbed matrix to be in $S$. We do this for the sake of generality, since such an assumption is not strictly necessary. Bear in mind, however, that in most cases of interest the unperturbed matrix will belong to $S$.

Instances of fully nongeneric families in the literature are scarce: several structures were explored in [13] in the context of comparing structured and unstructured condition numbers. It was shown, for instance, that whenever $A \in S$ and $S$ is either of the classes of Toeplitz, of Hankel matrices [13, §3.3], of complex symmetric or of persymmetric matrices [13, §3.4], then $S$ cannot possibly be fully nongeneric.

More specifically, the general framework in which most of the results in [13] were obtained is that of structures defined through (eventually indefinite) scalar products. This includes several classes of matrices (symmetric, skew-symmetric, Hermitian, Hamiltonian, …) of relevance in applications. To be more precise, let $M$ be a real orthogonal matrix, either symmetric or skew-symmetric, and define both sets

$$J = \{ A \in \mathbb{C}^{n \times n}; \ A^T M A \}$$ (6)

of self-adjoint matrices and
of skew-adjoint matrices with respect to the scalar product \( \langle x, y \rangle = x^T M y \) defined by \( M \). If we denote by \( I_n \) the identity matrix of order \( n \), then

\[
\mathbb{L} = \left\{ A \in \mathbb{C}^{n \times n} : A^T M = -MA \right\}
\]

\( \text{(7)} \)

\( F_n = \begin{bmatrix}
0 & \cdots & 0 & 1 \\
0 & \cdots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 0 & 0
\end{bmatrix}
\]

\( \text{(8)} \)

the reverse identity (or ‘flip’ matrix) of order \( n \), by

\[
\Sigma_{p, q} = \begin{bmatrix}
I_p & 0 \\
0 & -I_q
\end{bmatrix}
\]

the \((p, q)\)-sign matrix \( p + q = n \), and define

\[
J_{2n} = \begin{bmatrix}
0 & I_n \\
-I_n & 0
\end{bmatrix}
\]

then,

- the sets of complex symmetric and complex skew-symmetric matrices are, respectively, the sets \( \mathbb{J} \) and \( \mathbb{L} \) generated by taking \( M = I_n \),
- pseudo-symmetric and pseudo-skew-symmetric matrices are the ones generated by taking \( M = \Sigma_{p, q} \),
- persymmetric and per-skew-symmetric correspond to choosing \( M = F_n \), and
- if we restrict ourselves to real matrices, the choice \( M = J_{2n} \) gives rise to the classes of real Hamiltonian and real skew-Hamiltonian matrices.

It turns out that whenever the relevant matrix structure \( S \) is defined in this way, and the un-}

perturbed matrix \( A \) belongs to \( S \), the full nongenericity of \( S \) for \( A \) and \( \lambda \) requires very restrictive conditions on the eigenvalue \( \lambda \) and its multiplicities. The following proposition gives a summarized version of the results obtained in [13, Theorems 3.7 & 3.8] for structures defined through bilinear forms:

**Proposition 1.2.** Let \( \lambda \) be an eigenvalue of a matrix \( A \) belonging to a class \( S \) of structured matrices, and let \( \kappa_S(\lambda) = (n_S, \alpha_S) \) and \( \kappa_A(\lambda) = (n_1, \alpha_1) \) be, respectively, the structured and the unstructured condition numbers of \( \lambda \) with respect to any unitarily invariant norm, as defined in, (3) and (4). Let \( M \) be an orthogonal symmetric matrix.

(i) If \( S \) is the set (6) of self-adjoint matrices with respect to the scalar product defined by \( M \), then \( \kappa_S(A, \lambda) = \kappa_A(A, \lambda) \).

(ii) If \( S \) is the set (7) of skew-adjoint matrices with respect to the scalar product defined by \( M \), then \( n_S < n_1 \) only if \( \lambda = 0 \), \( n_1 \) is odd, and there is one single Jordan block of \( A \) corresponding to \( \lambda = 0 \).

If we consider structures

\[
\mathcal{J} = \left\{ A \in \mathbb{C}^{n \times n} : A^H M = MA \right\}, \quad \mathcal{L} = \left\{ A \in \mathbb{C}^{n \times n} : A^H M = -MA \right\}
\]

associated to sesquilinear forms \( \langle x, y \rangle = x^H M y \), where \( H \) stands for the conjugate transpose, the situation is simpler, and much less interesting from our point of view, since no structure defined in this way can be fully nongeneric [13, Theorem 3.10]:

**Proposition 1.3.** Let \( \lambda \) be an eigenvalue of a matrix \( A \in \mathbb{C}^{n \times n} \), let \( M \) be a real orthogonal matrix, either symmetric or skew-symmetric, and define \( S = \{ A \in \mathbb{C}^{n \times n} : A^H M = \gamma MA \} \) for a fixed \( \gamma \in \{-1, 1\} \). Let
κ_S(A, λ) = (n_S, α_S) and κ(A, λ) = (n_1, α_1) be, respectively, the structured and the unstructured condition numbers of λ with respect to the spectral norm, as defined in (3) and (4). Then κ_S(A, λ) = κ(A, λ).

Therefore, in what follows we will focus on the case described in item (ii) of Proposition 1.2, namely, that of a zero eigenvalue of a skew-adjoint matrix with one single largest Jordan block whose dimension is odd. Since the original analysis was made with the spectral norm, this will be our choice of matrix norm in Sections 4 and 5, where we deal with condition numbers. Therefore, unless otherwise stated, the symbol ∥·∥ denotes from now on the spectral matrix norm

\[ ∥A∥ = \sqrt{\lambda_{\text{max}}(A^H A)} , \]

although occasionally, as in Lemma 4.4, we will resort to the Frobenius matrix norm

\[ ∥A∥_F = \sqrt{\text{trace}(A^H A)} \]

(A^H denotes the conjugate transpose of A). By the equivalence of norms in finite-dimensional spaces, most results in this paper still hold for arbitrary matrix norms, up to some appropriate constant factor.

Our goal is to determine the most likely leading exponents γk and leading coefficients αk in the expansion (2) for structured perturbations for such an eigenvalue, and our main tool to do it will be the Newton polygon, an elementary geometric construction first proposed by Isaac Newton (see [3], [1, Appendix A7], [17]).

The paper is organized as follows: once the basic notation is set in Section 2, we use in Section 3 the Newton diagram, applied to arbitrary perturbation matrices E, as a tool to find out which coefficients in the characteristic polynomial of the perturbed matrix A + εE are more likely to determine the leading exponent and leading coefficient in the asymptotic expansion (2). The main result, Theorem 3.3, shows that these characteristic coefficients depend in general on both first (i.e., eigenvectors) and second vectors in the longest Jordan chains associated with the eigenvalue under study. In Section 4 we specialize the results obtained for arbitrary perturbations in Section 3 to the case of fully nongeneric perturbations. Both asymptotic expansions and bounds on the structured condition numbers will be obtained. Finally, Section 5 presents extensions to matrix pencils of the results in Section 4. Two classes of structured pencils are considered, symmetric/skew-symmetric and palindromic, under appropriate assumptions on the eigenvalue, which guarantee full nongenericity.

2. Notation

We introduce in this section the notation to be used in Sections 2–4 of the paper: let λ be an eigenvalue of A ∈ C^{n×n}. The Jordan canonical form of A can be written as

\[ \begin{bmatrix} J & 0 \\ 0 & \hat{J} \end{bmatrix} = \begin{bmatrix} Q & \hat{P} \end{bmatrix} A \begin{bmatrix} P \\ \hat{P} \end{bmatrix} \]  

(9)

with

\[ \begin{bmatrix} Q \\ \hat{Q} \end{bmatrix} \begin{bmatrix} P \\ \hat{P} \end{bmatrix} = I, \]  

(10)

where J corresponds to λ and \( \hat{J} \) is the part of the Jordan form corresponding to the other eigenvalues of A. Let

\[ J = \text{Diag}(\Gamma_1^{1}, \ldots, \Gamma_1^{r_1}, \ldots, \Gamma_q^{1}, \ldots, \Gamma_q^{r_q}) , \]  

(11)
where, for \( j = 1, \ldots, q, \)

\[
\Gamma_j^1 = \cdots = \Gamma_j^r_j = \begin{bmatrix}
\lambda & 1 \\
& \ddots & 1 \\
& & \ddots & 1 \\
& & & \lambda
\end{bmatrix}
\]

is a Jordan block of dimension \( n_j \), repeated \( r_j \) times, and ordered so that

\[ n_1 > n_2 > \cdots > n_q. \]

The \( n_j \) are called the partial multiplicities for \( \lambda \). The eigenvalue \( \lambda \) is semisimple (nondefective) if \( q = n_1 = 1 \) and nonderogatory if \( q = r_1 = 1 \).

Most of our results will depend only on the information associated with the two largest sizes \( n_1 > n_2 \) of Jordan blocks associated with \( \lambda \). Therefore, it will be useful to single out matrices of left and right eigenvectors, corresponding to Jordan blocks of sizes \( n_1 \) and \( n_2 \). Let

\[
X_1 = \begin{bmatrix}Pe_1, Pe_{n_1+1}, \ldots, Pe_{(r_1-1)n_1+1} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_1},
\]

\[
Y_1 = \begin{bmatrix}Q^H e_{n_1}, Q^H e_{2n_1}, \ldots, Q^H e_{r_1n_1} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_1},
\]

\[
Z_1 = \begin{bmatrix}Pe_{r_1n_1+1}, Pe_{r_1n_1+n_2+1}, \ldots, Pe_{r_1n_1+(r_2-1)n_2+1} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_2},
\]

\[
W_1 = \begin{bmatrix}Q^H e_{r_1n_1+n_2}, Q^H e_{r_1n_1+2n_2}, \ldots, Q^H e_{r_1n_1+r_2n_2} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_2},
\]

where, for every index \( k \), \( e_k \) stands for the \( k \)-th column of the identity matrix. Therefore, the matrices \( X_1 \) and \( Y_1 \) contain, respectively, the right and the left eigenvectors associated with all Jordan blocks of size \( n_1 \) corresponding to \( \lambda \), while the matrices \( Z_1 \) and \( W_1 \) contain, respectively, the right and the left eigenvectors associated with all Jordan blocks of size \( n_2 \).

