Orthogonally additive polynomials and applications

Polinomios ortogonalmente aditivos y aplicaciones

Pablo Linares Briones

Directores de Tesis:

Alberto Ibort Latre
José Luis González Llavona

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Departamento de Análisis Matemático.
Universidad Complutense de Madrid.

Contents

Resumen.

En toda teoría de operadores, tanto lineales como no lineales, es fundamental disponer de teoremas de representación que nos permitan el manejo de los mismos. Un buen ejemplo de este hecho lo encontramos en el marco del Análisis Funcional lineal con el Teorema de Riesz para la representación integral de los funcionales lineales y continuos sobre un espacio $C(K)$.

En los últimos años han sido muchos los matemáticos interesados en el estudio de operadores no lineales entre espacios de Banach, en particular de polinomios. De nuevo, siempre es deseable disponer de teoremas de representación que nos permitan estudiar su comportamiento y propiedades. Recientemente, Benyamini, Lassalle y Llavona obtienen en [BLLl] un teorema de representación para la clase de polinomios ortogonalmente aditivos sobre retículos de Banach. La potencia de este teorema radica en que todo polinomio n-homogéneo de esta clase se representa a través de un funcional lineal y continuo sobre la n-concavificación del espacio, es decir, su estudio se reduce a un problema lineal. Este es el punto de partida de esta tesis doctoral.

Siguiendo el enfoque mencionado, hemos obtenido numerosas aplicaciones a campos tan diversos como la teoría espectral de operadores en espacios de Hilbert, resultados de estructura de retículos de Hilbert, desarrollo de una teoría similar a la de polinomios ortogonales pero aplicada a una clase más amplia de funciones poniendo especial atención en espacios $C[0,1]$ y L^p. Finalmente, hemos desarrollado la teoría clásica de momentos de Hausdorff, tanto en el caso real del intervalo $[0,1]$ como en el caso trigonométrico del círculo unidad $\mathbb{T}$, estudiando el problema multilineal.

El capítulo 1 se dedica a notaciones y conceptos preliminares.

En el capítulo 2 introducimos los polinomios ortogonalmente aditivos sobre retículos de Banach. Sundaresan fue el primer matemático que estudió la representación de polinomios ortogonalmente aditivos en [S2]. Sus resul-

tados se centran en los espacios ℓ_p y L^p de funciones. Recientemente, como ya hemos indicado, Benyamini, Lassalle y Llavona prueban el caso general en [BLLl]. Han aparecido otros trabajos para el caso $C(K)$ (ver [PGV] y [CLZ1]) y para C^*-álgebras [PPV].

En este capítulo presentamos una nueva prueba del teorema de representación para el caso ℓ_p que aparece en [ILLl3]:

Teorema 1. *Sea $1 \leq p < \infty$. El espacio de polinomios ortogonalmente aditivos, n-homogéneos $\mathcal{P}_o(^n\ell_p)$ es isométricamente isomorfo a ℓ_∞ para $1 \leq p \leq n$ y a $\ell_{p/p-n}$ para $n < p < \infty$.*

Nuestra prueba es totalmente independiente de las anteriores y usa la teoría de productos tensoriales.

Exploramos además las aplicaciones del teorema de representación en retículos de Hilbert probando el siguiente resultado:

Teorema 2. *Sea H un espacio de Hilbert real y separable. Si H es un retículo de Banach con orden $\leq$. Entonces, H es orden isométrico a $L^2(\mu)$ para cierta medida μ.*

Este teorema puede verse como un caso particular del clásico teorema de representación de Kakutani para espacios L^p abstractos.

Finalmente encontramos otra aplicación del teorema de polinomios ortogonalmente aditivos que nos permite obtener una nueva prueba del teorema espectral de operadores autoadjuntos en espacios de Hilbert. Este estudio aparece en [ILL14].

El capítulo 3 se centra en el estudio de polinomios ortogonalmente aditivos en espacios de Riesz.

Siguiendo las ideas del estudio de la ortogonalidad aditiva mediante los productos tensoriales, R. Ryan nos indicó la dirección del estudio de la ortogonalidad en espacios de Riesz. Dedicamos un capítulo a la extensión de la ortogonalidad aditiva en este tipo de espacios. Relacionaremos nuestro concepto con el de ortosimetría, ya existente en la literatura, y probaremos el siguiente teorema de representación donde la clave deja de ser la continuidad:

Teorema 3. *Sean E, F espacios de Riesz Arquimedianos con F uniformemente completo y $(\bigodot_n E, \odot_n)$ la n-potencia de E. El espacio $\mathcal{P}_o^+(^nE, F)$ de polinomios n-homogéneos, positivos y ortogonalmente aditivos en E es Riesz isomorfo al espacio $\mathcal{L}^+(\bigodot_n E, F)$ de aplicaciones lineales y positivas de $\bigodot_n E$ a F.*

En el capítulo 4 extendemos la teoría clásica de polinomios ortogonales a una nueva clase de funciones, que contiene a los polinomios, que denominamos polinomios P-ortogonales. En la teoría clásica de polinomios ortogonales, la ortogonalidad se define con respecto al funcional lineal llamado de momentos. La identificación entre los polinomios ortogonalmente aditivos n-homogéneos P y las formas lineales T_P sobre la n-concavificación, señalada anteriormente, nos permite definir una nueva ortogonalidad con respecto al polinomio P, usando la ortogonalidad clásica respecto a T_P. Esta idea simple se manifiesta especialmente interesante en el caso de los espacios $C[0,1]$ y L^p. Extendemos a este contexto todos los resultados clásicos de la teoría como son el Teorema de Favard, los resultados sobre relaciones de recurrencia a tres términos, teoremas de existencia, etc.

Recientemente, entre los especialistas de la teoría de polinomios ortogonales, se le está concediendo especial relevancia a ortogonalidad con respecto a una forma bilineal. En particular, se ha estudiado los polinomios ortogonales de Sobolev, polinomios ortogonales con respecto al producto escalar de Sobolev:

$$(f,g)_S = \sum_{j=0}^{k} \int f^{(j)} g^{(j)} \mathrm{d}\mu_j$$

donde μ_j son medidas de Borel positivas y las derivadas se consideran en sentido débil (véase por ejemplo [MAR], [MMB], [MF1], [MF2], [M] y sus referencias).

Usando un potente teorema de Boas [B], Durán prueba en [Dur2] un teorema que tiene aplicación a la teoría de polinomios ortogonales y muestra cuando la ortogonalidad con respecto a una forma bilineal se puede reducir a la ortogonalidad clásica. Para terminar el capítulo 4 extendemos este resultado a productos escalares definidos en polinomios algebraicos sobre $\mathbb{R}$ obteniendo el siguiente teorema:

Teorema 4. *Sea B un producto escalar definido en el espacio de polinomios algebraicos reales sobre $\mathbb{R}$. Son equivalentes:*

- *El operador multiplicación por t es simétrico y acotado con respecto de B.*

- *Existe una medida de Borel regular en un compacto de $\mathbb{R}$ tal que*

$$B(p,q) = \int p(t)q(t)\mathrm{d}\mu(t).$$

Por último, dedicamos dos capítulos al problema multilineal de momentos. Cada capítulo extiende el problema de momentos clásico al caso multilineal. Estudiamos dichos problemas para el caso de Hausdorff, es decir cuando los polinomios están definidos en $I = [0,1]$ y también para el caso de polinomios trigonométricos definidos en el círculo unidad $\mathbb{T}$.

El funcional de momentos L asociado a una sucesión μ_k, $k \in \mathbb{N}$ de números reales, es el elemento del dual (algebraico) del espacio de polinomios $\mathbb{R}[t]$ definido por:

$$L(p) = \sum_{k \geq 0} p_k \mu_k,$$

donde $p(t) = \sum_{k \geq 0} p_k t^k \in \mathbb{R}[t]$ es un polinomio arbitrario. Dado un intervalo $I \subset \mathbb{R}$ el problema clásico de momentos se traduce en encontrar la integralidad de dicho operador, es decir, bajo que condiciones, existe una medida regular μ en $I \subset \mathbb{R}$ tal que $L(t^k) = \int_I t^k d\mu(t)$, $k = 0, 1 \ldots$. Si $I = [0,1]$ el problema de momentos se conoce como problema de momentos (clásico) de Hausdorff.

La solución del problema clásico de Hausdorff (ver por ejemplo [ST] y sus referencias) establece que la medida μ existe si para cierta constante C:

$$\sum_{m=0}^{k} |\lambda_{(k,n)}| < C,$$

para todo $k = 0, 1, \ldots$, y $0 \leq m \leq k$, donde

$$\lambda_{(k,m)} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m, \quad 0 \leq m \leq k,$$

y Δ denota el operador:

$$\Delta \mu_n = \mu_n - \mu_{n-1},$$

que cumple:

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m}.$$

La solución original de este problema, basada en propiedades de compacidad del conjunto de funciones de variación acotada, fue utilizada para obtener varias versiones del teorema de Riesz (ver por ejemplo [Hil2]). La extensión de este problema al caso de varias variables fue resuelta por Hildebrant y Shoenberg y por Haviland en los años treinta [Hil2, Hav1, Hav2] bajo la suposición general de positividad del funcional de momentos L. Recíprocamente, el teorema de Riesz se utilizó para dar una prueba alternativa del problema de momentos de Hausdorff [Hil1].

Estos resultados sugieren una extensión natural del problema de momentos al caso multilinear. Sea $\mathbf{k} = (k_1, \dots, k_n)$ un multiíndice no negativo de longitud n, es decir, $k_1, \dots, k_n = 0, 1, \dots$. Una familia $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^n$ de reales, se denomina una sucesión de multimomentos o simplemente multimomentos. Denotaremos por L al funcional n-lineal definido por los multimomentos $\mu_{\mathbf{k}}$ en el espacio de polinomios $\mathbb{R}[t]$ mediante la formula:

$$L(t^{k_1}, \dots, t^{k_n}) = \mu_{k_1,\dots,k_n} \quad \forall k_1, \dots, k_n = 0, 1, \dots.$$

En este contexto, el problema clásico de momentos admite dos formulaciones. La primera sería, dada una sucesión de multimomentos $\mu_{\mathbf{k}}$, encontrar condiciones necesarias y suficientes para la existencia de una medida de Borel regular μ en $[0, 1]$ tal que

$$L(p_1(t), \dots, p_n(t)) = \int_0^1 p_1(t) \cdots p_n(t) d\mu(t), \quad \forall p_1(t), \dots, p_n(t) \in \mathbb{R}[t].$$

Este problema lo hemos denominado problema (multilineal) fuerte de Hausdorff.

La segunda, más débil que la anterior, que denominamos problema multilineal clásico de momentos de Hausdorff por su similitud con el caso lineal, consiste en encontrar una medida μ en $[0, 1]^n \subset \mathbb{R}^n$ tal que:

$$L(p_1(t_1), \dots, p_n(t_n)) = \int_{I^n} p_1(t_1) \cdots p_n(t_n) d\mu(t_1, \dots, t_n).$$

Ejemplos particulares de este problema consisten en la caracterización de funcionales bilineales que se pueden representar bien por una medida en $[0, 1]$ o por una medida en $[0, 1] \times [0, 1]$. Resultados parciales en esta dirección se obtienen mediante funcionales bilineales semiclásicos en términos de polinomios biortogonales, en relación con la teoría de matrices aleatorias y sistemas integrables (ver por ejemplo [Ber], [BlK], [AMV]).