Finally, we collect in the matrices \( X_2 \) (resp. \( Y_2 \)) the second vectors in the right (resp. left) Jordan chains of length \( n_1 \), i.e.,

\[
X_2 = \begin{bmatrix}Pe_2, Pe_{n_1+2}, \ldots, Pe_{(r_1-1)n_1+2} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_1},
\]

\[
Y_2 = \begin{bmatrix}Q^H e_{n_1-1}, Q^H e_{2n_1-1}, \ldots, Q^H e_{r_1n_1-1} \end{bmatrix} \in \mathbb{C}^{n_1 \times r_1}.
\]

Now, for every additive perturbation \( A + \varepsilon E \) corresponding to a perturbation matrix \( E \in \mathbb{C}^{n \times n} \), we define matrices

\[
\Phi_1 = Y_1^H EX_1 \in \mathbb{C}^{r_1 \times r_1}, \quad \Phi_2 = \begin{bmatrix}Y_1^H & W_1^H\end{bmatrix} E \begin{bmatrix}X_1 & Z_1\end{bmatrix} \in \mathbb{C}^{(r_1 + r_2) \times (r_1 + r_2)},
\]

\[
\Phi_{12} = Y_1^H EX_2 \in \mathbb{C}^{r_1 \times r_1}, \quad \text{and} \quad \Phi_{21} = Y_2^H EX_1 \in \mathbb{C}^{r_1 \times r_1}.
\]

Notice that the matrices defined in (18) and (19) are just submatrices of \( \tilde{E} = QEP = P^{-1}EP \), for the matrices \( P, Q \) given in (9); if we partition the upper left corner of \( \tilde{E} \) conformally with the Jordan blocks of sizes \( n_1 \) and \( n_2 \), the matrix \( \Phi_1 \) corresponds to choosing all lower left entries of the \( r_1 \times r_1 \) blocks of size \( n_1 \), while \( \Phi_2 \) corresponds to choosing the lower left corners of all \( (r_1 + r_2) \times (r_1 + r_2) \) blocks of sizes larger than or equal to \( n_2 \). The matrix \( \Phi_{12} \) (resp. \( \Phi_{21} \)) corresponds to choosing the entries which are immediately on top of (resp. to the right of) the lower left corners of the blocks of size \( n_1 \). For instance, suppose we have a \( 10 \times 10 \) matrix having one single eigenvalue \( \lambda \) with \( n_1 = 3, r_1 = 2 \) and \( n_2 = 2, r_2 = 2 \). Then the matrix \( P^{-1}EP \) can be represented as follows.
\[P^{-1}EP = \begin{bmatrix}
\star & \star & \star & \star & \star \\
\spadesuit_1 & \spadesuit & \spadesuit_2 & \spadesuit & \star \\
\heartsuit_1 & \heartsuit & \heartsuit_2 & \heartsuit & \diamondsuit_1 & \diamondsuit_2 \\
\spadesuit_3 & \spadesuit_4 & \star & \star & \star & \star \\
\spadesuit_5 & \spadesuit_6 & \diamondsuit_7 & \diamondsuit_8 & \star & \star \\
\spadesuit_9 & \spadesuit_{10} & \diamondsuit_{11} & \diamondsuit_{12} & \\
\end{bmatrix}, \quad (20)
\]

and the matrices defined in (18) and (19) are just

\[\Phi_1 = \begin{bmatrix}
\heartsuit_1 & \heartsuit_2 \\
\spadesuit_3 & \spadesuit_4 \\
\end{bmatrix}, \quad \Phi_{12} = \begin{bmatrix}
\spadesuit_1 & \spadesuit_2 \\
\spadesuit_3 & \spadesuit_4 \\
\end{bmatrix}, \quad \Phi_{21} = \begin{bmatrix}
\spadesuit_1 & \spadesuit_2 \\
\spadesuit_3 & \spadesuit_4 \\
\end{bmatrix}, \quad \Phi_2 = \begin{bmatrix}
\heartsuit_1 & \heartsuit_2 & \diamondsuit_1 & \diamondsuit_2 \\
\spadesuit_3 & \spadesuit_4 & \diamondsuit_3 & \diamondsuit_4 \\
\spadesuit_5 & \spadesuit_6 & \diamondsuit_7 & \diamondsuit_8 \\
\spadesuit_9 & \spadesuit_{10} & \diamondsuit_{11} & \diamondsuit_{12} \\
\end{bmatrix}.
\]

We conclude this preliminary section by noting that Lidskii’s classical perturbation theory \([14,17]\) states that, whenever the matrix \(\Phi_1 = YH_1E X_1\) is invertible, the leading exponent \(\gamma_k\) in the expansion (2) is equal to \(1/n_1\), and the leading coefficients \(\alpha_k\) are just the eigenvalues of \(\Phi_1\). Actually, one can show (see \([17, p. 809]\)) that, even if \(\Phi_1\) is singular, every nonzero eigenvalue of \(\Phi_1\) leads to \(n_1\) expansions of this form. Therefore, the only way for the degeneracy condition \(n_S < n_1\) to be fulfilled for a class of matrices \(S\) is that all the eigenvalues of \(\Phi_1 = YH_1E X_1\) are zero for every matrix \(E \in S\).

For instance, example (5) above shows that the class of complex skew-symmetric matrices is fully nongeneric for \(\lambda = 0\), since

\[X_1 = \begin{bmatrix}
1 \\
0 \\
i
\end{bmatrix}, \quad Y_1 = \begin{bmatrix}
1 \\
0 \\
i
\end{bmatrix}
\]

is one possible choice of eigenvectors in the Jordan form, and one can easily check that \(Y_1H_1E X_1 = 0\) for any complex skew-symmetric perturbation \(E\).

Finally, when required we will make use of the customary notation for submatrices: given an \(m \times n\) matrix \(A\) and two index sets \(\alpha \in \{1, \ldots, m\}\) and \(\beta \in \{1, \ldots, n\}\), we will denote by

\[A(\alpha, \beta)\]

the submatrix of \(A\) that lies in the rows indexed by \(\alpha\) and the columns indexed by \(\beta\). If the submatrix is principal and lies in the rows and columns indexed by a set \(\alpha\), we denote it simply by \(A(\alpha)\).

3. Characteristic coefficients for arbitrary perturbations: the Newton diagram approach

Our ultimate goal is to identify the most likely leading exponent and leading coefficient for the expansion (2) in the special situation described in item (ii) of Proposition 1.2. To do that, we will first need some auxiliary results for the general case when the perturbation is arbitrary. We collect in this section all the results we need, which will be applied to fully nongeneric perturbations in the next section.

Therefore, consider any matrix \(A \in \mathbb{C}^{m \times n}\) with a multiple eigenvalue. With no loss of generality, we may assume that this unperturbed eigenvalue is zero (otherwise, it suffices to make a shift in the
eigenvalues). Furthermore, we may assume for the sake of simplicity that zero is the only eigenvalue of $A$, i.e. $\hat{J}$ in (9) is empty. The general case may be reduced to this one using appropriate Riesz projections (we refer to [14, pp. 83–84], or [1, §3.9.1] for more details). We look at the eigenvalues of the perturbed matrix $A + \varepsilon E$ as roots of its characteristic polynomial,

$$P(\lambda, \varepsilon) = \det(\lambda I - A - \varepsilon E) = \lambda^n + \alpha_1(\varepsilon)\lambda^{n-1} + \cdots + \alpha_{n-1}(\varepsilon)\lambda + \alpha_n(\varepsilon), \quad (23)$$

which is a monic polynomial in $\lambda$ whose coefficients depend on $\varepsilon$. Notice that $\lambda$ is now a parameter, not the eigenvalue under study as in the previous section. Let

$$\alpha_k(\varepsilon) = \tilde{\alpha}_k \varepsilon^{a_k} + \cdots, \quad k = 1, \ldots, n,$$

i.e., $a_k$ is the leading exponent and $\tilde{\alpha}_k$ is the leading coefficient of $\alpha_k(\varepsilon)$ (hence, $\tilde{\alpha}_k \neq 0$ and no term of order lower than $a_k$ appears in the expansion of $\alpha_k(\varepsilon)$).

It is well known [1,12] that the roots of (23) are given by expansions in fractional powers of $\varepsilon$. The leading exponents of these expansions can be easily found through the following elementary geometrical construction: we plot the values $a_k$ versus $k$ for $k \in \{1, \ldots, n\}$ together with the point $(0, 0)$ corresponding to $\lambda^n$ (if $\alpha_j(\varepsilon) = 0$ for some $j$, the corresponding point is disregarded). Then, we draw the segments on the lower boundary of the convex hull of the plotted points. These segments constitute the so-called Newton polygon, or Newton diagram, associated with $P(\lambda, \varepsilon)$. The slopes of the different segments in the Newton diagram are precisely the leading powers of the $\varepsilon$-expansions of the roots $\lambda = \lambda(\varepsilon)$ of (23). The number of roots corresponding to each slope equals the length of the projection on the horizontal axis of the segment with that particular slope, and the leading coefficients can be obtained as the solution of a certain low-order polynomial equation whose coefficients are obtained from the coefficients in $P(\lambda, \varepsilon)$ of the terms associated with the endpoints of that segment (see [17, §3], [1, Appendix A7], [3] for more details). For example, let

$$P(\lambda, \varepsilon) = \lambda^5 - \varepsilon^2\lambda^4 + (\varepsilon - 2\varepsilon^2)\lambda^3 + 3\varepsilon^2\lambda - \varepsilon^3.$$

Its Newton diagram, depicted in Fig. 1, shows that $P(\lambda, \varepsilon)$ has four roots of order $\varepsilon^{1/2}$ and one of order $\varepsilon$.

As mentioned at the end of Section 2, classical perturbation theory shows that when the perturbation matrix $E$ is arbitrary, the smallest slope in the Newton diagram is generically $1/n_1$, corresponding to the segment joining the origin with the point $P_1 = (n_1 r_1, r_1)$ (see Fig. 2 below). This point appears in the Newton diagram if and only if the matrix $\Phi_1 = Y_1^H E X_1$ defined in (18) is invertible, and this invertibility is precisely the genericity condition which guarantees this most likely behavior. For fully nongeneric perturbations, this generic behavior cannot take place, as explained at the end of Section 2. In fact, even if $\Phi_1$ is singular, any nonzero eigenvalue of $\Phi_1$ would produce a point such as $P_2^J$ in Fig. 2, which in turn would give rise to asymptotic expansions of order $\varepsilon^{1/n_1}$ (see [17, p. 809]...
for more details on this). Therefore, all the eigenvalues of $\Phi_1$ must be zero for fully nongeneric perturbations, and no point such as $P_2^1$ can possibly appear in the Newton diagram if the perturbation belongs to a fully nongeneric family.

To determine the smallest possible slope among all structured perturbations, we will just assume that $P_1$ is not present in the diagram, and then examine the points on the Newton diagram which are most likely candidates to produce the smallest possible slope when connecting them with the origin. Since they are the grid points closest to $P_1$, the most likely candidates are (see Fig. 2)

- the point $P_{\text{left}} = (n_1 r_1 - 1, r_1)$, which is closest to $P_1$ on the left,
- the point $P_{\text{up}} = (n_1 r_1 + r_1 + 1)$, which is right on top of $P_1$, and
- the points $P_{\text{right}, j} = (n_1 r_1 + j n_2, r_1 + j)$, $j = 1, \ldots, r_2$ on the segment of slope $1/n_2$ with $P_1$ as its left endpoint,

where $n_1, n_2, r_1, r_2$ are the quantities introduced in (11). Of course, the last kind of points $P_{\text{right}, j}$ can only appear if $q > 1$, i.e., if there are Jordan blocks of at least two different sizes.

First, notice that if we denote by $m_{\text{left}}$, $m_{\text{up}}$, respectively, the slopes of the segments connecting the origin $(0, 0)$ with each of these points, then

$$m_{\text{left}} = \frac{r_1}{n_1 r_1 - 1} \leq m_{\text{up}} = \frac{r_1 + 1}{n_1 r_1} \iff r_1 (n_1 - 1) \geq 1,$$

which is always the case, except in the trivial situation $n_1 = 1$. Therefore, we may restrict ourselves to comparing the slopes determined in the Newton diagram by $P_{\text{left}}$ and the points $P_{\text{right}, j}, j = 1, \ldots, r_2$:

**Lemma 3.1.** Consider the points $P_{\text{left}} = (n_1 r_1 - 1, r_1)$ and $P_{\text{right}, j} = (n_1 r_1 + j n_2, r_1 + j)$, $j \in \{1, \ldots, r_2\}$, and let $m_{\text{left}}$ and $m_{\text{right}, j}$ be, respectively, the slopes of the segments connecting the origin $(0, 0)$ with each of these points. Then, for every $j \in \{1, \ldots, r_2\}$,

$$m_{\text{left}} \leq m_{\text{right}, j} \text{ if and only if } n_1 - n_2 \geq \frac{r_1 + j}{r_1 j}.$$ 

**Proof.** The proof is elementary, and is left to the reader. □

Now, we concentrate on the coefficients in the characteristic polynomial of the terms associated with these two kinds of points. In order to determine them, we first try to understand, for every
possible point \((p, k)\) in the Newton diagram, what is the connection of the perturbation matrix \(E\) with the coefficient of the corresponding term in \(\lambda^{n-p}e^k\) of the characteristic polynomial \((23)\). The following auxiliary lemma answers that question:

**Lemma 3.2.** Let \(p \in \{1, \ldots, n\}\) and \(k \leq p\), let \(P(\lambda, \varepsilon)\) be the characteristic polynomial \((23)\) and set \(\widetilde{E} = P^{-1}EP = QEP\), where \(E\) is any \(n \times n\) perturbation matrix, and \(P\) and \(Q\) are defined in \((9)\). Then, up to its sign, the coefficient of the term in \(\lambda^{n-p}e^k\) of \(P(\lambda, \varepsilon)\), corresponding to a possible point \((p, k)\) in the Newton diagram, is equal to the sum of all possible \(k\)-dimensional minors of \(E\) corresponding to submatrices obtained by removing from any of the principal \(p\)-dimensional submatrices of \(E\) exactly \(p - k\) rows and columns corresponding to entries where the canonical form \(J\) in \((9)\) has a supradiagonal 1, with the restriction that these supradiagonal entries must be chosen from at most \(k\) Jordan blocks of \(J\).

**Proof.** First, we observe that

\[
P(\lambda, \varepsilon) = \det(\lambda I - A - \varepsilon E) = \det(\lambda I - J - \varepsilon \tilde{E}),
\]

so \(P(\lambda, \varepsilon)\) is just a linear combination with coefficients \(\pm 1\) of all possible products of \(n\) entries of the matrix \(\lambda I - J - \varepsilon \tilde{E}\), with the restriction that no two factors can be on the same row or column. Recall that the simplifying assumptions at the beginning of this section imply that the Jordan canonical form \(J\) is nilpotent, so every entry of \(\lambda I - J - \varepsilon \tilde{E}\) contains a multiple of \(\varepsilon\), and for a few of its entries an \(\varepsilon\)-independent term must be added, either a \(\lambda\) for diagonal entries, or a \(-1\) for the supradiagonal positions where \(J\) has a 1. It is clear that the only way to obtain a term in \(\lambda^{n-p}e^k\) with a product of \(n\) entries is to choose exactly \(n - p\) diagonal \(\lambda\)'s among them. Hence, the \(p\) remaining factors in the product must be chosen from a \(p\)-dimensional principal submatrix of \(J + \varepsilon \tilde{E}\). Moreover, one must choose among them exactly \(p - k\) supradiagonal minus ones. Therefore, the product of the remaining \(k\) factors corresponds, as claimed, to a minor of dimension \(k\) of \(\tilde{E}\).

Moreover, we are not free to make whatever choices we want for the minus ones since, due to the special position of the \(\varepsilon\)-independent terms, every time we choose a minus one we are, at the same time, excluding from the product those lambdas which lie on the same row or the same column. The number of lambdas we remove depends on the number of Jordan blocks we sample the minus ones from, since the first minus one we choose from any given block excludes two lambdas, while any further minus one from the same block removes only one lambda. Suppose the \(p - k\) minus ones are taken from \(s\) different blocks. Then, these \(p - k\) choices exclude \(p - k + s\) lambdas, which, together with the \(n - p\) lambdas which were actually chosen in the product, cannot exceed the total number \(n\) of available lambdas. We conclude that \(s \leq k\), i.e., we are allowed to sample minus ones from at most \(k\) Jordan blocks. \(\Box\)

We are now in the position to describe the coefficients of the terms in the characteristic polynomial corresponding to both kinds of possible points \(P_{\text{left}}\) and \(P_{\text{right}, j}\) in the Newton diagram.

**Theorem 3.3.** Let \(A \in \mathbb{C}^{n \times n}\) have a single zero eigenvalue with Jordan canonical form \(J\) as in \((11)\) and let \(P(\lambda, \varepsilon)\) be the characteristic polynomial of \(A + \varepsilon E\) for an arbitrary \(E \in \mathbb{C}^{n \times n}\). Let

\[
P_{\text{left}} = (n_1 r_1 - 1, r_1) \quad \text{and} \quad P_{\text{right}, j} = (n_1 r_1 + j n_2, r_1 + j), \quad j \in \{1, \ldots, r_2\},
\]

and let \(\Phi_1, \Phi_2, \Phi_{12}\) and \(\Phi_{21}\) be the matrices defined in \((18)\) and \((19)\). For every \(j \in \{1, \ldots, r_1\}\), let \(\Phi_{1,j}^{\text{row}}\) (resp., \(\Phi_{1,j}^{\text{col}}\)) be the \(r_1 \times r_1\) matrix obtained by replacing the \(j\)-th row (resp., the \(j\)-th column) of \(\Phi_1\) with the \(j\)-th row of \(\Phi_{21}\) (resp., the \(j\)-th column of \(\Phi_{12}\)). Then:

\begin{enumerate}[(i)]
  \item If either \(q = 1\) in \((11)\), or \(q > 1\) with \(n_1 - n_2 \geq 2\), the coefficient in \(P(\lambda, \varepsilon)\) of the term in \(\lambda^{n-n_1 r_1+1}e^{r_1}\) corresponding to \(P_{\text{left}}\) is \(C_{\text{left}, 1} = (-1)^{r_1} \sum_{j=1}^{r_1} \det \Phi_{1,j}^{\text{row}} + \sum_{j=1}^{r_1} \det \Phi_{1,j}^{\text{col}}\).
\end{enumerate}
(ii) If \( n_2 = n_1 - 1 \), the coefficient in \( P(\lambda, \varepsilon) \) of the term in \( \lambda^{n_1 - r_1 + 1} \varepsilon^{r_1} \) corresponding to \( P_{\text{left}} \) is

\[
C_{\text{left}, 2} = C_{\text{left}, 1} + (-1)^{r_1} \left( \sum_{j=1}^{r_1} \sum_{k=1}^{r_2} \det \Phi_2(\{1, \ldots, j - 1, j + 1, \ldots, r_1, r_1 + k\}) \right),
\]

where \( C_{\text{left}, 1} \) is the coefficient described in item (i) above.

(iii) For each \( j \in \{1, \ldots, r_2\} \), the coefficient in \( P(\lambda, \varepsilon) \) of the term in \( \lambda^{n_1 - r_1 - jn_2} \varepsilon^{r_1 + j} \) corresponding to \( P_{\text{right}, j} \) is the sum of all \((r_1 + j)\)-dimensional principal minors of \( \Phi_2 \) containing \( \Phi_1 \) multiplied by \((-1)^{r_1 + j}\).

**Proof.** First, notice that either of \( C_{\text{left}, 1} \) or \( C_{\text{left}, 2} \) correspond to taking \( p = n_1 r_1 - 1 \) and \( k = r_1 \) in Lemma 3.2. Consequently, they are sums of all possible \( r_1 \)-dimensional minors of \( \tilde{E} = P^{-1} EP \) obtained by

(R1) first, removing \( n - n_1 r_1 + 1 \) rows and columns from \( \tilde{E} \) corresponding to diagonal positions, and

(R2) then, removing \( (n_1 - 1) r_1 - 1 \) rows and columns corresponding to supradiagonal positions chosen from at most \( r_1 \) Jordan blocks of \( J \).

Let us analyze in how many ways can we make the second choice R2: suppose the \((n_1 - 1)r_1 - 1\) supradiagonal positions are chosen from \( r_1 - s \) Jordan blocks of size \( n_1 \) and \( s \) Jordan blocks of size \( n_2 \) or smaller, and let \( n_{ij} < n_1, j = 1, \ldots, s \), be the sizes of those \( s \) Jordan blocks. Then the \( r_1 \) chosen blocks contain exactly

\[
(r_1 - s)(n_1 - 1) + \sum_{j=1}^{s} (n_{ij} - 1)
\]

available supradiagonal positions, and this number should not be smaller than the number \((n_1 - 1)r_1 - 1\) of supradiagonal positions to be chosen. Since \( n_{ij} < n_1 \) for all \( j = 1, \ldots, s \), this can only happen if either \( s = 0 \), or \( s = 1 \) and \( n_2 = n_1 - 1 \). The first case, \( s = 0 \), corresponds to item (i) in the statement.

(i) If \( s = 0 \), we have to choose in step R2 all but one of the supradiagonal positions available in the \( r_1 \) Jordan blocks of largest size \( n_1 \). Consequently, the choice R1 reduces to exactly one row and column corresponding to one single diagonal position from one of those \( r_1 \) largest Jordan blocks. But that diagonal position cannot be chosen arbitrarily, since when we remove the corresponding row and column we cannot afford to remove more than one supradiagonal position. Thus, we must choose either the top left or the bottom right diagonal position in the Jordan block (any other choice would remove two different supradiagonal positions). One can easily check that removing any top left diagonal position leads to minors containing entries of \( \Phi_1 \) and \( \Phi_{12} \), while removing the right bottom one leads to minors containing entries of \( \Phi_1 \) and \( \Phi_{21} \).