Recientemente, se demostró [BV1] que el espacio de funcionales continuos n-lineales en el espacio de funciones continuas en un compacto está en correspondencia biunívoca con el espacio de polimedidas de Borel regular numerablemente aditivas. Esto nos llevó a considerar una versión más débil del problema de momentos multilineal clásico de Hausdorff. Si $\mu_{\mathbf{k}}$ es una sucesión de multimomentos y L su funcional de momentos n-lineal asociado, diremos que $\mu_{\mathbf{k}}$ satisface el problema (multilineal) débil de momentos de Hausdorff si existe una polimedida γ en $\mathrm{Bo}[0,1] \times \cdots \times \mathrm{Bo}[0,1]$ tal que:

$$\mu_{k_1,\ldots,k_n} = \int_{I^n} (t^{k_1}, \ldots, t^{k_n}) d\gamma$$

para todo $k_1, \ldots, k_n = 0, 1, \ldots$.

En el capítulo 5 de esta tesis resolvemos de forma satisfactoria los problemas débil, clásico y fuerte de Hausdorff en su versión multilineal.

En efecto, probamos, ver Teorema 5.5.2, que si K es un compacto de $\mathbb{R}$ y $\mu_{\mathbf{k}}$ una sucesión de multimomentos, existe una medida μ en K que resuelve el problema fuerte si y solo si $\mu_{\mathbf{k}}$ es una sucesión de Hänkel (es decir $\mu_{\mathbf{k}+\mathbf{1}_l} = \mu_{\mathbf{k}+\mathbf{1}_{l+1}}$, donde $(\mathbf{1}_l)_j = \delta_{lj}$, para cada $l = 1, \ldots, n$).

El problema débil tiene respuesta positiva, ver Teorema 5.2.5, si y solo si, la sucesión de multimomentos $\mu_{\mathbf{k}}$ es débilmente acotada (definición 5.2.4).

Finalmente, la condición necesaria para resolver el problema clásico (Teorema 5.3.1) resulta ser que la sucesión de multimomentos sea acotada (definición 5.2.3).

Esta tesis finaliza con el capítulo 6 en el que abordamos el problema multilineal de momentos en el caso trigonométrico.

En el caso lineal, el problema de momentos (clásico) trigonométrico, consiste en determinar condiciones necesarias y suficientes para la existencia de una medida positiva de Borel regular en el círculo $\mathbb{T}$ tal que para una sucesión de números complejos c_k, $k \in \mathbb{Z}$ tenemos:

$$c_k = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta), \quad k \in \mathbb{Z}. \tag{1}$$

Consideramos la transformada de Fourier-Stieltjes de la medida μ, es decir la función $\hat{\mu}$ en $\mathbb{Z}$ (el dual del grupo $\mathbb{T}$) definida como

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta)$$

para todo $k \in \mathbb{Z}$. El problema trigonométrico de momentos es equivalente a encontrar una función c en $\mathbb{Z}$ tal que $c = \hat{\mu}$.

En 1911, Riesz y Herglotz [Rie, Her] probaron independientemente que una condición necesaria y suficiente para la existencia de tal medida es que la función $c(k) = c_k$ sea definida positiva, es decir, para cualquier entero N, la forma cuadrática

$$\sum_{k,l=0}^{N} c(k-l)\bar{\xi}_k \xi_l$$

definida en $\mathbb{C}^{N+1}$ es positiva.

El problema de momentos se puede extender a medidas de Borel con signo. En este caso, las funciones $c(k)$ para las que existe la medida que cumple la condición indicada por la ecuación (1) son combinaciones lineales finitas de funciones definidas positivas (ver por ejemplo [Rud]).

Este problema de momentos, también se extendió al caso de varias variables sin mayores dificultades como puede leerse en los artículos [Hil2], [Hav1], [Hav2]. En ese caso, el problema consiste en encontrar una medida positiva de Borel regular μ en $\mathbb{T}^n$ tal que

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} e^{-i\mathbf{k}\cdot\boldsymbol{\theta}} \mu(\mathrm{d}\boldsymbol{\theta}), \quad \mathbf{k} \in \mathbb{Z}^n, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n.$$

dada una sucesión de números complejos $c_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. La solución puede establecerse de nuevo en términos de una condición de positividad.

En cualquier caso, el teorema de representación de Riesz juega un papel fundamental. La situación es diferente si consideramos el problema en el caso multilineal, ya que el teorema de representación no es válido.

Dada una sucesión de multimomentos, $c_{\mathbf{k}}$, definimos el funcional multilineal L_c en el espacio de polinomios trigonométricos como

$$L_c(p_1(\theta_{k_1}), \ldots, p_n(\theta_{k_n})) = \sum p_{k_1} \cdots p_{k_n} c_{-\mathbf{k}},$$

donde para $j = 1, \ldots, n$, $p_j(\theta_{k_j}) = \sum p_{k_j} e^{ik_j\theta_{k_j}}$.

Utilizaremos de nuevo el teorema de representación de Bombal y Villanueva , Teorema 5.2.2, que identifica funcionales multilineales acotados con polimedidas. La extensión natural del problema trigonométrico al caso multilineal buscará condiciones necesarias y suficientes para garantizar la existencia de una polimedida regular γ en el toro n-dimensional $\mathbb{T}^n$ tal que:

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} (e^{-ik_1\theta_1}, \ldots, e^{-ik_n\theta_n})\, \gamma(\mathrm{d}\boldsymbol{\theta}), \quad \forall \mathbf{k} \in \mathbb{Z}^n,$$

dada la familia de números complejos $c_{\mathbf{k}}$ indexada por el multiíndice $\mathbf{k} \in \mathbb{Z}^n$.

De acuerdo con nuestra terminología para el problema de Hausdorff, este problema será denominado problema trigonométrico de momentos débil multilineal.

Para su resolución, introduciremos la norma $\|\cdot\|_w$, definida en el espacio de funciones c de $\mathbb{Z}^n$, por:

$$\|c\|_w = \sup_{\substack{\|\varphi_l\|_\infty \leq 1,\, l=1,\ldots,n \\ \varphi_l(\theta) = \sum a_{k_l} e^{-ik_l\theta},\, N \in \mathbb{N}}} \left| \sum_{|\mathbf{k}| \leq N} a_{k_1} \cdots a_{k_n} c(k_1, \ldots, k_n) \right|,$$

siendo $|\mathbf{k}| = \sum_{l=1}^n |k_l|$.

Analizaremos propiedades de esta norma y presentaremos varios ejemplos mostrando que es estrictamente más débil que la norma de Fréchet y la de la convergencia absoluta. Esta norma nos permite resolver el problema débil de momentos:

Teorema 5. *Una sucesión de multimomentos $c_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$ es una solución del problema débil trigonométrico multilineal de momentos si y solo si $||c||_w < \infty$. Además $||c||_w = ||L||$ siendo L un funcional lineal y acotado que verifique $c = \hat{L}$.*

Chapter 1

Introduction.

The general frame of our thesis can be settled in functional analysis, more concretely in the theory of polynomials on Banach spaces.

The theory of polynomials on Banach spaces, now a major area of research in functional analysis, appears related to the study of holomorphy on an infinite dimensional setting at the beginning of the past century.

According to Dineen ([D], pages 74-77), at the beginning of the twentieth century, first Koch and then Hilbert, studying properties holomorphic functions on infinite many variables worked with polynomials to expand such functions. In fact, the main motivation to study polynomials in infinitely many dimensions was the relations with the theory of holomorphy on infinite dimensions.

The study of polynomials attracted the attention of other important mathematicians such as Fréchet who, in his paper [Fr1] appeared in 1909, generalizes the property $f(x+y)-f(x)-f(y)$ of linear functions, to give a definition based on differences of a polynomial on n-variables. His study leads to a definition of polynomial on $\mathbb{R}^{\mathbb{N}}$.

Fréchet generalizes his definition afterwards to, what he called, "functionelle d'ordre entiere n" ([Fr2] page 204) a polynomial on the space of continuous functions $C(J)$ with J a real compact interval. Finally, he studies a generalization of Riesz Representation Theorem for bilinear functionals in [Fr3].

The path started by Fréchet, relating polynomials with multilinear functions was followed by Michal in the decade of the thirties of the past century. By this time many important properties of polynomials, including polarization formula were known.

Another important line in the study of polynomials on Banach spaces uses the tensor product theory. This line has its starting point in the work of Grothendieck that was introduced by Gupta in the theory of holomorphy on infinite dimensions. A milestone is the Ph.D. Thesis by Ryan [R1] appeared in 1980 who proved that the space of n-homogeneous polynomials on a Banach space is the dual of the completed n-fold symmetric tensor product with respect to the π-topology.

On the last decades, the study of polynomials on Banach spaces has gain more importance as an independent field of research rather than linked to the theory of holomorphy.

Following once more the book by Dineen [D], we can separate the study of polynomials in three lines, the first one relates the polynomials and the theory of multilinear functionals. This will be our main viewpoint of the theory in this work. The second defines the polynomials with the theory of tensor products. We will use this approach in Section 2.3 on Chapter 2 and on Chapter 3 as well. And the third studies the theory of polynomials on the light of the theory of locally convex spaces. This approach won't be followed here.

All these lines of research has to overcome a major problem in the theory of polynomials: it is difficult to conclude general properties of the space of homogeneous polynomials on a given Banach space since this space can be too large.

One way to deal with this nuance, has been to consider subsets of the space of polynomials such that compact polynomials, weakly continuous on bounded sets, integral polynomials and so on. This families relies on topological properties of the polynomials and considering only polynomials having those important properties interesting conclusions have been reached.

We will follow that line of reasoning and we will consider a special subset of polynomials, the orthogonally additive polynomials. This kind of polynomials are defined on Banach lattices, which does not implies a crucial limitation in the space since the main Banach spaces, including L^p, ℓ_p and $C(K)$, can be viewed as Banach lattices. Furthermore, its defining property is not topological but algebraic: we will say that a polynomial is orthogonally additive if

$$P(x+y) = P(x) + P(y)$$

whenever x, y are disjoint elements.

The first mathematician who studied orthogonally additive polynomials was Sundaresan, who in his article [S2] in 1991 proved the following representation theorem for these polynomials on the spaces L^p and ℓ_p:

Theorem 6 (Sundaresan [S2]).

1. *Let* $X = \ell_p$ *with* $1 \leq p < \infty$ *the space* $P_o({}^nX)$ *is isometrically isomorphic to* ℓ_∞ *for* $1 \leq p \leq n$ *and to* $\ell_{p/p-n}$ *for* $n < p < \infty$. *The isometry takes* P *to* $(a_j = P(e_j))_j$.