As a way of illustration, consider, for instance, a \( 6 \times 6 \) Jordan matrix \( J \) with a single zero eigenvalue and two Jordan blocks of size 3. Three supradiagonal positions must be chosen in step R2, and this can only be achieved if the diagonal position chosen in step R1 is either the top left or the right bottom entry of one of the two Jordan blocks. If we write

\[
\tilde{E} = \begin{bmatrix}
\ast & \ast & \ast & \ast & \ast & \ast \\
\spadesuit_1 & \ast & \spadesuit_2 & \ast & \ast \\
\heartsuit_1 & \heartsuit_1 & \ast & \ast & \ast & \ast \\
\spadesuit_3 & \ast & \ast & \ast & \ast & \ast \\
\heartsuit_3 & \ast & \heartsuit_4 & \ast & \ast \\
\spadesuit_3 & \ast & \ast & \heartsuit_4 & \ast & \ast
\end{bmatrix}
\]

then, in this particular example,
– choosing the (1, 1) diagonal element in step R1 leads to \( \det \begin{bmatrix} \heartsuit_1 & \spadesuit_2 \\ \spadesuit_3 & \spadesuit_4 \end{bmatrix} \).

– choosing the (3, 3) diagonal element in step R1 leads to \( \det \begin{bmatrix} \spadesuit_1 & \spadesuit_2 \\ \spadesuit_3 & \spadesuit_4 \end{bmatrix} \).

– choosing the (4, 4) diagonal element in step R1 leads to \( \det \begin{bmatrix} \spadesuit_1 & \spadesuit_2 \\ \spadesuit_3 & \spadesuit_4 \end{bmatrix} \).

– choosing the (6, 6) diagonal element in step R1 leads to \( \det \begin{bmatrix} \spadesuit_1 & \spadesuit_2 \\ \spadesuit_3 & \spadesuit_4 \end{bmatrix} \).

In general, any entry of \( \Phi_2 \) beyond the diagonal blocks of largest size \( n_1 \) is removed, since all diagonal positions in those blocks must be chosen in step R1. Moreover, we have that, in general, for any \( j \in \{1, \ldots, r_1\} \),

– choosing in step R1 the top left diagonal entry from the \( j \)-th Jordan block leads to the determinant of an \( r_1 \times r_1 \) matrix obtained by replacing the \( j \)-th column of \( \Phi_1 \) with the \( j \)-th column of \( \Phi_{12} \), and

– choosing in step R1 the bottom right diagonal entry from the \( j \)-th Jordan block leads to the determinant of an \( r_1 \times r_1 \) matrix obtained by replacing the \( j \)-th row of \( \Phi_1 \) with the \( j \)-th row of \( \Phi_{21} \).

This proves our claim in item (i) of the statement, except for the sign. The sign is obtained from multiplying the sign \((-1)^{n_1 r_1 - 1}\) of the coefficient in \( P(\lambda, \varepsilon) \) of the term \( \lambda^{n_1 r_1 + 1} \) by the sign \((-1)^{(n_1 - 1) r_1 - 1} \) which corresponds to removing \((n_1 - 1)r_1 - 1\) rows and columns corresponding to supradiagonal positions.

(ii) If \( n_2 = n_1 - 1 \), all the choices analyzed above for both steps R1 and R2 remain valid, but there are additional admissible choices, namely, choosing \( n_2 = 1 \) of the supradiagonal positions in step R2 from one single \( n_2 \)-dimensional diagonal block, say the \( k \)-th one for some \( k \in \{1, \ldots, r_2\} \), together with all supradiagonal positions in all but one of the \( n_1 \)-dimensional ones, say all but the \( j \)-th block for some \( j \in \{1, \ldots, r_1\} \). In that case, the only way to complete the number of diagonal positions to be chosen in step R1 is to exhaust all those in the \( j \)-th \( n_1 \)-dimensional block, together with all those in every \( n_2 \)-dimensional diagonal block except the \( k \)-th one. The corresponding minor which is left is just the determinant of the principal submatrix \( \Phi_2((1, \ldots, j - 1, j + 1, \ldots, r_1, r_1 + k)) \) of \( \Phi_2 \) obtained by removing its \( j \)-th row and column and appending the \((r_1 + k)\)-th ones.

As an illustration, consider, for instance, a \( 10 \times 10 \) Jordan matrix \( J \) with a single zero eigenvalue and four Jordan blocks, two of size 3 and two of size 2. Three supradiagonal positions must be chosen in step R2, two of them from a 3-dimensional diagonal block and one from a 2-dimensional one. This can only be achieved if the diagonal positions chosen in step R1 exhaust all diagonal positions in the two remaining diagonal blocks. If we write

\[
\begin{bmatrix}
* & * & * & * & * & * & * & * & * & * \\
♥_1 & ♠_1 & ♠_2 & ♠_3 & ♠_4 & ♠_5 & ♠_6 & ♠_7 & ♠_8 & ♠_9 \\
* & * & * & * & * & * & * & * & * & * \\
♣_1 & ♣_2 & ♣_3 & ♣_4 & ♣_5 & ♣_6 & ♣_7 & ♣_8 & ♣_9 & ♣_{10} \\
* & * & * & * & * & * & * & * & * & * \\
♦_1 & ♦_2 & ♦_3 & ♦_4 & ♦_5 & ♦_6 & ♦_7 & ♦_8 & ♦_9 & ♦_{10} \\
\end{bmatrix},
\]

then, for instance, choosing the supradiagonal positions in step R2 from the second block of size 3 (i.e., from all but the first one) and from the first block of size 2 leads to the minor

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The argument is the same in the general case, leading to item (ii) in the statement.

(iii) Finally, $C_{\text{right}, j}$ corresponds to taking $p = n_1 r_1 + j n_2$, $k = r_1 + j$ in Lemma 3.2. Consequently, $C_{\text{right}, j}$ is equal to the sum of all possible $(r_1 + j)$-dimensional minors of $\tilde{E} = P^{-1} EP$ obtained by removing from $\tilde{E}$

(R1) the rows and columns corresponding to all diagonal entries from diagonal blocks of size less than $n_1$, except for $j$ blocks of size $n_2$, and

(R2) the rows and columns corresponding to all supradiagonal positions contained in all $r_1$ diagonal blocks of size $n_1$ and in the $j$ diagonal blocks of size $n_2$ not covered in step R1.

One can immediately check that this leads to our claim in item (iii) of the statement (the sign is deduced similarly to (i) and (ii)). □

4. First order expansions for fully nongeneric structures

We are now in the position to apply the results of the previous section to the case we are interested in, namely that of fully nongeneric perturbations to a zero eigenvalue of a skew-adjoint matrix with one single largest Jordan block whose dimension is odd. The fact that there is one single largest Jordan block means that $\Phi_1, \Phi_2$ and $\Phi_3$ are $1 \times 1$ matrices, which greatly simplifies the formulas. Also, we may assume that $n_1 \neq 1$, otherwise the situation is trivial.

First of all, we need an auxiliary result before we identify the smallest slope in the Newton diagram for each possible situation. This result is the following.

Lemma 4.1. Let $M$ be a symmetric, real and orthogonal matrix and let $S = \{A \in \mathbb{C}^{n \times n}; A^T M = -M A\}$. Let $A \in S$ have Jordan form (9) with $\lambda = 0$, $r_1 = 1$, $n_1$ odd, $n_2 \neq 1$, and $n_2 = n_1 - 1$. Then, the coefficients $C_{\text{right}, j}$ described in (iii) of Theorem 3.3 are zero for all $j \in \{1, \ldots, r_2\}$ and for any perturbation $E \in S$.

Proof. Using the structured canonical forms of families of skew-adjoint matrices ([16, Theorem 7.3], see also [13, Corollary A.4]) one can show that the structure $S$ induces the following relationships between right and left generalized eigenvectors:

\begin{itemize}
  \item Since $n_1$ is odd and $r_1 = 1$, we have $y_1 = M x_1$, where $x_1$ and $y_1$ are column vectors, defined, respectively, as in (12), (13) (recall that $r_1 = 1$).
  \item Since $n_2 = n_1 - 1$, then $n_2$ is even and $r_2$ is also even, so $W_1 = M Z_1 \begin{bmatrix} 0 & I_{\frac{n_2}{2}} \\ -I_{\frac{n_2}{2}} & 0 \end{bmatrix}$, where $Z_1$ and $W_1$ are defined as in (14), (15), respectively.
\end{itemize}

Using these relationships and the fact that $(M E)^T = -M E$ for all $E \in S$, it is easy to rewrite the matrix $\Phi_2$ in (18) as

$$\Phi_2 = \begin{pmatrix} 0 & v & w \\ w^T & A & B \\ -v^T & C & A^T \end{pmatrix}$$

where $\Phi_2 \in \mathbb{C}^{(1+r_2) \times (1+r_2)}$, $v, w \in \mathbb{C}^{1 \times \frac{n_2}{2}}$, and $A, B, C \in \mathbb{C}^{\frac{n_2}{2} \times \frac{n_2}{2}}$, where $B$ and $C$ are skew-symmetric matrices, and $A$ has no particular structure. We want to prove that the sum of all $k$-dimensional principal minors of $\Phi_2$ containing the first row and column of $\Phi_2$ is zero for all $k \in \{2, 3, \ldots, r_2 + 1\}$. To do that, we first prove that the determinant of a matrix with the structure (24) is zero because the matrix $\Phi_2$ can be decomposed as:

$$\Phi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -I_{\frac{n_2}{2}} \\ 0 & I_{\frac{n_2}{2}} & 0 \end{pmatrix} \begin{pmatrix} 0 & v & w \\ -v^T & C & A^T \\ -w^T & -A & -B \end{pmatrix}$$
so \( \det(\Phi_2) = 0 \) because the second factor in the matrix product above is skew-symmetric with odd size, and therefore singular.