2. *If* $X = L^p[0,1]$ *with Lebesgue measure* μ *and* $1 \leq p < \infty$ *we have:*

   - $1 \leq n < p$: $P \in P_o({}^nX)$ *if and only if there is a unique* $\xi \in L^{\frac{p}{p-n}}$ *such that* $P(x) = \int_0^1 \xi x^n \mathrm{d}\mu$
   - $n = p$: $P \in P_o({}^nX)$ *if and only if there exists a unique* $\xi \in L^\infty$ *such that* $P(x) = \int_0^1 \xi x^n \mathrm{d}\mu$
   - $n > p$: $P_o({}^nX) = \{0\}$.

However the main reason for focusing our study on this kind of polynomials is the Representation Theorem by Benyamini, Lassalle and Llavona ([BLLl] Theorem 2.3) which generalizes the previous result, giving a very useful characterization of orthogonally additive polynomials:

Theorem 7 (Theorem 2.3 [BLLl]). *Let* X *be a Banach lattice of functions and let* Y *be a Banach space. Fix* $n \in \mathbb{N}$. *Then the map* $T \longmapsto \mathcal{P}_T$, *given by* $\mathcal{P}_T(x) = T(x^n)$, *is a linear isometry of* $\mathcal{L}(X_{(n)}, Y)$ *onto* $\mathcal{P}_o({}^nX, E)$.
In particular, when E *is the scalar field, the map* $\varphi \longmapsto \mathcal{P}_\varphi$ *is a surjective linear isometry between* $(X_{(n)}, \|\!| \cdot |\!\|)^*$ *and* $\mathcal{P}_o({}^nX)$.

Recently, there have been several works on this field, Pérez-García and Villanueva in 2005 [PGV] and Carando, Lassalle and Zalduendo in 2006 [CLZ1], proved independently the representation theorem for the case $C(K)$. Moreover, Palazuelos, Peralta and Villanueva have studied in [PPV] orthogonally additive polynomials on C^*-algebras. More recently, Carando, Lassalle and Zanlduendo have studied in [CLZ2] the orthogonality additivity for holomorphic functions.

Our main objectives in this thesis can be summarized as follows. On the one hand, we will present a theory of orthogonally additive polynomials which will turn out to be very flexible in the sense that allows to be studied from several complementary points of view gaining better insight each time.

Moreover, we are not just confined to polynomials on Banach lattices but we will see that we can have the theory on the setting of Riesz spaces as well. On the other hand we would like also to show that this flexibility appears as well in applications.

In fact, the theory of orthogonally additive polynomials will appear in connection with several classical fields of mathematics such that the Spectral Theorem for self-adjoint operators, Orthogonal Polynomials or the Moment Problem among others.

The thesis is structured as follows: Chapter 2 is devoted to a general introduction on the theory of orthogonally additive polynomials and their representation. We will also give a new proof for the representation theorem of orthogonally additive polynomials on ℓ_p:

Theorem 8. *Let $1 \leq p < \infty$. The space of orthogonally additive, n-homogeneous polynomials $\mathcal{P}_o(^n\ell_p)$ is isometrically isomorphic to ℓ_∞ for $1 \leq p \leq n$ and to $\ell_{p/p-n}$ for $n < p < \infty$.*

Our proof is completely independent of the other existent proofs and uses the theory of tensor product and its relations with the theory of polynomials on the spirit of Ryan's classical result. This result can be seen in [ILLl3].

Then we will explore the applications of the representation theorem on a Hilbert lattice. We will prove first the following result:

Theorem 9. *Let H be a separable real Hilbert. Suppose H is a Banach lattice with order $\leq$. Then H is order isometric to $L^2(\mu)$ for certain measure μ.*

Which can be seen as a particular case of Kakutani's representation theorem of abstract L^p spaces. Our proof will appear as an application of the Representation theorem of orthogonally additive polynomials.

Surprisingly enough we will find another application of the representation theorem of orthogonally additive polynomials which will allow us to present a new proof of the Spectral Theorem of self-adjoint operators on Hilbert spaces. The applications of the theory to the spectral theory can be seen in [ILL14].

Following the ideas of the study of orthogonality additivity by means of tensor product. R. Ryan pointed us in the direction of the study of orthogonality on Riesz spaces. On these spaces there have been several works about orthosymmetric operators, a concept related with orthogonally additive polynomials but defined for multilinear functions.

Chapter 3 will be devoted to an extension of the concept of orthogonally additive polynomials to Riesz spaces. We will relate our concept with the orthosymmetry that exists in the literature and we will prove there a representation theorem for those polynomials. Note that in Riesz spaces the key is no longer the continuity of the polynomials but its positivity:

Theorem 10. *Let E, F be Archimedean Riesz spacies with F uniformly complete and let $(\bigodot_n E, \odot_n)$ be the n-power of E. The space $\mathcal{P}_o^+({}^nE, F)$ of positive orthogonally additive n-homogeneous polynomials on E is Riesz isomorphic to the space $\mathcal{L}^+(\bigodot_n E, F)$ of positive linear applications from $\bigodot_n E$ to F.*

Then we turn our attention to the classical theory of orthogonal polynomials and the multilinear problem of moments. In Chapter 4 we will review this theory and we will present a new kind of functions, called P-orthogonal polynomials relating the classical theory of orthogonal polynomials and the orthogonality of this new functions. The way of making this relation is thanks to the Representation theorem of orthogonally additive polynomials which links a linear functional and an orthogonally additive polynomial. This relation will allow us to translate theorems of the classical theory of orthogonal polynomials to this new setting.

The study of P-orthogonal polynomials will be focused on the spaces $C(K)$ and L^p. The first case will be done straightforward, but the second will need the introduction of a regularization function in order to deal with those new functions.

Recently, among the specialists on the theory of orthogonal polynomials, there has been a special concern on the theory of orthogonality with respect to a bilinear form. In particular, it has been studied Sobolev orthogonal polynomials with respect to Sobolev's inner product:

$$(f, g)_S = \sum_{j=0}^{k} \int f^{(j)} g^{(j)} \mathrm{d}\mu_j$$

vhere μ_j are Borel positive measures and the derivatives are taken in the weak sense. (see for instance [MAR], [MMB], [MF1], [MF2], [M] and their references).

Using a strong theorem by Boas [B], Durán proved in [Dur2] a theorem with applications to the theory of orthogonal polynomials showing when the orthogonality with respect to a bilinear form can be reduced to the classic orthogonality. To conclude chapter 4, we will extend this result to inner products defined on algebraic polynomials over $\mathbb{R}$ proving the following theorem:

Theorem 11. *Let B be an inner product defined on the space of real algebraic polynomials on $\mathbb{R}$. The following are equivalent:*

- *The operator multiplication by t is symmetric and bounded with respect to B.*
- *There exists a Borel regular measure on a compact of $\mathbb{R}$ such that*

$$B(p,q) = \int p(t)q(t)\mathrm{d}\mu(t).$$

inspired in a theorem by Durán appeared in [Dur2].

In Chapter 5 we will generalize the classical Hausdorff moment problem to the multilinear case.

This moment problem is a part of a vast theory of problems of moment that began in the nineteenth century consisting on finding a regular Borel measure μ in a given interval $I \subseteq \mathbb{R}$ such that

$$\mu_n = \int_I t^n \mathrm{d}\mu$$

where $(\mu_n) \subset \mathbb{R}$ is a given sequence.

The moment problem takes different names depending on the subset I, if $I = (0,\infty)$, it is called Stieltjes moment problem, Hamburger moment problem if $I = \mathbb{R}$. The case of our study, the Hausdorff moment problem arises when $I = [0,1]$

Our contribution is a generalization of this problem to a multilinear setting. As it turns out, when we study the multilinear case there is not just one problem but there are different instances of the same problem.

The first case is the strong multilinear moment problem which looks for necessary and sufficient conditions for the existence of a regular Borel measure μ on $[0,1]$ such that, for a given sequence of multimoments $\mu_{\mathbf{k}}$[1].

$$L_\mu(p_1(t),\dots,p_n(t)) = \int_0^1 p_1(t)\cdots p_n(t)d\mu(t)$$

[1]A family of numbers $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^n$ where $\mathbf{k} = (k_1,\dots,k_n)$ is a nonnegative multi-index of length n will be called multimoment sequence. L_μ is the moment functional associated to $\mu_{\mathbf{k}}$.

where $p_1(t), \ldots, p_n(t) \in \mathbb{R}[t]$ and L_μ is the n-linear moment functional defined by the multimoments $\mu_{\mathbf{k}}$ on the space of polynomials $\mathbb{R}[t]$ by means of the formula:

$$L_\mu(t^{k_1}, \ldots, t^{k_n}) = \mu_{k_1,\ldots,k_n} \text{ for } k_1, \ldots, k_n = 0, 1, \ldots.$$

We can also consider a less restrictive condition and study necessary and sufficient conditions so that there exists a Borel measure μ on $[0,1]^n \subset \mathbb{R}^n$ such that:

$$L_\mu(p_1(t_1), \ldots, p_n(t_n)) = \int_{I^n} p_1(t_1) \cdots p_n(t_n) d\mu(t_1, \ldots, t_n)$$

for a given sequence of multimoments $\mu_{\mathbf{k}}$.

Due to the similarities with the classical result, this problem will be called classical multilinear Hausdorff moment problem.

Finally, we would consider as well a moment problem where the integration is done with respect to a polymeasure instead of a measure, what leads to the weak multilinear moment problem: find necessary and sufficient conditions for the existence of a polymeasure γ on $\mathrm{Bo}[0,1] \times \cdots \times \mathrm{Bo}[0,1]$ such that, for a given sequence of multimomens $\mu_{\mathbf{k}}$:

$$\mu_{k_1,\ldots,k_n} = \int_{I^n} (t^{k_1}, \ldots, t^{k_n}) d\gamma$$

for every $k_1, \ldots, k_n = 0, 1, \ldots$.

The reason to consider this problem is the relations of (classical) Hausdorff moment problem and Riesz representation theorem. In fact, in the classical theory, they can be seen as equivalent. As it is well known, there are not a generalization of the Riesz representation theorem to a multilinear setting where the representation is done by means of measures. Howevey, Bombal and Villanueva proved in [BV1], Theorem 7, a Riesz representation theorem for multilinear functions using polymeasures. We will study the relations of this theorem with the weak multilinear moment problem.

These results appear in [ILL11].

This thesis ends with Chapter 6 where we study the multilinear trigonometric moment problem.

In the linear setting, the classical trigonometric problem of moments consists in determining necessary and sufficient conditions for the existence of a positive regular Borel measure μ on the circle $\mathbb{T}$ such that for a given sequence of complex numbers c_k, $k \in \mathbb{Z}$ we have:

$$c_k = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta), \quad k \in \mathbb{Z}. \tag{1.1}$$

Consider the Fourier-Stieltjes transform of the measure μ, that is the function $\hat{\mu}$ on $\mathbb{Z}$ (the dual group of $\mathbb{T}$) defined as

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta)$$

for all $k \in \mathbb{Z}$. The trigonometric moment problem is equivalent to find a function c defined on $\mathbb{Z}$ such that $c = \hat{\mu}$.

In 1911, Riesz and Herglotz [Rie, Her] showed independently that a necessary and sufficient condition for the existence of such a measure is that the function $c(k) = c_k$ is positive definite, that is, if for any positive integer N the quadratic form

$$\sum_{k,l=0}^{N} c(k-l)\bar{\xi}_k \xi_l$$

defined on $\mathbb{C}^{N+1}$ is positive.