Now, notice that, for every \( k \in \{2, 3, \ldots, r_2 + 1\} \), any \( k \)-dimensional principal minor of \( \Phi_2 \) containing the first row and column of \( \Phi_2 \) must satisfy one of the two following properties, which derive from the special structure of the matrix \( \Phi_2 \):

- either the \( k \)-dimensional principal submatrix has the same form (24) of \( \Phi_2 \), and therefore the corresponding minor is zero,
- or the corresponding \( k \)-dimensional principal minor can be paired in a one-to-one correspondence with another \( k \)-dimensional principal minor containing the first row and column of \( \Phi_2 \) whose value is its exact opposite. To be more precise, for every minor \( |\Phi_2(1, i_1, \ldots, i_k-1)| \) which is not of the form (24) there is another \( k \)-dimensional principal minor \( |\Phi_2(1, j_1, \ldots, j_k-1)| \) whose value is the opposite of \( |\Phi_2(1, i_1, \ldots, i_k-1)| \). It is easy to check that the relation between the indices \( j_s \) and \( i_s \) for \( s \in \{1, \ldots, k-1\} \) and \( k \in \{2, 3, \ldots, r_2\} \)

\[
J_s = \begin{cases} 
  i_s + \frac{r_2}{2} & \text{if } 2 \leq i_s \leq \frac{r_2}{2} + 1, \\
  i_s - \frac{r_2}{2} & \text{if } \frac{r_2}{2} + 1 < i_s \leq r_2 + 1.
\end{cases}
\]

Therefore, using these two properties, we may directly conclude that the sum of all \( k \)-dimensional principal minors of \( \Phi_2 \) containing the first row and column of \( \Phi_2 \) is zero for all \( k \in \{2, 3, \ldots, r_2 + 1\} \).

For example if \( r_2 = 6 \) and \( k = 3 \), there are \( \binom{6}{2} = 15 \) principal minors containing the first row and column of the matrix \( \Phi_2 \) and they have the following properties:

- the minors \( \Phi_2(1, 2, 5), \Phi_2(1, 3, 6) \) and \( \Phi_2(1, 4, 7) \) have the same structure (24) as \( \Phi_2 \), so they all vanish, and
- the remaining twelve minors satisfy the following pairings: \(|\Phi_2(1, 2, 3)| = -|\Phi_2(1, 5, 6)|, |\Phi_2(1, 2, 4)| = -|\Phi_2(1, 5, 7)|, |\Phi_2(1, 2, 6)| = -|\Phi_2(1, 3, 5)|, |\Phi_2(1, 2, 7)| = -|\Phi_2(1, 4, 5)|, |\Phi_2(1, 3, 7)| = -|\Phi_2(1, 4, 6)|, and \( |\Phi_2(1, 3, 4)| = -|\Phi_2(1, 6, 7)| \). Therefore, they cancel each other in the global sum of \( 3 \)-dimensional principal minors. \( \Box \)

We now make use of Lemma 3.1 and Lemma 4.1 in order to identify the smallest slope in the Newton diagram for each possible situation and, consequently, the leading exponent in the asymptotic expansion (2) when the perturbation matrix \( E \) is restricted to a fully nongeneric family \( S \). Our first result determines, for each situation, the first term in the asymptotic expansion corresponding to that minimal slope.

**Theorem 4.2.** Let \( M \) be a symmetric, real and orthogonal matrix and let \( S = \{ A \in \mathbb{C}^{n \times n}; \ A^T M = -MA \} \). Let \( A \in S \) have Jordan form (9) with \( \lambda = 0 \), \( r_1 = 1 \) and \( n_1 \) odd, \( n_1 \neq 1 \). Let the scalars \( \Phi_1, \Phi_{12} \) and \( \Phi_{21} \), and the matrix \( \Phi_2 \in \mathbb{C}^{(r_2 + 1) \times (r_2 + 1)} \) be defined as in (18) and (19). Then, for any perturbation \( E \in S \):

(i) If either \( q = 1 \) in (11), or \( q > 1 \) with \( n_1 - n_2 \geq 2 \), the smallest slope in the Newton diagram associated with the zero eigenvalue is \( 1/(n_1 - 1) \), corresponding to the segment joining the origin with \( P_{\text{left}} = (n_1 - 1, 1) \) in the Newton diagram, provided that

\[
C_{\text{left,1}} = - (\Phi_{12} + \Phi_{21}) \neq 0.
\]

In that case, the perturbed matrix \( A + \varepsilon E \) has at least \( n_1 - 1 \) eigenvalues of order \( \varepsilon^{1/(n_1 - 1)} \). Moreover, (i.1) if either \( q = 1 \) in (11), or \( q > 1 \) with \( n_1 - n_2 > 2 \), then the perturbed matrix \( A + \varepsilon E \) has exactly \( n_1 - 1 \) eigenvalues of the form

\[
\hat{\lambda}(\varepsilon) = (C_{\text{left,1}})^{1/(n_1 - 1)} \varepsilon^{1/(n_1 - 1)} + o(\varepsilon^{1/(n_1 - 1)}),
\]
(i.2) if \( q > 1 \) with \( n_1 - n_2 = 2 \), the perturbed matrix \( A + \varepsilon E \) has exactly \( n_1 + n_2 = 2n_1 - 2 \) eigenvalues of order \( \varepsilon^{1/(n_1-1)} \) provided that the sum \( C_{\text{right},1} \) of all 2-dimensional principal minors of \( \Phi_1 \) is not zero. Furthermore, these \( 2n_1 - 2 \) perturbed eigenvalues can be written in the form

\[
\hat{\lambda}(\varepsilon) = \left( C^\pm \right)^{1/(n_1-1)} \varepsilon^{1/(n_1-1)} + o\left( \varepsilon^{1/(n_1-1)} \right),
\]

where \( C^\pm \) denotes either of the two constants \( C^+ \) and \( C^- \) given by

\[
C^\pm = \frac{C_{\text{left},1} \pm \sqrt{C_{\text{left},1}^2 - 4C_{\text{right},1}}}{2}
\]

for \( C_{\text{left},1} \) as in (25).

(ii) If \( q > 1 \) and \( n_2 = n_1 - 1 \), the smallest slope in the Newton diagram associated with the zero eigenvalue is \( 1/(n_1-1) \), corresponding to the segment joining the origin with \( P_{\text{left}} = (n_1 - 1, 1) = (n_2, 1) \) in the Newton diagram, provided that

\[
C_{\text{left},2} = -(\Phi_{12} + \Phi_{21} + \text{trace}(\Phi_2)) \neq 0.
\]

In that case, the perturbed matrix \( A + \varepsilon E \) has exactly \( n_1 - 1 \) eigenvalues of the form

\[
\hat{\lambda}(\varepsilon) = (C_{\text{left},2})^{1/(n_1-1)} \varepsilon^{1/(n_1-1)} + o\left( \varepsilon^{1/(n_1-1)} \right).
\]

**Proof.** Under the conditions in the statement, we know that the structure \( S \) is fully nongeneric for \( \lambda = 0 \) and \( A \), so there cannot be any point on the Newton diagram lying on the segment joining the origin with \( P_1 = (n_1 r_1, r_1) = (n_1, 1) \).

- If \( q = 1 \), i.e., the zero eigenvalue is nonderogatory, there are no points \( P_{\text{right},j} \) in the Newton diagram, and the smallest slope is trivially given by \( P_{\text{left}} \), provided that the corresponding coefficient in the characteristic polynomial is not zero. That coefficient is equal to \( -(\Phi_{12} + \Phi_{21}) \), according to part (i) of Theorem 3.3.

- Now, let \( q > 1 \). Since \( r_1 = 1 \), Lemma 3.1 implies that

\[
m_{\text{left}} \leq m_{\text{right},j} \iff n_1 - n_2 \geq \frac{j + 1}{j}
\]

with equality on the left if and only if there is equality on the right. However, since \( n_1 - n_2 \) is an integer, there can only be equality on the right for \( j = 1 \), i.e., if \( n_1 - n_2 = 2 \). Therefore:

- If \( q > 1 \) and \( n_1 - n_2 > 2 \), then \( m_{\text{left}} < m_{\text{right},j} \) for every \( j \), so the smallest possible slope is \( 1/(n_1 - 1) \), provided that the corresponding coefficient \( C_{\text{left},1} \) in the characteristic polynomial, given again by item (i) in Theorem 3.3, is not zero. Moreover, in that case, the leading term in the perturbation expansion (2) is given by the polynomial equation involving only the terms corresponding to the two endpoints of the segment with the smallest slope \( 1/(n_1 - 1) \) (see [17] for more details), i.e.,

\[
\lambda^n - C_{\text{left},1} \lambda^{n-1} \varepsilon = \lambda^{n-n_1+1} \left( \lambda^{n_1-1} - C_{\text{left},1} \varepsilon \right) = 0.
\]

- If \( q > 1 \) and \( n_1 - n_2 = 2 \), then the three points \( (0, 0) \), \( P_{\text{left}} = (n_1 - 1, 1) \) and \( P_{\text{right},1} = (n_1 + n_2, 2) = (2n_1 - 2, 2) \) are aligned on the Newton diagram, so the smallest slope is \( 2/(2n_1 - 2) = 1/(n_1 - 1) \) again, and there will be \( n_1 + n_2 = 2(n_1 - 1) \) perturbed eigenvalues of order \( \varepsilon^{1/(n_1-1)} \), provided the coefficient \( C_{\text{right},1} \), given by item (iii) in Theorem 3.3, is not zero. Furthermore, in that case, the leading term in the perturbation expansion (2) is given by the polynomial equation involving only the terms corresponding to the three points \( (0, 0) \), \( P_{\text{left}} \), \( P_{\text{right},1} \) on the segment with smallest slope \( 1/(n_1 - 1) \), i.e.,

\[
\lambda^n - C_{\text{left},1} \lambda^{n-1} \varepsilon + C_{\text{right},1} \lambda^{n-n_1-n_2} \varepsilon^2 = 0.
\]
However, if we replace \( n_2 \) with \( n_1 - 2 \), and factor out the lowest power \( \lambda^{n_2-2n_1+2} \) of \( \lambda \), we obtain

\[
\lambda^{2n_1-2} - C_{\text{left},1}\lambda^{n_1-1} + C_{\text{right},1}\epsilon^2 = 0,
\]

which can be reduced to a quadratic equation

\[
z^2 - C_{\text{left},1}\epsilon z + C_{\text{right},1}\epsilon^2 = 0
\]

through the change of variable \( z = \lambda^{n_1-1} \). The coefficients (26) are just the solutions to this quadratic equation.