The problem of moments can be extended to signed Borel measures. In such case, functions $c(k)$ for which a signed Borel measure exists satisfying equation (1.1) are finite linear combinations of positive definite functions (see for instance [Rud]).

This moment problem was also extended to the multivariate case without further difficulties as it can be seen in the papers [Hil2], [Hav1], [Hav2]. It consists in finding a positive regular Borel measure on μ on $\mathbb{T}^n$ such that

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} e^{-i\mathbf{k}\cdot\boldsymbol{\theta}} \mu(\mathrm{d}\boldsymbol{\theta}), \quad \mathbf{k} \in \mathbb{Z}^n, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n.$$

given a sequence of complex numbers $c_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Its solution can be settled in terms of an analogous positivity condition. The proof of this can be obtained by using M. Riesz extension principle for positive functionals (see for instance [Akh]).

Even in the more general situation of signed measures, the solution comes easily using the Jordan decomposition of signed measures.

In all cases, Riesz representation theorem plays a fundamental role. The situation is different if we consider the problem through a multilinear approach since that theorem, as we saw in the previous chapter, is no longer valid now.

Given a sequence of multimoments $c_{\mathbf{k}}$, we will define the multilinear functional L_c defined on the space of trigonometric polynomials as

$$L_c(p_1(\theta_{k_1}), \ldots, p_n(\theta_{k_n})) = \sum p_{k_1} \cdots p_{k_n} c_{-\mathbf{k}},$$

where for $j = 1, \ldots, n$, $p_j(\theta_{k_j}) = \sum p_{k_j} e^{ik_j \theta_{k_j}}$.

We will use again the representation theorem by Bombal and Villanueva, Theorem 5.2.2 that identifies bounded multilinear functionals with polymeasures. Then, the natural extension of the trigonometric problem of moments to the multilinear case, will study necessary and sufficient conditions to guarantee the existence of a regular polymeasure γ on the n-dimensional torus $\mathbb{T}^n$ such that:

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} (e^{-ik_1\theta_1}, \ldots, e^{-ik_n\theta_n})\, \gamma(\mathrm{d}\boldsymbol{\theta}), \quad \forall \mathbf{k} \in \mathbb{Z}^n,$$

where the family complex numbers $c_{\mathbf{k}}$ indexed by the multiindex $\mathbf{k} \in \mathbb{Z}^n$ is given.

According to our terminology in the previous chapter, this problem will be called the Weak Trigonometric Moment Problem. In order to solve it, we will introduce a norm, weaker than the supremum norm on the space of functions c on $\mathbb{Z}^n$:

$$\|c\|_w = \sup_{\substack{\|\varphi_l\|_\infty \leq 1, l = 1, \ldots, n \\ \varphi_l(\theta) = \sum a_{k_l} e^{-ik_l\theta}, N \in \mathbb{N}}} \left| \sum_{|\mathbf{k}| \leq N} a_{k_1} \cdots a_{k_n} c(k_1 \cdots k_n) \right|,$$

with $|\mathbf{k}| = \sum_{l=1}^n |k_l|$.

We will analyze the properties of this norm presenting some examples to show that the norm is strictly weaker than Fréchet norm and the absolute convergence norm. The norm $\| \cdot \|_w$ gives the condition we needed to solve the weak moment problem:

Theorem 12. *A sequence of multimoments $c_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$ is a solution of the weak trigonometric multilinear moment problem if and only if $\|c\|_w < \infty$.*

Moreover $\|c\|_w = \|L\|$ *where* L *is a bounded multilinear functional such that* $c = \hat{L}$.

The content of this chapter can be found in [ILL12].

1.1 Notations and preliminaries.

We will use a standard notation in this work. Banach spaces as well as Banach lattices will be denoted by X and Y. We will reserve letters E and F to denote Riesz spaces in Chapter 3. We will denote as usual by $C(K)$ the Banach space of continuous functions over K a Hausdorff compact space, and by ℓ_p the space of sequences (x_n) such that

$$\sum_{n\in\mathbb{N}} |x_n|^p \leq \infty.$$

If (Ω, Σ, μ) is a measure space, we will denote by $L^p(\Omega, \Sigma, \mu)$ the space of (classes of equivalence of) functions integrable to the power p. It will be abbreviated by $L^p(\Omega)$, $L^p(\mu)$ or just by L^p if there is no possible confusion.

Recall that a function $A : X_1 \times \ldots \times X_n \longrightarrow Y$, is said to be multilinear or n-linear if it is linear for every variable separately. The space of continuous n-linear functions from $X_1 \times \ldots \times X_n$ to Y is denoted by $\mathcal{L}^n(X_1, \ldots, X_n; Y)$. If $X_1 = \cdots = X_n = X$ we will write $\mathcal{L}(^nX, Y)$

We present now the definition of polynomial on a Banach space.

Definition 1. *Let* X *and* Y *be two Banach spaces over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$. *An application* $P : X \to Y$ *is said to be a continuous* n*-homogeneous polynomial if there exists a* n*-linear function* $A \in \mathcal{L}(^nX, Y)$ *such that*

$$P(x) = A(x, \ldots, x).$$

The space of continuous n-homogeneous polynomials from X to Y will be denoted by $\mathcal{P}(^nX, Y)$ it is a Banach space with the norm

$$\|P\| = \sup\{\|P(x)\| : x \in X, \|x\| \leq 1\}$$

If $Y = \mathbb{K}$ we will write $\mathcal{P}(^nX)$.

In general the multilinear function A defining the polynomial P is not unique since given a multilinear function A, A^s defined by

$$A^s(x_1, \ldots, x_n) = \frac{1}{n!} \sum_{\sigma\in S_n} A(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$$

defines the same polynomial.

We have to restrict to the space of symmetric continuous multilinear functions, $\mathcal{L}_s(^nX, Y)$. Recall that a multilinear function is said to be symmetric if

$$A(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = A(x_1, \ldots, x_n)$$

for any permutation $\sigma \in S_n$.

Now, due to the Polarization Formula[2]:

Theorem 13 (Polarization Formula). *Let $A \in \mathcal{L}^s(^nX, Y)$. Then for every $x_1, \ldots, x_n \in X$*

$$A(x_1, \ldots, x_n) = \frac{1}{n!2^n} \sum_{\epsilon_j = \pm 1} \epsilon_1 \cdots \epsilon_n A(\epsilon_1 x_1 + \cdots + \epsilon_n x_n)^n$$

We have that

Theorem 14. *The application $\hat{}: \mathcal{L}^s(^nX, Y) \to \mathcal{P}(^nX, Y)$; $A \mapsto \hat{A}$, where $\hat{A}(x) = A(x, \ldots, x)$ is a topological isomorphism.*

This theorem is a direct consequence of the Polarization Formula since this Formula leads to the following inequalities between norms:

$$\|\hat{A}\| \leq \|A\| \leq \frac{n^n}{n!} \|\hat{A}\|.$$

We will consider P to be the polynomial generated by A meaning that A is the unique symmetric multilinear form such that $\hat{A} = P$. On the other way round, the inverse of this application will be called $\check{}$ and given a polynomial P, $\check{P}$ will denote the unique symmetric multilinear function A such that $A(x, \ldots, x) = P(x)$.

In general, we will work over the field of real numbers although our results could be extended as well to the complex field.

In order not to extend this introduction and make the lecture of the following chapters as self-contained as possible, other notations and introductions will be presented in the moment we will need to use them.

[2]The notation $A(x)^n$ stands for $A(x, \ldots, x)$

Chapter 2

Orthogonally Additive Polynomials in Banach Lattices.

2.1 Introduction.

As we have remarked in the previous chapter, given two Banach spaces X and Y the space of n-homogeneous polynomials $\mathcal{P}(^n X, Y)$ may be "big" enough to obtain interesting results concerning every polynomial in the space. Due to this reason, one line of research in the theory of n-homogeneous polynomials in Banach spaces has been focused on the study of polynomials with remarkable properties such that finite type, approximable, compact, weakly continuous on bounded sets, nuclear or integral polynomials. The interested reader can be referred to [D] for a thorough study.

Our interest will be centered on a type of polynomials, the orthogonally additive ones, defined on Banach lattices. For these polynomials there is a representation theorem that brings together different areas of analysis.

In this chapter we will make a brief survey of the known theory of orthogonally additive polynomials and their representation. We provide as well a new proof of the representation theorem for the case $X = \ell_p$ based on tensor product theory.

Orthogonally additive polynomials on Hilbert spaces will be studied obtaining a particular case of Kakutani's representation theorem as a consequence of the representation theorem of orthogonally additive polynomials.

Finally we present a new proof of the spectral theorem of bounded self-adjoint operators on Hilbert spaces which does not need to use the theory of spectral measures.

Before stating the definition of orthogonally additive polynomial, we need to recall the notion of Banach lattice. A classical reference for Banach lattice theory is the book by Lindenstrauss and Tzafriri [LT].

Definition 2.1.1. *Let $(X,\leq)$ be a real Banach space and a partial order relation. We will say that X is a Banach lattice if it verifies:*

(i) *If $x \leq y$ then $x + z \leq y + z$ for all $x, y, z \in X$.*

(ii) *$ax \geq 0$ for every $x \geq 0$ and every nonnegative real a.*

(iii) *Given $x, y \in X$, there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.*

(iv) *$\|x\| \leq \|y\|$ whenever $|x| \leq |y|$, where the absolute value $|x|$ is defined as $|x| = x \vee (-x)$.*

For most of the usual examples of Banach spaces there is a natural Banach lattice order. If $X = L^p(\Omega)$ or $X = C(K)$ we can define the order given by $f \leq g$ if and only if $f(x) \leq g(x)$ for almost every $x \in \Omega$ or for every $x \in K$ respectively. Another common example is the case $X = \ell_p$ where $x = (x_n) \leq y = (y_n)$ if and only if $x_n \leq y_n$ for every $n \in \mathbb{N}$.

A key notion in the definition of orthogonally additive polynomials is that of disjointness of two elements. We will say that $x, y \in X$ are disjoint (or orthogonal) if the greatest lower bound of their absolute values is zero, that is $|x| \wedge |y| = 0$. In the previous examples, two functions f and g in $L^p(\Omega)$ or in $C(K)$ are disjoint if $essmin(f) \cap essmin(g) = \emptyset$ where $supp(f)$ stands for the support of f. If $X = \ell_p$, $x = (x_n)$ and $y = (y_n)$ are disjoint if $x_n y_n = 0$ for each $n \in \mathbb{N}$.

Definition 2.1.2. *Let X a Banach lattice, Y Banach space. A n-homogeneous polynomial $P \in \mathcal{P}(^nX, Y)$ is said to be an orthogonally additive polynomial if*

$$P(x + y) = P(x) + P(y)$$

whenever x and y are disjoint elements of X.

The space of orthogonally additive n-homogeneous polynomials from X to Y is denoted by $\mathcal{P}_o(^nX, Y)$. In the case $Y = \mathbb{K}$ we will use the notation $\mathcal{P}_o(^nX)$.

Examples 2.1.3.

1. *Let* $X = C[0,1]$*,* $Y = \mathbb{K}$ *and* μ *a regular Borel measure in* $[0,1]$*, the polynomial defined by*
$$P(f) = \int_0^1 f^n \mathrm{d}\mu$$
is an orthogonally additive polynomial.

2. *Another simple example of orthogonally additive polynomial that we will use in Section 2.4 is the 2-homogeneous polynomial* $P(x) = \|x\|_{\ell_2}^2$ *in* ℓ_2*, where* ℓ_2 *is assumed to be real. We will see later that this result is true for the polynomial* $\|\cdot\|_H^2$ *where* H *is any real Hilbert space. The case* $H = \ell_2$ *it can be proved directly as* $P(x+y) = P(x) + 2\sum x_n y_n + P(y)$ *and* $x_n y_n = 0$ *if* x *and* y *are disjoint.*

2.2 The Representation Theorem of Orthogonally Additive Polynomials in Banach Lattices.

2.2.1 Introduction.

In 1965, Chacon and Friedman, motivated by some applications, studied in [CF] a possible extension of the Riesz representation theorem for the case of a functional, not necessarily linear. Working on $C[0,1]$, they introduce what they called additive functional (additive for disjoint elements) and proved a representation theorem. This result was extended for the case $C(S)$ with S a compact metric space in [FK1] and with S a compact topological space in [FK3]. The same result was also proved independently in [MM] but with an additional hypothesis.

There has been several generalizations, see [DO1], where the term orthogonally additive functional is used for the first time, [DO2], [FK2] or [S1].

The concept of orthogonally additive functional was generalized to the polynomial setting by Sundaresan who, in 1991, obtained the following result:

Theorem 2.2.1 (Sundaresan [S2]).

1. *Let* $X = \ell_p$ *with* $1 \le p < \infty$*, the space* $\mathcal{P}_o(^n X)$ *is isometrically isomorphic to* ℓ_∞ *for* $1 \le p \le n$ *and to* $\ell_{p/p-n}$ *for* $n < p < \infty$*. The isometry takes* P *to* $(a_j = P(e_j))_j$*.*

2. If $X = L^p[0,1]$ with Lebesgue measure μ and $1 \leq p < \infty$ we have:

- $1 \leq n < p$: *$P \in \mathcal{P}_o(^nX)$ if and only if there is a unique $\xi \in L^{\frac{p}{p-n}}$ such that $P(x) = \int_0^1 \xi x^n \mathrm{d}\mu$*
- $n = p$: *$P \in \mathcal{P}_o(^nX)$ if and only if there exists a unique $\xi \in L^\infty$ such that $P(x) = \int_0^1 \xi x^n \mathrm{d}\mu$*
- $n > p$: *$\mathcal{P}_o(^nX) = \{0\}$.*

In 2005, Pérez-García and Villanueva [PGV] obtained a representation theorem for the case $C(K)$ by means of representation of multilinear functionals with polymeasures. We will make use of these techniques on Chapter 5. In 2006 Benyamini, Lassalle and Llavona [BLLl] extended this result to every Banach lattice as we will see on the next section. Also in 2006, Carando, Lassalle and Zalduendo [CLZ1] found an independent proof for the case $C(K)$ using a linearization procedure. Finally, Palazuelos, Peralta and Villanueva [PPV] studied orthogonally additive polynomials in the context of C^*-algebras. More recently, Carando, Lassalle and Zalduendo have studied in [CLZ2] the orthogonality additivity for holomorphic functions.

2.2.2 The Representation Theorem.

In the present section, the representation theorem of orthogonally additive polynomials by Benyamini, Lassalle and Llavona which appears in [BLLl], is stated.

Let X be a Banach lattice. We will assume that X is a Banach lattice of functions defined in some measure space (Ω, Σ, μ). The order is given pointwise as in the case of L^p, that is $f \leq g$ if and only if $f(x) \leq g(x)$ for almost every $x \in \Omega$. Since every Banach lattice can be represented in such a way (see for instance [LT], chapter 1. b.), we are not restricting the class of spaces. On the other hand, this assumption is very convenient since it simplifies considerably the presentation of the theorem.

Given $f \in X$ and $\alpha > 0$, we define f^α as the function

$$f^\alpha(x) = sign(f(x))|f(x)|^\alpha.$$

The n-concavification of X is defined to be the linear space spanned by

$$X_{(n)} = \{f^n : f \in X\}$$

endowed with the algebraic operations and order defined from those of X. We define as well a natural quasi-norm given by

$$|||g||| = \|g^{1/n}\|^n.$$

The proof of the fact that $|||\cdot|||$ is a quasi-norm can be seen in [BLLl].

The n-concavification of a general lattice can be defined by means of functional calculus as it is explained in [LT].

Consider the map defined on X by $P(f) = T(f^n)$ where T is a continuous linear form on $(X_{(n)}, |||\cdot|||)$. P is in fact a polynomial defined by the symmetric n-linear form $A(f_1, \ldots, f_n) = T(f_1 \cdots f_n)$. As $f, g \in X$ are disjoint if $fg = 0$, for disjoint elements $(f + g)^n = f^n + g^n$ hence P is orthogonally additive. The representation theorem of orthogonally additive polynomials asserts that every such polynomial has this form:

Theorem 2.2.2 (Theorem 2.3 [BLLl]). *Let X be a Banach lattice of functions and let Y be a Banach space. Fix $n \in \mathbb{N}$. Then the map $T \longmapsto \mathcal{P}_T$, given by $\mathcal{P}_T(x) = T(x^n)$, is a linear isometry of $\mathcal{L}(X_{(n)}, Y)$ onto $\mathcal{P}_o({}^nX, E)$. In particular, when E is the scalar field, the map $\varphi \longmapsto \mathcal{P}_\varphi$ is a surjective linear isometry between $(X_{(n)}, |||\cdot|||)^*$ and $\mathcal{P}_o({}^nX)$.*

From now on we will refer to this result simply as the representation theorem. We are interested mostly in the scalar case.

The quasi-norm of the theorem turns out to be a norm in several cases. For instance if $X = \ell_p$ or $X = L^p$, $X_{(n)}$ is $\ell_{p/n}$ or $L^{p/n}$ respectively and $|||\cdot|||$ equals the p/n-norm. In these cases, we obtain the results of Theorem 2.2.1 by Sundaresan. Another favorable case is when $X = C(K)$. For this case $X_{(n)} = X$ and the quasi-norm is $\|\cdot\|_\infty$.

2.2.3 Orthogonally Additive Polynomials in order continuous Köthe function spaces.

As we shall see later, the representation theorem becomes useful if we know an appropriate representation of the dual of $(X_{(n)}, |||\cdot|||)$. This is the case of a wide class of spaces named Köthe function spaces with the property of being order continuous. We we will need the following definitions that can be found in [LT].

Definition 2.2.3. *Let X be a Banach lattice of equivalence classes (modulo equality almost everywhere) of locally integrable real valued functions in a* *measure space* (Ω, Σ, μ) *complete and* σ*-finite. We say that* X *is a Köthe function space if the following conditions hold:*

- *If* $g \in X$*,* f *is measurable and* $|f(x)| \leq |g(x)|$ *almost everywhere then* $f \in X$ *and* $\|f\| \leq \|g\|$*.*

- $\chi_E \in X$ *for every* $E \in \Sigma$ *with* $\mu(E) < \infty$*.*

We will be interested in those space that are order continuous:

Definition 2.2.4. *A Banach lattice* X *will be said to be order continuous if every downward directed set* $\{x_\alpha : \alpha \in A\}$ *such that* $\inf\{x_\alpha : \alpha \in A\} = 0$ *verifies* $\lim_\alpha \|x_\alpha\| = 0$*. It will be said to be* σ*-order continuous if it verifies the same definition for sequences.*

For those spaces, it is proved in [BLLl] the following Corollary:

Corollary 2.2.5. *Let* X *be an order continuous Köthe function space. Then every* n*-homogeneous orthogonally additive polynomial* $P \in \mathcal{P}_o(^nX)$ *can be represented as*

$$P(f) = \int f^n \xi \mathrm{d}\mu$$

for some measurable function ξ *on* Ω *such that* $f\xi \in L^1(\Omega, \Sigma, \mu)$ *for every* $f \in X$*.*

Remark 2.2.6. *The previous Corollary does not include the case* $X = C(K)$*, since these spaces does not satisfies in general the second property of definition 2.2.3. Nevertheless both [CLZ1] and [PGV] obtain that for every polynomial* n*-homogeneous, orthogonally additive* P *on* $C(K)$*, there is a Borel regular measure* μ *such that*

$$P(f) = \int_K f^n \mathrm{d}\mu.$$

This result, although not contained in the previous Corollary, can also be obtained as a direct consequence of Theorem 2.2.2 and Riesz Representation Theorem.

2.3 The Representation Theorem in ℓ_p.

The aim of this section is to present a new proof of the Theorem 2.2.1 for the case $X = \ell_p$. As our main tool will be the theory of tensor products of Banach spaces, we introduce briefly the basic facts that we will use. For more information about tensor products, the reader is referred to the book by Ryan [R2].

For our purposes, the best way of introducing the tensor product of Banach spaces is as a linearizing space for multilinear forms. Let $X_1, \dots, X_n$ be Banach spaces. For $x_1 \in X_1, \dots, x_n \in X_n$ we define the tensor $x_1 \otimes \dots \otimes x_n$ as the linear form on the space of n-linear maps $\mathcal{L}^n(X_1, \dots, X_n)$ given by

$$A \longmapsto x_1 \otimes \dots \otimes x_n(A) = A(x_1, \dots, x_n).$$

The tensor product $X_1 \otimes \dots \otimes X_n$ can be defined as the subspace of the algebraic dual of $\mathcal{L}^n(X_1, \dots, X_n)$ spanned by the elements $x_1 \otimes \dots \otimes x_n$.

Denote by $\otimes$ the multilinear mapping $(x_1, \dots, x_n) \longmapsto x_1 \otimes \dots \otimes x_n$. It can be seen (see Proposition 1.4 in [R2]) that for every multilinear map $A : X_1 \times \dots \times X_n \longrightarrow Y$ there exists a unique linear map $\tilde{A} : X_1 \otimes \dots \otimes X_n \longrightarrow Y$ such that $A(x_1, \dots, x_n) = \tilde{A}(x_1 \otimes \dots \otimes x_n)$. The correspondence taking A to $\tilde{A}$ is a isomorphism from $\mathcal{L}^n(X_1, \dots, X_n; Y)$ to $\mathcal{L}(X_1 \otimes \dots \otimes X_n; Y)$. One of the most interesting properties of the tensor product of Banach spaces is the uniqueness up to isomorphism of this property for the pair $(X_1 \otimes \dots \otimes X_n, \otimes)$ see for instance [R2] Proposition 1.5.