Finally, if \( q > 1 \) with \( n_2 = n_1 - 1 \), then we know from Lemma 4.1 that all coefficients corresponding to \( P_{\text{right},j} \) are zero. Therefore, in this case, the smallest possible slope is again \( 1/(n_1 - 1) \), provided that the corresponding coefficient

\[
C_{\text{left},2} = -(\Phi_{12} + \Phi_{21} + \text{trace}(\Phi_2))
\]

in the characteristic polynomial, given by item (ii) in Theorem 3.3, is not zero. In principle, there might be further points on the Newton diagram along the straight line through the origin with slope \( 1/(n_1 - 1) \): notice that the segment joining together all points \( P_{\text{right},j} \) is parallel to that line, and one unit away from it, so any point of the form \((n_1 + jn_2 - 1), j + 1\)\) is one unit to the left of \( P_{\text{right},j} \), could be, in principle, on the Newton diagram. On closer inspection, however, one can check that the appearance of these points is precluded by the fact that \( r_1 = 1 \): indeed, consider, for instance, the first of these points, \((n_1 + n_2 - 1, 2) = (2(n_1 - 1), 2)\), which corresponds to the term in \( \lambda^{n_2-2n_1+2}\epsilon^2 \) in the characteristic polynomial: according to Lemma 3.2, the coefficient of that term is the sum of all possible two-dimensional minors of \( \tilde{E} \) obtained by removing from any \( 2n_1 \)-dimensional principal minor of \( \tilde{E} \) exactly \( 2n_1 - 2 \) rows and columns where \( j \) has a supradiagonal 1, with the restriction that these supradiagonal positions must be chosen from at most two Jordan blocks of \( J \). However, since \( r_1 = 1 \), there is no way of choosing \( 2n_1 - 2 \) supradiagonal 1s from less than three blocks, since there is only one Jordan block of size \( n_1 \). Hence, the coefficient of this term is zero, and the point \((2(n_1 - 1), 2)\) is not plotted in the process of constructing the Newton diagram. The same happens with the remaining points \((j + 1)(n_1 - 1), j + 1\).

Therefore, \( P_{\text{left}} \) is the only point from the Newton diagram on the segment of slope \( 1/(n_1 - 1) \) and there are exactly \( n_1 - 1 \) perturbed eigenvalues of order \( \epsilon^{1/(n_1 - 1)} \), and the leading term in the perturbation expansion is obtained from the polynomial equation involving only the terms corresponding to the two endpoints of the segment with the smallest slope, in this case

\[
\lambda^n - C_{\text{left},2}\lambda^{n-n_1+1}\epsilon = 0.
\]

What Theorem 4.2 tells us is that the effect of a fully nongeneric structure on the asymptotic perturbation expansion (2) is always to increase the leading exponent from its generic value of \( 1/n_1 \) to \( 1/(n_1 - 1) \). Furthermore, the leading coefficient no longer depends only on eigenvectors as in the generic case, but also on the second vectors (left and right) in the single longest Jordan chain.

Moreover, the simplicity of the formulas for the leading coefficients in parts (i) and (ii) of Theorem 4.2 allows us to obtain new formulas for the structured condition number whenever \( n_2 \neq n_1 - 2 \) (finding bounds for the case \( n_2 = n_1 - 2 \) seems to be technically much more difficult, and is currently under study). This will ultimately lead to tight bounds on the second component of \( \kappa_S \) in those cases:

**Theorem 4.3.** Let \( M \) be a symmetric, real and orthogonal matrix and let \( S = \{A \in \mathbb{C}^{n \times n} : A^T M = -M A\} \). Let \( A \in S \) have Jordan form (9) with \( \lambda = 0 \), \( r_1 = 1 \) and \( n_1 \) odd, \( n_1 \neq 1 \). Let the scalars \( \Phi_1, \Phi_{12} \) and \( \Phi_{21} \), and the matrix \( \Phi_2 \in \mathbb{C}^{(r_1+1) \times (r_1+1)} \) be as defined in (18) and (19), let \( x_1, x_2 \) be, respectively, the first and second vector in the right Jordan chain of length \( n_1 \) in the Jordan canonical form (9), and let \( Z_1 \) be the matrix collecting all right eigenvectors associated with Jordan blocks of size \( n_2 \) in the Jordan canonical form (9). Then, the structured condition number of the zero eigenvalue of \( A \) is...
Here

\[ \kappa_S(A, 0) = (n_1 - 1, \alpha_S), \]

where

(i) if either \( q = 1 \) in (11), or \( q > 1 \) with \( n_1 - n_2 > 2 \), then

\[
\alpha_S = \sup_{\|E\| \leq 1} \left| \text{trace} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^T & ME \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} \right|; \tag{27}
\]

(ii) if \( q > 1 \) and \( n_2 = n_1 - 1 \), then

\[
\alpha_S = \sup_{\|E\| \leq 1} \left| \text{trace} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^T & ME \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{bmatrix} \right| + \text{trace} (\Phi_2). \tag{28}
\]

**Proof.** (i) We know from Theorem 4.2 that if either \( q = 1 \) in (11), or \( q > 1 \) with \( n_1 - n_2 \geq 2 \), the smallest possible leading exponent in (2) for perturbations in \( S \) is \( 1/(n_1 - 1) \). Furthermore, the second entry of the structured condition number is

\[
\alpha_S = \sup_{\|E\| \leq 1} \left| \Phi_{12} + \Phi_{21} \right|.
\]

Since \( r_1 = 1 \), the matrices \( X_1, X_2, Y_1, Y_2 \) in (12) and (13) are just column vectors, which we denote by \( x_1, x_2, y_1, y_2 \). We may write

\[
\Phi_{12} + \Phi_{21} = y_1^H Ex_2 + y_2^H Ex_1 = \text{trace} \begin{bmatrix} y_1^H \\ y_2^H \end{bmatrix} E \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \tag{29}
\]

Moreover, using the structured canonical forms of families of skew-adjoint matrices [16, Theorem 7.3], one can show that the structure \( \mathcal{S} \) induces the relationship

\[
\begin{bmatrix} y_1^H \\ y_2^H \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} M,
\]

between the vectors in left and right Jordan chains. Our claim follows from replacing this identity in (29).

(ii) If \( n_2 = n_1 - 1 \) then by part (ii) of Theorem 4.2, the smallest possible leading exponent in (2) for perturbations in \( S \) is \( 1/(n_1 - 1) \) and

\[
\alpha_S = \sup_{\|E\| \leq 1} \left| \Phi_{12} + \Phi_{21} + \text{trace}(\Phi_2) \right|. \tag{30}
\]

Similarly to (i) and using the structured canonical forms of families of skew-adjoint matrices [16, Theorem 7.3] and the result [13, Corollary A.4] we can see that

\[
\Phi_{12} + \Phi_{21} + \text{trace}(\Phi_2) = \Phi_{12} + \Phi_{21} + \text{trace}(W_1^H EZ_1)
\]

\[
= \text{trace} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} ME \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} Z_1.
\]

Our claim follows from replacing this identity in (30). □
Finally, Theorem 4.3 allows us to obtain either an explicit formula for the condition number in Frobenius norm, or upper and lower bounds on the structured condition number in spectral norm, which roughly determine the magnitude of the constant \(\alpha_S\) within a factor \(\sqrt{n}\).

Lemma 4.4. Let \(M\) be a symmetric, real and orthogonal matrix and let \(S = \{A \in \mathbb{C}^{n \times n}: A^T M = -M A\}\). Let \(A \in S\) have Jordan form \((9)\) with \(\lambda = 0, r_1 = 1, n_1\) odd, \(n_1 \neq 1\) and \(n_2 \neq n_1 - 2\). Let \(x_1, x_2\) be, respectively, the first and second vector in the right Jordan chain of length \(n_1\) in the Jordan canonical form \((9)\), and let the matrix \(Z_1 \in \mathbb{C}^{n \times n}\) be as defined in \((14)\). Let

\[
\kappa_S(A, 0) = (n_S, \alpha_S)
\]

be the structured condition number associated with the zero eigenvalue, either in the Frobenius or in the spectral matrix norm. Let the matrix \(K \in \mathbb{C}^{n \times n}\) be defined as

\[
(i) \quad K = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} \text{ if } q = 1 \text{ in } (11), \text{ or } q > 1 \text{ and } n_1 - n_2 > 2,
\]

\[
(ii) \quad K = \begin{bmatrix} x_1 & x_2 \end{bmatrix} Z_1 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \\ Z_1^T \end{bmatrix} \text{ if } q > 1 \text{ in } (11) \text{ and } n_2 = n_1 - 1.
\]

Then, \(\alpha_S = \|K\|_F\) if the condition number is taken with respect to the Frobenius norm, or

\[
\frac{\|K\|_F^2}{\|K\|} \leq \alpha_S \leq \sqrt{n}\|K\|_F
\]

if the condition number is taken with respect to the spectral norm \(\|\cdot\|\).

Proof. (i) If \(q = 1\) in \((11)\), or \(q > 1\) and \(n_1 - n_2 \geq 2\), we know from Theorem 4.3 that \(\alpha_S\) can be written as \((27)\). Since the trace is invariant under cyclic permutations,

\[
\text{trace} \left( \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} ME \begin{bmatrix} x_1 & x_2 \end{bmatrix} \right) = \text{trace} \left( \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} ME \right)
\]

which is just \(\text{trace}(KME)\). We now apply the Cauchy–Schwarz inequality

\[
|\text{trace}(KME)| \leq \|K\|_F \|ME\|_F
\]

for the matrix inner product to obtain

\[
\alpha_S = \sup_{\|E\|_F \leq 1} \text{trace}(KME) \leq \|K\|_F \sup_{\|E\|_F \leq 1} \|ME\|_F.
\]

- If \(E\) is such that \(\|E\|_F \leq 1\), then \(\|ME\|_F \leq 1\) due to the unitary invariance of the Frobenius norm, and \(\alpha_S \leq \|K\|_F\). The reverse inequality is obtained from taking the perturbation matrix \(E = 1/\|K\|_F M\tilde{K}\), which belongs to \(S\) and satisfies \(\|E\|_F = 1\).

- If \(E\) is such that \(\|E\| \leq 1\) for the spectral norm, then \(\|ME\|_F \leq \sqrt{n}\|ME\| \leq \sqrt{n}\) due to unitary invariance, so \(\alpha_S \leq \sqrt{n}\|K\|_F\). The reverse inequality is obtained from taking the perturbation matrix \(E = 1/\|K\|_F M\tilde{K}\), which belongs to \(S\) and satisfies \(\|E\| = 1\).

(ii) Analogous to (i). \(\square\)

As an illustration, consider the structure \(S = \{A \in \mathbb{C}^{3 \times 3}: A\Sigma_3 = -\Sigma_3 A^T\}\) for the real symmetric orthogonal matrix

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\[ \Sigma_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

and let

\[ A = \begin{bmatrix} \sqrt{2} & i & 0 \\ i & 0 & i \\ 0 & i & -\sqrt{2} \end{bmatrix} \in S, \]

which has a triple eigenvalue \( \lambda = 0 \) with a single Jordan block of size 3, i.e., we are in the case \( q = 1 \).