Up to now, we have seen that the tensor product space of Banach spaces linearizes the space of multilinear maps in an algebraic fashion. It is natural to look for a norm in $X_1 \otimes \dots \otimes X_n$ to extend this linearization to a continuous setting. This objective is attained by the so called projective norm or π-norm. The π-norm is defined for a tensor

$$u = \sum_{i=1}^{m} x_{1,i} \otimes \dots \otimes x_{n,i}$$

as

$$\pi(u) = \inf \left\{ \sum_{i=1}^{m} \|x_{1,i}\| \cdots \|x_{n,i}\| : u = \sum_{i=1}^{m} x_{1,i} \otimes \dots \otimes x_{n,i} \right\}$$

The tensor product $X_1 \otimes \dots \otimes X_n$ endowed with the π-norm is denoted by $X_1 \otimes_\pi \dots \otimes_\pi X_n$. It is in general not complete. Its completion with the

π-norm is denoted by $X_1\hat{\otimes}_\pi \ldots \hat{\otimes}_\pi X_n$ and it is known as the projective tensor product. If $X = X_1 = \cdots = X_n$ we will write $\widehat{\bigotimes}_{n,\pi} X$ to denote the n-fold projective tensor product.

It can be proved (see Theorem 2.9 in [R2]) that given

$$B : X_1 \times \ldots \times X_n \longrightarrow Y$$

bounded n-linear mapping there exists a unique operator

$$\tilde{B} : X_1\hat{\otimes}_\pi \ldots \hat{\otimes}_\pi X_n \longrightarrow Y$$

verifying $B(x_1, \ldots, x_n) = \tilde{B}(x_1 \otimes \ldots \otimes x_n)$. Furthermore, the correspondence $B \longmapsto \tilde{B}$ is an isometric isomorphism between the Banach spaces $\mathcal{L}^n(X_1, \ldots, X_n; Y)$ and $\mathcal{L}(X_1\hat{\otimes}_\pi \ldots \hat{\otimes}_\pi X_n; Y)$.

If we take $Y = \mathbb{K}$ this result identifies

$$\mathcal{L}^n(X_1, \ldots, X_n) \cong (X_1\hat{\otimes}_\pi \ldots \hat{\otimes}_\pi X_n)^*.$$

Using this duality, we have a new formula for the π-norm

$$\pi(u) = \sup\{|\langle u, B\rangle| : B \in \mathcal{L}^n(X_1, \ldots, X_n), \|B\| \leq 1\}$$

where $\langle u, B\rangle$ is defined for $u = \sum_{i=1}^m x_{1,i} \otimes \ldots \otimes x_{n,i}$ as

$$\langle u, B\rangle = \sum_{i=1}^m B(x_{1,i}, \ldots, x_{n,i}).$$

Consider now the case $X = X_1 = \cdots = X_n$. The polynomials on X are identified with the symmetric multilinear forms on X. Hence, the problem of a linearization of the space of polynomials as the dual of some subspace of the tensor product arises naturally. This problem was solved by Ryan in his Thesis [R1] introducing the symmetric tensor product.

Let s be the map in $\bigotimes_n X$ given by

$$s(x_1 \otimes \ldots \otimes x_n) = \frac{1}{n!} \sum_{\sigma \in S_n} x_{\sigma(1)} \otimes \ldots \otimes x_{\sigma(n)}.$$

We define the n-fold symmetric tensor product $\bigotimes_{n,s} X$ as the subspace of $\bigotimes_n X$ generated $s(x_1 \otimes \ldots \otimes x_n)$ with $x_1, \ldots, x_n \in X$. If we complete the symmetric tensor product with respect to the π-norm to get $\widehat{\bigotimes}_{n,s,\pi} X$, Ryan's result reads as follows:

$$\mathcal{P}(^n X) = \left(\widehat{\bigotimes}_{n,s,\pi} X \right)^*$$

where the identification $L \longmapsto P$ satisfies $P(x) = L(x \otimes \ldots \otimes x)$.

Our proof of the representation theorem for the case ℓ_p follows the same path. We will look for a subspace of the symmetric tensor product whose dual equals the space of orthogonally additive polynomials.

Note that if A is the n-linear symmetric form associated to $P \in \mathcal{P}_o(^n \ell_p)$ then $A(e_{k_1}, \ldots, e_{k_n})$ unless $e_{k_1} = \cdots = e_{k_n} = 0$. The idea is that the essential information of P is obtained by evaluating the linearization of its associated symmetric form A in tensors of the form $e_k \otimes \ldots \otimes e_k$. Hence, we define the tensor diagonal, $D_{n,p}$ as the closed subspace of $\widehat{\bigotimes}_{n,s,\pi} \ell_p$ generated by $\underbrace{e_k \otimes \ldots \otimes e_k}_{n}$, $k \in \mathbb{N}$.

We will give a characterization of this subspace and we will prove that its dual is isometrically isomorphic to $\mathcal{P}_o(^n \ell_p)$. In Ryan's book, it is proved that for the case $n = 2$,

Example 2.3.1 (Example 2.23 [R2]). *The tensor diagonal $D_{2,p}$ in $\ell_p \hat{\otimes}_\pi \ell_p$ satisfies:*

$$D_{2,p} = \begin{cases} \ell_1 & \text{if } 1 \leq p \leq 2 \\ \ell_{p/2} & \text{if } 2 < p < \infty \end{cases}$$

The main tool was the following result where r_i stands for the Rademacher functions:

Proposition 2.3.2 (Rademacher Averaging, Lemma 2.22 [R2]). *Let E and F be vector spaces and let $x_1 \ldots x_n \in E$ and $y_1, \ldots y_n \in F$. Then*

$$\sum_{i=1}^{m} x_i \otimes y_i = \int_0^1 \left(\sum_{i=1}^{m} r_i(t) x_i \right) \otimes \left(\sum_{i=1}^{m} r_i(t) y_i \right) \mathrm{d}t.$$

In order to generalize the description of the tensor diagonal in $\widehat{\bigotimes}_{n,s,\pi} \ell_p$, we will need a generalization of the Rademacher Averaging which involves the generalized Rademacher functions introduced by Aron and Globevnik in [AG]:

Definition 2.3.3 (Definition 1.1 [AG]). *Fix $n \in \mathbb{N}$, $n \geq 2$ and let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_n$ denote the n^{th} roots of unity. Let $r_1 : [0,1] \longrightarrow \mathbb{C}$ be the step function taking the value α_j on $(\frac{j-1}{n}, \frac{j}{n})$ for $j = 1, \ldots, n$. Assuming that r_{k-1} has been defined, define r_k in the following way: fix any of the n^{k-1} sub-intervals I of $[0,1]$ used in the definition of r_{k-1}. Divide I into n equal intervals $I_1, \ldots, I_n$ and set $r_k(t) = \alpha_j$ if $t \in I_j$.*

We will also need the following

Lemma 2.3.4 (Lemma 1.2 [AG]). *For each $n = 2, 3, \ldots$ the associated functions r_k satisfy the following properties:*

- *$|r_k(t)| = 1$ for all $k \in \mathbb{N}$ and all $t \in [0,1]$.*
- *For any choice of $k_1, \ldots, k_n$*

$$\int_0^1 r_{k_1}(t) \cdots r_{k_n}(t) \mathrm{d}t = \begin{cases} 1 \text{ if } k_1 = \cdots = k_n \\ 0 \text{ otherwise.} \end{cases}$$

The next lemma is a generalization of Lemma 2.22 in [R2]:

Lemma 2.3.5 (Rademacher Averaging). *Let $X_1, \ldots X_n$ vector spaces and let $x_{1,1}, \ldots, x_{1,k} \in X_1, \ldots,$ and $x_{n,1}, \ldots, x_{n,k} \in X_n$. Then*

$$\sum_{i=1}^{k} x_{1,i} \otimes \cdots \otimes x_{n,i} = \int_0^1 \left(\sum_{i=1}^{k} r_i(t) x_{1,i} \right) \otimes \cdots \otimes \left(\sum_{i=1}^{k} r_i(t) x_{n,i} \right) \mathrm{d}t.$$

Proof. Just expand the integral and use the second property of Lemma 2.3.4. □

Theorem 2.3.6 (Ibort, Linares and Llavona [ILLl3]). *Let $1 \leq p < \infty$. The tensor diagonal $D_{n,p}$ in $\widehat{\bigotimes}_{n,s,\pi} \ell_p$ is isometrically isomorphic to $\ell_{p/n}$ if $n < p < \infty$ and to ℓ_1 if $1 \leq p \leq n$.*

Proof.

1. $n < p < \infty$.
   Let $u = \sum_{i=0}^{k} a_i e_i \otimes \cdots \otimes e_i \in D_{n,p}$. Using Lemma 2.3.5, we write

$$u = \int_0^1 \left(\sum_{i=1}^{k} sign(a_i) |a_i|^{1/n} r_i(t) e_i \right) \otimes \cdots \otimes \left(\sum_{i=1}^{k} |a_i|^{1/n} r_i(t) e_i \right) \mathrm{d}t$$

and, like in [R2] (pages 34-35) we get:

$$\begin{aligned}\pi(u) &\le \sup_{0\le t\le 1}\left\|\sum_{i=1}^{k} sign(a_i)|a_i|^{1/n} r_i(t)e_i\right\|_p \cdots \left\|\sum_{i=1}^{k}|a_i|^{1/n} r_i(t)e_i\right\|_p = \\ &= \left(\sum_{i=1}^{k}|a_i|^{p/n}\right)^{n/p} = \|(a_i)\|_{p/n}\end{aligned}$$

To prove the identity, define a n-linear form on ℓ_p by $B(x_1,\ldots,x_n) = \sum b_i x_{1,i}\cdots x_{n,i}$ where $x_j = (x_{j,k})$ and $b_i = sign(a_i)|a_i|^{p/n-1}$. Using Hölder's inequality, it is easy to see that $\|B\| \le (\sum_{i=1}^{k}|a_i|^{p/n})^{1-p/n}$ and then

$$\sum_{i=1}^{k}|a_i|^{p/n} = |\langle u,B\rangle| \le \pi(u)(\sum_{i=1}^{k}|a_i|^{p/n})^{1-p/n}.$$

Hence $\|(a_i)\|_{p/n} \le \pi(u)$ and therefore $D_{n,p}$ is isometrically isomorphic to $\ell_{p/n}$.

2. $1 \le p \le n$.
Let be $u = \sum_{i=0}^{k} a_i e_i \otimes \cdots \otimes e_i \in D_{n,p}$, then $\pi(u) \le \sum_{i=0}^{k}|a_i|$. Reciprocally, define $B(x_1,\ldots,x_n) = \sum_{i=0}^{k} sign(a_i)x_{1,i}\cdots x_{n,i}$, we have that $|B(x_1,\ldots,x_n)| \le \|x_1\|_n\cdots\|x_n\|_n \le \|x_1\|_p\cdots\|x_k\|_p$ and so $\|B\| \le 1$. Then

$$\pi(u) \ge \langle u,B\rangle = \sum_{i=1}^{\infty}|a_i|$$

and we are done.

□

Remark 2.3.7. *Definition 2.3.3 gives the classical Rademacher functions for the case $n=2$ and these results are those in Example 2.3.1.*

Our proof is based in the fact that the orthogonally additive polynomials are isometrically isomorphic to the dual of the tensor diagonal $D_{k,p}$. We need a previous lemma:

Lemma 2.3.8. *Let $1 \leq p < \infty$. The dual of the tensor diagonal $D_{n,p}^*$ is isometrically isomorphic to ℓ_∞ if $1 \leq p \leq n$ and to $\ell_{p/p-n}$ if $n < p < \infty$ in the sense that for every $F \in D_{n,p}^*$, $(F(e_i \otimes \cdots \otimes e_i))$ is in ℓ_∞ for the first case and in $\ell_{p/p-n}$ for the second.*

Proof. The proof is standard, observe that $\ell_{p/p-n}$ is the dual of $\ell_{p/n}$ and carry on the same proof of $\ell_q^* = \ell_{q'}$ for $1/q + 1/q' = 1$ with the identification of the projective norm shown in Theorem 2.3.6. □

We are now ready to prove the theorem:

Theorem 2.3.9 (Ibort, Linares and Llavona [ILLl3]). *Let $1 \leq p < \infty$. The space of orthogonally additive, n-homogeneous polynomials $\mathcal{P}_o(^n\ell_p)$ is isometrically isomorphic to ℓ_∞ for $1 \leq p \leq n$ and to $\ell_{p/p-n}$ for $n < p < \infty$.*

Proof. The proof consists of showing that $\mathcal{P}_o(^n\ell_p)$ is isometrically isomorphic to $D_{n,p}^*$. We will suppose that $n < p < \infty$. The other case is analogous.

Let $F \in D_{n,p}^*$, the correspondence is established by associating F to the polynomial $P(x) = \widetilde{F}(x \otimes \cdots \otimes x)$ where $\widetilde{F} \in (\widehat{\bigotimes}_{n,s,\pi}\ell_p)^*$ is defined by

$$\widetilde{F}(e_{k_1} \otimes \cdots \otimes e_{k_n}) = \begin{cases} F(e_{k_1} \otimes \cdots \otimes e_{k_n}) \text{ if } k_1 = \cdots = k_n \\ 0 \text{ otherwise} \end{cases}$$

To see that $\widetilde{F}$ is well defined let $x = \sum x_k e_k \in \ell_p$, then

$$\begin{aligned} |\widetilde{F}(x \otimes \cdots \otimes x)| &\leq \sum |x_{k_1} \cdots x_{k_n} \widetilde{F}(e_{k_1} \otimes \cdots \otimes e_{k_n})| \\ &= \sum |x_k^n F(e_k \otimes \cdots \otimes e_k)| \\ &\leq \|(x_k^n)\|_{p/n} \|(F(e_k \otimes \cdots \otimes e_k))\|_{p/p-n} \\ &\leq \|F\| \|x\|_p^n = \|F\| \pi(x \otimes \cdots \otimes x) \end{aligned}$$

hence $\widetilde{F}$ is continuous and $\|\widetilde{F}\| \leq \|F\|$, then we have the equality since $\widetilde{F}$ was an extension of F.

To see that $P(x) = \widetilde{F}(x \otimes \cdots \otimes x)$ is orthogonally additive, note that is enough to check that the n-linear symmetric form ϕ associated to P, verifies that $\phi(e_{k_1}, \ldots, e_{k_n})$ is zero whenever at least two of its entries will be different and this is true by definition of $\widetilde{F}$.

As the correspondence between polynomials and the dual of the symmetric tensor product is an isometric isomorphism, $\|P\| = \|\widetilde{F}\| = \|F\|$ which completes the proof. □

Finally we will see that there is another possible proof of Theorem 2.3.9 using a result by Zalduendo (see Corollary 1 of [Z]):

Lemma 2.3.10 ([Z]). *Let $n < p$ and let ϕ a continuous n-linear form on ℓ_p. Then*

$$(\phi(e_k, \ldots, e_k)) \in \ell_{p/p-n}.$$

From this result, it can be shown as well that the space of orthogonally additive polynomials is isometrically isomorphic to $\ell_{p/p-n}$ if $n < p$. We don't repeat the proof since the ideas are essentially the same, however the proof presented here is self-contained.

Remark 2.3.11. *This method will also be valid for $1 \leq p \leq n$ since in this case, trivially $(\phi(e_k, \ldots, e_k)) \in \ell_\infty$. It is shown in [Z] that this is the best characterization.*

2.4 Orthogonally Additive Polynomials in Hilbert lattices.

In this section we will work with H a real separable Hilbert space. Its inner product will be denoted by $(\cdot, \cdot)_H$ although we will only write the subscript H to avoid confusion.

If (e_n) is an orthonormal basis of H, as in the case of ℓ_2, we can define the coordinate order in H: $x \leq y$ for $x, y \in H$ if $x_n = (x, e_n) \leq y_n = (y, e_n)$. On the other hand, if H is a space of functions defined in some measure space, it can be defined as well a pointwise order given by $x \leq y$ if and only if $x(t) \leq y(t)$ for almost every t. These two orders can coincide as in the case of ℓ_2 but in general they are different as in the case of $L^2[0, 1]$ where the coordinate order is given by the Rademacher functions.

A Hilbert space together with a Banach lattice order will be called a Hilbert lattice. Note that the coordinate order is always a Banach lattice order but in general, the pointwise order is not. This is the content of the following example:

Example 2.4.1. *Let $H = W^{1,2} = \{f \in L^2 : f' \in L^2\}$ the Sobolev space of square integrable functions with weak derivatives square integrable as well. For details about the theory of Sobolev spaces, see [A]. This space is a Hilbert space and its inner product is:*

$$(f,g)_{1,2} = \int fg + f'g' \mathrm{d}\mu \tag{2.1}$$

Lets assume that there exists a (Banach lattice) pointwise order. In that case, the polynomial $P(f) = (f,f)_{1,2} = \|f\|_{1,2}^2$ *would be orthogonally additive and the representation theorem would imply the existence of a continuous linear form* L *such that* $P(f) = L(f^2)$. *Then* $(f,g)_{1,2} = L(fg)$ *where* fg *is the usual product of functions. This leads to a contradiction because the inner product (2.1) is not symmetric with respect to the product of functions that is* $(fh,g)_{1,2} \neq (f,gh)_{1,2}$ *for arbitrary* $f,g,h \in W^{1,2}$.

Thus the representation theorem in a Hilbert space, provides a criterion to determine if in a Hilbert space of functions a pointwise order inducing a Banach lattice structure can be defined. The key here is that it is necessary that the inner product would be symmetric with respect to the usual product of functions since from the representation theorem there is a linear functional L such that $(f,g) = L(fg)$ and the second member has the symmetry property.

We begin our study of the theory of orthogonally additive polynomials in Hilbert lattices with a straightforward generalization of Sundaresan's results to the case of a general Hilbert lattice:

Proposition 2.4.2 (Ibort, Linares and Llavona [ILL14]). *Given* H *a separable Hilbert space,* $(\cdot,\cdot)$ *the inner product defining its norm and* (e_n) *an orthonormal basis which induces in* H *a coordinate order, then every* m*-homogeneous orthogonally additive polynomial* $P \in \mathcal{P}_o({}^mH)$, $m \geq 2$ *has the form*

$$P(x) = \sum a_n x_n^m$$

with (a_n) *in* ℓ_∞ *and* $x_n = (x,e_n)$ *the Fourier coefficients of* x.

Proof. The proof follows immediately from the representation theorem since if P is orthogonally additive in H and Φ is the natural isometric isomorphism from H to ℓ_2 defined by $\Phi(x) = \{x_n = (x,e_n)\}_n$, we have that $P_\Phi(\{x_n\}) := P(\sum x_n e_n)$ is m-homogeneous and orthogonally additive in ℓ_2 and the representation theorem says that those polynomials take the form

$$P_\Phi(\{x_n\}) = \sum a_n x_n^m$$

for some $a = (a_n) \in \ell_\infty$. □

Note that in particular, the polynomial $P(x) = \|x\|^2$ is orthogonally additive. This result can be generalized for any Hilbert lattice since if $x, y \in H$ are disjoint respect to any Hilbert lattice order, then $|x+y| = |x-y|$ and hence $\|x+y\| = \|x-y\|$. Consequently $(x, y) = \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2)$. This proof can be seen for instance in Corollary 2.7.5 of [MN].

Remark 2.4.3. *In Chapter 4 we will be interested in that result for the case* $m = 2$. *From the proof of the representation theorem, it is not difficult to check that if* $P \in \mathcal{P}_o(^2H)$ *there is a linear continuous application* L *defined in* $\{x = \sum x_n e_n : (x_n) \in \ell_1\}$ *verifying* $\mathcal{P}(x) = L(x^2)$ *and its associated bilinear form* B *satisfies* $B(x, y) = L(x \cdot y)$ *every* $x, y \in H$.

Note that the product $x \cdot y$ *is the product of the Fourier coefficients or Hadamard product, that is:*

$$x \cdot y = \sum x_n y_n e_n \tag{2.2}$$

which belongs to H *thanks to the Cauchy-Schwarz inequality.*

The relation between the polynomial and the linear form becomes clearer from the proposition 2.4.2. If the polynomial $\mathcal{P}$ *is given by the sequence* $a = (a_n) \in \ell_\infty$, *then* L *is defined by*

$$L(x) = \sum a_n x_n$$

and obviously $\|L\| \leq \|a\|_\infty$.

We also have

Proposition 2.4.4 (Ibort, Linares and Llavona [ILL14]). *Let* $H, (\cdot, \cdot)$ *be a Hilbert space and* B *a positive definite continuous bilinear application. Assume that* $T : H \longrightarrow H$ *is a bounded self-adjoint operator with respect to* B *(that is* $B(Tx, y) = B(x, Ty)$*). Then the polynomial* $P(f) = B(f, f)$ *is orthogonally additive with respect to the order induced by any orthonormal basis of eigenvectors of* T *containing at least three eigenvectors associated to three different eigenvalues.*

Proof. Let (e_n) be an orthonormal basis of eigenvectors of T, we have $Te_n = \lambda_n e_n$.

Since $P(x + y) = P(x) + 2B(x, y) + P(y)$, we just need to show that for disjoint x and y (in the order given by the basis) $B(x, y) = 0$. As $B(x, y) =$

$\sum_{n,m=0}^{\infty} x_n y_m B(e_n, e_m)$ and $x_n y_n = 0$, is enough to prove that $B(e_n, e_m) = 0$ if $n \neq m$.

Suppose, on the first place, that $\lambda_m \neq \lambda_n \neq 0$, then

$$\begin{aligned} B(e_n, e_m) &= \frac{1}{\lambda_n} B(\lambda_n e_n, e_m) = \frac{1}{\lambda_n} B(Te_n, e_m) \\ &= \frac{1}{\lambda_n} B(e_n, Te_m) = \frac{\lambda_m}{\lambda_n} B(e_n, e_m) \end{aligned}$$

and as $\lambda_n \neq \lambda_m$, necessarily $B(e_n, e_m) = 0$. If $\lambda_n = 0$ the result is symmetric. Let now $\lambda_m = \lambda_n$, choosing k such that $\lambda_k \neq 0, \lambda_m$ then $B(e_n, e_m) = B(e_n - e_k, e_m) + B(e_k, e_m)$ and the result follows like above for each member.