One can easily check that the unstructured condition number of \( \lambda = 0 \) is
\[ \kappa(A, 0) = (3, 4) \]
for both the spectral and the Frobenius norms. It turns out that any matrix in \( S \) is of the form

\[ E = \begin{bmatrix} x & y & 0 \\ z & 0 & y \\ 0 & z & -x \end{bmatrix}, \quad x, y, z \in \mathbb{C}, \]

and any perturbation \( A + \epsilon E \) with \( E \in S \) has one simple zero eigenvalue, and two nonzero perturbed eigenvalues \( \hat{\lambda}_k(\epsilon) \) of the form
\[ \hat{\lambda}_k(\epsilon) = (2iz + 2iy + 2\sqrt{2}x)^{1/2} \epsilon^{1/2} + o(\epsilon^{1/2}), \quad k = 1, 2. \]

Therefore, \( n_S = n_1 - 1 = 2 \) and the structured condition number for the Frobenius norm is
\[ \kappa_S(A, 0) = \left(2, \sup_{\|E\|_F \leq 1} |2iz + 2iy + 2\sqrt{2}x| \right). \]

One can check via Lagrange multipliers that the supremum above is equal to \( 2\sqrt{2} \), so
\[ \kappa_S(A, 0) = (2, 2\sqrt{2}). \]

On the other hand, one can check that \( x_1 = [1, \sqrt{2}i, -1]^T, \quad x_2 = [\sqrt{2}, i, 0]^T, \quad so \)
\[ K = \begin{bmatrix} 1 & \sqrt{2}i & 0 \\ \sqrt{2}i & i & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i & -\sqrt{2} \\ -i & 0 & -1 \\ \sqrt{2} & i & 0 \end{bmatrix}, \]

whose Frobenius norm is equal to \( 2\sqrt{2} \), as expected.

5. Extension to structured pencils

Most of the results in the previous section can be translated to the context of matrix pencils due to the coincidence of structured canonical forms between certain classes of pencils and certain classes of matrices: let \( A - \lambda B \) be a square and regular pencil, i.e. the matrices \( A \) and \( B \) are square and \( \det(A - \lambda B) \) is not identically zero. In this section, we denote the regular pencil \( A - \lambda B \) by \((A, B)\). For a finite eigenvalue \( \lambda \) of \((A, B)\), the Kronecker–Weierstraß form implies
\[ \begin{bmatrix} \frac{J}{0} & 0 \\ 0 & \frac{J_A}{} \end{bmatrix} = \begin{bmatrix} \frac{Q}{O} \quad A \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \frac{J_B}{} \end{bmatrix} = \begin{bmatrix} \frac{Q}{O} \quad B \end{bmatrix} \begin{bmatrix} \frac{P}{O} & \frac{\tilde{P}}{} \end{bmatrix} \]

where \( \begin{bmatrix} \frac{Q}{O} \end{bmatrix} \) and \( \begin{bmatrix} \frac{P}{O} & \frac{\tilde{P}}{} \end{bmatrix} \) are invertible and \( J \) contains all Jordan blocks associated with \( \lambda \). Let
\[ J = \text{Diag}(\Gamma_1^1, \ldots, \Gamma_r^1, \ldots, \Gamma_q^1, \ldots, \Gamma_q^{q_r}), \]

where, for \( j = 1, \ldots, q \), each \( \Gamma_j^k, k = 1, \ldots, r_j \), is a Jordan block of dimension \( n_j \), ordered so that
\[ n_1 > n_2 > \cdots > n_q. \]

Similarly, for an infinite eigenvalue of \((A, B)\), we have
\[
\begin{bmatrix}
1 & 0 \\
0 & \bar{J}_A
\end{bmatrix} = \begin{bmatrix}
\frac{Q}{Q} \\
0
\end{bmatrix} \begin{bmatrix}
\bar{P}
\end{bmatrix}.
\]
\[
\begin{bmatrix}
N & 0 \\
0 & \bar{J}_B
\end{bmatrix} = \begin{bmatrix}
\frac{Q}{Q} \\
0
\end{bmatrix} \begin{bmatrix}
\bar{P}
\end{bmatrix}.
\]

where \(N\) contains all the \(r_1 + \cdots + r_q\) nilpotent blocks of nilpotency indices \(n_1 > n_2 > \cdots > n_q\). We collect the first and second generalized right and left eigenvectors contained in \(P\) and \(Q\) associated with the block corresponding to Jordan blocks of sizes \(n_1\):

\[
X_1 = \left[ Pe_1, Pe_{n_1+1}, \ldots, Pe_{(r_1-1)n_1+1} \right] \in \mathbb{C}^{n \times r_1},
\]
\[
Y_1 = \left[ Q^H e_1, Q^H e_{n_1}, \ldots, Q^H e_{r_1n_1} \right] \in \mathbb{C}^{n \times r_1},
\]
\[
X_2 = \left[ Pe_2, Pe_{n_1+2}, \ldots, Pe_{(r_1-1)n_1+2} \right] \in \mathbb{C}^{n \times r_1},
\]
\[
Y_2 = \left[ Q^H e_{n_1+1}, Q^H e_{2n_1-1}, \ldots, Q^H e_{r_1n_1-1} \right] \in \mathbb{C}^{n \times r_1}.
\]

Finally, we collect the right and left eigenvectors associated with the Jordan blocks of sizes \(n_2\) in
\[
Z_1 = \left[ Pe_{r_1n_1+1}, Pe_{r_1n_1+n_2+1}, \ldots, Pe_{r_1n_1+(r_2-1)n_2+1} \right] \in \mathbb{C}^{n \times r_2},
\]
\[
W_1 = \left[ Q^H e_{r_1n_1+n_2}, Q^H e_{r_1n_1+2n_2}, \ldots, Q^H e_{r_1n_1+r_2n_2} \right] \in \mathbb{C}^{n \times r_2},
\]

where, for every index \(k\), the vector \(e_k\) is the \(k\)-th column of the identity matrix. Therefore, the matrices \(X_1\) and \(Y_1\) contain, respectively, the right and left eigenvectors associated with all Jordan blocks of size \(n_1\) corresponding to \(\lambda\), while \(Z_1\) and \(W_1\) contain, respectively, the right and left eigenvectors associated with all Jordan blocks of size \(n_2\).

It is well known that perturbed eigenvalues of pencils are also given by fractional expansions, which are either of the form (2) for finite eigenvalues, or of the form
\[
\frac{1}{\lambda_k} = \xi_k^{1/n_k} \varepsilon^{1/n_k} + o(\varepsilon^{1/n_k}), \quad k = 1, \ldots, m,
\]
for infinite ones. The following theorem summarizes results taken from [6], describing the first order expansions of perturbed eigenvalues for the generical situation when the perturbations to the pencil are arbitrary:

**Theorem 5.1.** Let \(\lambda\) be a finite eigenvalue of a regular matrix pencil \((A, B)\) with Kronecker form (32) and let \((E, F) \in \mathbb{C}^{n \times n} \times \mathbb{C}^{n \times n}\) be such that \(Y^H(E - \lambda F)X_1\) is invertible, where \(X_1\) and \(Y_1\) are defined as in (35) and (36). Then there are \(n_1, r_1\) eigenvalues \(\xi_k\) of the perturbed pencil \((A + \varepsilon E, B + \varepsilon F)\) admitting a perturbation expansion
\[
\hat{\lambda}_k = \lambda + (\xi_k)^{1/n_1} \varepsilon^{1/n_1} + o(\varepsilon^{1/n_1}), \quad k = 1, \ldots, r_1,
\]
where \(\xi_1, \ldots, \xi_{r_1}\) are the eigenvalues of \(Y^H(E - \lambda F)X_1\). For an infinite eigenvalue of \((A, B)\) with Kronecker form (34), let \(F \in \mathbb{C}^{n \times n}\) be such that \(Y^HFX_1\) is invertible. Then there are \(n_1, r_1\) eigenvalues \(\hat{\lambda}_k\) of the perturbed pencil \((A + \varepsilon E, B + \varepsilon F)\) admitting a perturbation expansion
\[
\frac{1}{\hat{\lambda}_k} = (\xi_k)^{1/n_1} \varepsilon^{1/n_1} + o(\varepsilon^{1/n_1}), \quad k = 1, \ldots, r_1,
\]
where the \(\xi_k\) are the eigenvalues of \(Y^HFX_1\).
Theorem 5.1 describes the generic behavior of a (finite or infinite) eigenvalue of the pencil \((A, B)\) under arbitrary perturbations. However, if the pencil \((A, B)\) has a certain particular structure, the behavior of the perturbation expansions above may be very different when the pencil is subject to perturbations which preserve that same structure. Thus, it makes sense to define Hölder-type structured condition numbers to account for that possible disparity in sensitivities: in view of (42) and (43), we may define the unstructured Hölder condition number of \(\lambda\) as an eigenvalue of the pencil \((A, B)\) as

\[
\kappa(A, B, \lambda) = (n_1, \alpha_1),
\]

where, as before, \(n_1\) is the largest size of Jordan blocks associated with \(\lambda\), and \(\alpha_1^{1/n_1} > 0\) is the largest possible magnitude of the coefficient of \(\varepsilon^{1/n_1}\) either in (42) or in (43) for conveniently normalized pencil perturbations (see [13, §4.1.1] for a more formal definition and further details on the normalization).

Our next result provides explicit first order expansions of the perturbed eigenvalues for symmetric/skew-symmetric and palindromic. More specifically:

Case 1: Symmetric/skew-symmetric pencils: Let \(S_1 = \{A \in \mathbb{C}^{n \times n} | MA = A^T M\}\) and \(S_2 = \{B \in \mathbb{C}^{n \times n} | MB = -B^T M\}\), for a real symmetric and orthogonal matrix \(M\). If \(\lambda = \infty\) is an eigenvalue of \((A, B) \in S_1 \times S_2\) with \(n_1\) odd and \(r_1 = 1\), then \(n_2 < n_1\).

Case 2: Palindromic pencils: Let \(S = \{(A, A^T) : A \in \mathbb{C}^{n \times n}\}\). If \(\lambda = 1\) is an eigenvalue of \((A, A^T) \in S\) with \(n_1\) odd and \(r_1 = 1\), then \(n_2 < n_1\).

Our next result provides explicit first order expansions of the perturbed eigenvalues for symmetric/skew-symmetric pencils:

**Theorem 5.2.** Let \(M\) be a real symmetric and orthogonal matrix and let \(S = S_1 \times S_2\), where \(S_1 = \{A \in \mathbb{C}^{n \times n} | MA = A^T M\}\) and \(S_2 = \{B \in \mathbb{C}^{n \times n} | MB = -B^T M\}\). Let \(\lambda = \infty\) be an eigenvalue of \((A, B) \in S\) with canonical form (34) such that its largest nilpotency index \(n_1\) is odd, and corresponds to a single Jordan block (i.e., \(r_1 = 1\)). Let \(q\) be the number of different nilpotency indices of the infinite eigenvalue, let \((E, F) \in S\) and let \(x_1, x_2\) and \(Z_1\) be the matrices defined in (35), (37) and (39). Then, there are at least \(n_1 - 1\) eigenvalues \(\lambda_k\) of order \(\varepsilon^{1/(n_1-1)}\) of the perturbed pencil \((A + \varepsilon E, B + \varepsilon F)\). Moreover, they can be written as

\[
\frac{1}{\lambda_k} = \xi^{1/n_1 - 1} \varepsilon^{1/n_1 - 1} + o(\varepsilon^{1/n_1 - 1}),
\]

where

(i) if \(q = 1\), or \(q > 1\) with \(n_1 - n_2 > 2\), then

\[
\xi = C_{\text{left},1} = -\text{trace}(G),
\]

where \(G\) is the \(2 \times 2\) matrix

\[
G = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} M F \begin{bmatrix} x_1 & x_2 \end{bmatrix};
\]
(ii) if \( q > 1 \) and \( n_1 - n_2 = 2 \), then \( \xi \) is either of the two roots \( C^+ \), \( C^- \) of the quadratic equation \( z^2 - C_{\text{left},1} z + C_{\text{right},1} = 0 \), where \( C_{\text{left},1} \) is as in (45) above, and \( C_{\text{right},1} \) is the sum of all two-dimensional principal minors of

\[
\Phi_2 = \begin{bmatrix} x_1^T \\ Z_1^T \end{bmatrix} M F \begin{bmatrix} x_1 & Z_1 \end{bmatrix} \in \mathbb{C}^{(r_2+1) \times (r_2+1)}
\]

containing its (1, 1) entry;

(iii) if \( q > 1 \) and \( n_1 - n_2 = 1 \), then \( \xi \) is minus the trace of the \( (r_2 + 2) \times (r_2 + 2) \) matrix

\[
G = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{r_2} \\
0 & 0 & I_{r_2} & 0
\end{bmatrix} \begin{bmatrix}
x_1^T \\
x_2^T \\
Z_1^T
\end{bmatrix}
\]

\[\times \begin{bmatrix} x_1 & x_2 & Z_1 \end{bmatrix} \in \mathbb{C}^{(r_2+1) \times (r_2+1)}.
\]

**Proof.** The structured canonical form of a pencil \( (A, B) \in \mathbb{S}_1 \times \mathbb{S}_2 \) for an infinite eigenvalue imposes the same structure on the matrices \( x_1, y_1, x_2, y_2, W_1 \) and \( Z_1 \) defined in (35)–(40) as on the matrices corresponding to a zero eigenvalue of a matrix \( B \in \mathbb{S}_2 \), see [13, Corollary A.4]. Hence, the first order expansion of an infinite eigenvalue of a pencil \( (A, B) \in \mathbb{S}_1 \times \mathbb{S}_2 \) is the same we would obtain for a zero eigenvalue of the matrix \( B \in \mathbb{S}_2 \), perturbed by \( F \in \mathbb{S}_2 \) under fully nongeneric conditions. Therefore, the argument in the proof Theorem 4.2 can be used in the same way for the infinite eigenvalue of fully nongeneric symmetric/skew-symmetric pencil structure. \( \square \)

The same can be done for palindromic pencils:

**Theorem 5.3.** Let \( \lambda = 1 \) be an eigenvalue of a palindromic matrix pencil \((A, A^T)\) with canonical form (32), and suppose there is only one Jordan block of odd largest size \( n_1 \) among the Jordan blocks associated with 1. Let \( q \) be the number of different sizes of Jordan blocks associated with \( \lambda = 1 \), let \((E, E^T) \in \mathbb{S}\) and let \( x_1, x_2 \) and \( Z_1 \) be the matrices defined in (35), (37) and (39). Then, there are at least \( n_1 - 1 \) eigenvalues \( \hat{\lambda}_k \) of order \( \varepsilon^{1/(n_1-1)} \) of the perturbed pencil \((A + \varepsilon E, A^T + \varepsilon E^T)\). Moreover, they can be written as

\[
\hat{\lambda}_k = 1 + \xi \varepsilon^{1/(n_1-1)} + o(\varepsilon^{1/(n_1-1)}), \quad k = 1, \ldots, n_1 - 1,
\]

where

(i) if \( q = 1 \) or \( q > 1 \) with \( n_1 - n_2 > 2 \), then

\[
\xi = C_{\text{left},1} = - \text{trace}(G),
\]

where \( G \) is the \( 2 \times 2 \) matrix

\[
G = \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix} \begin{bmatrix} x_1^T \\ x_2^T \end{bmatrix} (E - E^T) \begin{bmatrix} x_1 & x_2 \end{bmatrix};
\]

(ii) if \( q > 1 \) and \( n_1 - n_2 = 2 \), then \( \xi \) is either of the two roots \( C^+ \), \( C^- \) of the quadratic equation \( z^2 - C_{\text{left},1} z + C_{\text{right},1} = 0 \), where \( C_{\text{left},1} \) is as in (47) above, and \( C_{\text{right},1} \) is the sum of all two-dimensional principal minors of

\[
\Phi_2 = \begin{bmatrix} x_1^T \\ Z_1^T \end{bmatrix} (E - E^T) \begin{bmatrix} x_1 & Z_1 \end{bmatrix} \in \mathbb{C}^{(r_2+1) \times (r_2+1)}
\]

containing its (1, 1) entry;
(iii) the trace of the matrix

\[
G = \begin{bmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -I_{\frac{n}{2}} \\
0 & 0 & I_{\frac{n}{2}} & 0
\end{bmatrix}
\begin{bmatrix}
x^T_1 \\
x^T_2 \\
Z^T_1
\end{bmatrix}
\begin{bmatrix}
E - E^T
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
Z_1
\end{bmatrix}
\]

if \(n_1 - n_2 = 1\).

**Proof.** This result is a direct consequence of Theorem 5.2, since the structured canonical form for the eigenvalue 1 of a palindromic pencil \((A, A^T)\) can be extracted from the structured canonical form for the infinite eigenvalue of the associated symmetric/skew-symmetric pencil \((A + A^T, A - A^T)\). □

Finally, one can also obtain formulas for the corresponding structured condition numbers when \(n_2 - n_1 \neq 1\), as well as bounds on them, just as in Section 4:

**Theorem 5.4.**

(i) Let \(M\) be a real symmetric and orthogonal matrix and let \(S = S_1 \times S_2\), where \(S_1 = \{ A \in \mathbb{C}^{n \times n} \mid MA = A^T M \}\) and \(S_2 = \{ B \in \mathbb{C}^{n \times n} \mid MB = -B^T M \}\). Let \(\lambda = \infty\) be an eigenvalue of \((A, B) \in S\) with canonical form (34) such that its largest nilpotency index \(n_1\) is odd, and corresponds to a single Jordan block (i.e., \(r_1 = 1\)). Let \(q\) be the number of different nilpotency indices of the infinite eigenvalue, let \((E, F) \in S\) and let \(x_1, x_2\) and \(Z_1\) be the matrices defined in (35), (37) and (39). Then, the structured condition number of \(\lambda = \infty\) is \(\kappa_S(A, B, \infty) = (n_1 - 1, \alpha_S)\), where

(i.1) if either \(q = 1\), or \(q > 1\) with \(n_1 - n_2 > 2\), then

\[
\alpha_S = \sup_{\|F\| \leq 1 \atop F \in S_2} \left| \text{trace} \left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x^T_1 \\ x^T_2 \end{bmatrix} \right] \right|.
\]

(i.2) if \(q > 1\) with \(n_1 - n_2 = 1\), then

\[
\alpha_S = \sup_{\|F\| \leq 1 \atop F \in S_2} \left| \text{trace} \left[ \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{\frac{n}{2}} \\ 0 & 0 & I_{\frac{n}{2}} & 0 \end{bmatrix} \begin{bmatrix} x^T_1 \\ x^T_2 \\ Z^T_1 \end{bmatrix} \right] \right|.
\]

(ii) Let \(\lambda = 1\) be an eigenvalue of a palindromic matrix pencil \((A, A^T)\) with canonical form (32), and suppose there is only one Jordan block of odd largest size \(n_1\) among the Jordan blocks associated with 1. Let \(q\) be the number of different sizes of Jordan blocks associated with \(\lambda = 1\), let \((E, F) \in S\) and let \(x_1, x_2\) and \(Z_1\) be the matrices defined in (35), (37) and (39). Then, the structured condition number of \(\lambda = 1\) is \(\kappa(A, B, 1) = (n_1 - 1, \alpha_S)\) where

(ii.1) if either \(q = 1\) or \(q > 1\) with \(n_1 - n_2 > 2\), then

\[
\alpha_S = \sup_{\|E\| \leq 1 \atop E \in \mathbb{C}^{n \times n}} \left| \text{trace} \left[ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x^T_1 \\ x^T_2 \end{bmatrix} \right] \right|.
\]

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if \( n_1 - n_2 = 1 \), then

\[
\alpha_S = \sup_{\|E\| \leq 1} \text{trace} \left( \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{r_2}^Z \\ 0 & 0 & I_{r_2}^Z & 0 \end{bmatrix} \begin{bmatrix} X_1^T \\ X_2^T \\ Z_1^T \\ Z_2 \end{bmatrix} (E - E^T) \begin{bmatrix} X_1 & X_2 & Z_1 \end{bmatrix} \right).
\]

**Proof.** If we use the first order expansions (44) and (46) obtained in Theorems 5.2 and 5.3, respectively, then the formulas above for the condition number of infinite eigenvalues of symmetric/skew-symmetric pencils and for the condition number of the eigenvalue 1 of palindromic pencils are immediate. 

**Remark 5.5.** If we denote by \( \alpha_S \) the second component of the structured condition number in the Frobenius norm, either for \( \lambda = \infty \) in the symmetric/skew-symmetric case, or for \( \lambda = 1 \) in the palindromic one, it is easy to prove that

\[
\alpha_S = \|N\|_F,
\]

where the matrix \( N \) is defined as

(i) \( N = \begin{bmatrix} x_1 & x_2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T \) if either \( q = 1 \), or \( q > 1 \) with \( n_1 - n_2 > 2 \).

(ii) \( N = \begin{bmatrix} x_1 & x_2 \\ Z_1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{r_2}^Z \\ 0 & 0 & I_{r_2}^Z & 0 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & Z_1 \end{bmatrix}^T \) if \( n_1 - n_2 = 1 \).

Alternatively,

\[
\frac{\|N\|_F^2}{\|N\|} \leq \alpha_S \leq \sqrt{n} \|N\|_F
\]

if the condition number is computed in the spectral norm.

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**References**


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