That allows us to conclude that for any disjoint x and y, $B(x, y) = \sum_{n=0}^{\infty} x_n y_n = 0$ and so P is orthogonally additive. □

Now we turn our attention to the following problem:

Problem 2.4.5. *Is there a Hilbert lattice H different from L^2.*

In this context, we assume that two lattices are different if they are not order isomorphic. Recall that we say $T : X \longrightarrow Y$ where X and Y are Banach lattices is an order isomorphism if it is an isomorphism with preserves the lattice structure, that is

$$T(x \wedge y) = T(x) \wedge T(y) \text{ and } T(x \vee y) = T(x) \vee T(y).$$

We will need some results of the Banach lattice theory which can be found for instance in [LT].

Let H be a separable Hilbert space. We need to show that H is order isometric to a Köthe function space and then use Corollary 2.2.5.

From Theorem 1.b.14 in [LT], a separable Banach lattice is order isometric to a Köthe function space if an only if it is σ-order complete (that is, any order bounded sequence has a least upper bound). It is proved also in [LT] pages 3-4 that the dual of a Banach lattice (which turns to be a Banach lattice with the order given by $x^* \geq 0$ if and only if $x^*(x) \geq 0$ for every $x \geq 0$) has the property that every non-empty order bounded set has a least upper bound. Hence, as the canonical embedding from X into X^{**} is an order isometry ([LT] Proposition 1.a.2), we have that every reflexive lattice is σ-complete.

The next result solves our problem:

Theorem 2.4.6. *Let H be a separable real Hilbert. Suppose H is a Banach lattice with order $\leq$. Then H is order isometric to $L^2(\mu)$ for certain measure μ.*

Proof. As we have seen previously, H is order isometric to a Köthe function space X. A σ-complete Banach lattice with is not σ-order continuous contains a subspace isomorphic to ℓ_∞ (Proposition 1.a.7 [LT]). Then as H and hence X are separable, they are σ-order continuous and order continuous.

Then X is an order continuous Köthe function space over a measure space $(\Omega, \Sigma, \tilde{\mu})$. Furthermore, it is a Hilbert space. Denote $P(f) = \|f\|_X^2$. As we saw previously P is orthogonally additive. By Corollary 2.2.5 there is $\xi \geq 0$ measurable such that

$$P(f) = \int f^2 \xi \mathrm{d}\tilde{\mu}$$

hence $X = L^2(\mu)$ with $\mathrm{d}\mu = \xi \mathrm{d}\tilde{\mu}$. □

Remark 2.4.7. *This result is a particular case of the following Theorem by Kakutani:*

Theorem 2.4.8 (Kakutani [K]). *An abstract L^p space X, $1 \leq p < \infty$ is order isometric to an $L_p(\mu)$ space over some measure space (Ω, Σ, μ).*

Where an abstract L^p space is a Banach lattice X for which $\|x+y\|^p = \|x\|^p + \|y\|^p$ whenever $x, y \in X$ and $x \wedge y = 0$.

2.5 Spectral Theorem of bounded self-adjoint operators in Hilbert spaces.

Recall that a linear bounded operator $T : H \longrightarrow H$ is said to be self-adjoint if $(Tx, y) = (x, Ty)$ for every $x, y \in H$. The spectral theorem states that every such operator is, up to a change of variable given by an isometric isomorphism, a multiplication operator. There are many proofs of this classical result, the most widely known uses the theory of spectral measures and the integration with respect to a spectral measure (see for instance [W]). We present here a new proof which does not use that theory and relies only on the properties of orthogonally additive polynomials and the representation theorem:

Theorem 2.5.1. *Let T a bounded self-adjoint operator on a Hilbert space H. There is an isometric isomorphism $U\colon H \longrightarrow \bigoplus_\alpha L^2(\mu_\alpha)$ where μ_α is a positive Borelian measure on $\mathbb{R}$ with compact support and a multiplication operator $\tilde{T}_\alpha$, $\tilde{T}_\alpha f(s) = sf(s)$, $f \in L^2(\mu_\alpha)$, such that $T = U^{-1}(\bigoplus_\alpha \tilde{T}_\alpha)U$.*

Proof. As a self-adjoint operator in a Hilbert space splits as a sum of operators on disjoint cyclic Hilbert spaces, it suffices to prove the statement assuming that there exists a cyclic vector u for the operator T on H. This is, we will assume that $H = \overline{\mathcal{A}(T)u}$ where

$$\mathcal{A}(T) = \{p(T) = p_0 + p_1 T + \cdots + p_n T^n : n \in \mathbb{N}\}$$

is the Abelian algebra of operators generated by T and the closure is taken with respect to the operator norm.

Because $T\colon H \to H$ is a bounded self-adjoint operator, its spectrum $\sigma(T)$ is real and is contained on the interval $[-\|T\|, \|T\|]$. We shall denote by $K = \sigma(T)$ and by $\mathcal{P}(K)$ the set of algebraic polynomials on K.

The map $U\colon \mathcal{A}(T)u \to \mathcal{P}(K)$ defined by $U(p(T)u) = p(s)$ allows to define a pre-Hilbert structure on $\mathcal{P}(K)$ as

$$\langle p, q\rangle_0 = (p(T)u, q(T)u)_H.$$

The bilinear form $\langle \cdot, \cdot\rangle_0$ is continuous with respect to the uniform topology. In fact a simple computation shows that $|\langle p, q\rangle_0| \leq C\|p\|_\infty \|q\|_\infty$. Hence by Weierstrass theorem the bilinear form $\langle \cdot, \cdot\rangle_0$ can be extended to a bilinear form on the Banach lattice of continuous functions on K. If we denote by P_0 the 2-homogenous polynomial on $C(K)$ defined by $\langle \cdot, \cdot\rangle_0$, this is:

$$P_0(f) = \langle f, f\rangle_0,$$

then we will show that P_0 is orthogonally additive with respect to the pointwise order on $C(K)$, in other words we will show that if $|f| \wedge |g| = 0$, then $P_0(f + g) = P_0(f) + P_0(g)$. We shall consider two sequences of polynomials (f_n), (g_n) on K converging uniformly to f and g respectively. Thus $(f_n + g_n)$ will converge uniformly to $f + g$.

Now $P_0(f_n + g_n) = P_0(f_n) + P_0(g_n) + 2\langle f_n, g_n\rangle_0$. However by construction:

$$\begin{aligned} \langle f_n, g_n\rangle_0 &= (f_n(T)u, g_n(T)u)_H \\ &= (u, f_n(T)g_n(T)u)_H = \langle 1, f_n g_n\rangle_0, \end{aligned}$$

because T is self-adjoint. But now, it is clear that $f_n g_n \to fg$ uniformly because:

$$\|f_n g_n - fg\|_\infty \leq \|f_n - f\|_\infty \|g_n\|_\infty + \|f\|_\infty \|g_n - g\|_\infty.$$

Hence we conclude that:

$$P_0(f+g) = \lim_{n\to\infty} P_0(f_n + g_n) = P_0(f) + P_0(g) + 2 \lim_{n\to\infty} \langle f_n, g_n\rangle_0,$$

and

$$\lim_{n\to\infty} \langle f_n, g_n\rangle_0 = \lim_{n\to\infty} \langle 1, f_n g_n\rangle_0 = \langle 1, fg\rangle_0,$$

which vanishes if $|f| \wedge |g| = 0$, because then $fg = 0$.

As we saw in Remark 2.2.6, there will exists a Borel measure μ on $\mathbb{R}$ with support on K such that:

$$P_0(f) = \int_K f(s)^2 d\mu(s).$$

Notice that the continuous map U being defined on a dense subspace of H can be uniquely extended to a linear isometry from H to $L^2(\mu)$. Now because $U(Tp(T)u) = sp(s)$ for all polynomials $p(s)$ on K, we obtain the spectral representation of T, $T = U^{-1}\tilde{T}U$, where $\tilde{T}f(s) = sf(s)$. Then it is also obvious that the 2-homogenous polynomial $P_T(x) = (x, Tx)_H$ can be represented as

$$P_T(x) = \int f(s)^2 s d\mu(s),$$

where $U(x) = f \in L^2(\mu)$. □

Chapter 3

Orthogonally Additive Polynomials on Riesz Spaces.

In parallel to the study of the orthogonally additive polynomials in Banach lattices, in the last years there has been several authors interested in the study of an analogous concept, for bilinear forms on Riesz spaces.

In particular, one of the milestones of f-algebras, has been to prove that every such space is a commutative algebra. This result was the main motivaion for Buskes and van Rooij to introduce in 2000 the concept of orthosymmetric bilinear mapping in a Riesz space (see [BvR1]). As we will see in Section 3.2, they obtained a new proof of the commutativity of f-algebras by showing that the orthosymmetric positive mappings are symmetric.

In 2004 the same authors studied the representation of orthosymmetric mappings on Riesz spaces introducing the notion of the square of a Riesz space and giving several characterizations of it in their article [BvR3]. The square works in a similar way as the concavification for Banach lattices. In fact, one of the characterizations given by Buskes and van Rooij develops this idea.

The n-linear case was studied by Boulabiar and Buskes [BB] in 2006 who proved similar characterizations for the notion of the n-power of a Riesz space.

The aim of this Chapter is to provide a representation theorem for orthogonally additive polynomials on Riesz spaces independent of the representation theorem in Banach lattices.

We will consider the problem of defining a proper generalization of orthosymmetric n-linear form. As far as we know there are three different definitions, two of them appear in Loane's Thesis [L] and the definition of Boulabiar and Buskes. For our purposes, we need that the conditions of orthosymmetry and positiveness will be sufficient for the symmetry as in the linear case and that if P is a polynomial defined by a multilinear form A, the orthosymmetry for A will be equivalent to the fact that P is orthogonally additive. As we will see, the definitions in Loane's Thesis do not fulfill our requirements and the one of Boulabiar and Buskes does. However, the proof of this fact given in Theorem 3.2.9 will use the representation theorem in Banach lattices. Hence, we define the orthosymmetry in a new way that allows us to prove what we need without using the representation theorem.

3.1 Riesz spaces.

We begin this chapter with a brief introduction to the theory of Riesz spaces based mainly on the references [AB] and [JvR]. Our reference for the theory of polynomials on Riesz spaces is Loane's Thesis [L].

Let E be a real vector space and $\leq$ an order relation compatible with the algebraic operations in E, that is $x + z \leq y + z$ whenever $x \leq y$ and any $z \in E$, and $0 \leq ax$ for every $x \in E$ and nonnegative real a. Such E is called an ordered vector space.

Definition 3.1.1. *An ordered vector space E is called Riesz space, if for every $x, y \in E$ there exists a least upper bound $x \vee y$ and a greatest lower bound $x \wedge y$.*

Like in the case of Banach lattices, we define the absolute value of $x \in E$ as $|x| = x \vee (-x)$. An element $x \in E$ is said to be positive if $x \geq 0$. The positive cone is the space

$$E^+ = \{x \in E : x \geq 0\}.$$

Note that every $x \in E$ can be decomposed as the difference of two positive elements $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$. Furthermore, $|x| = x^+ + x^-$.

A Riesz space will be called Archimedean if for every $x, y \in E^+$, such that $0 \leq nx \leq y$ for every $n \in \mathbb{N}$ we have that $x = 0$.

Let E and F be two Riesz spaces a $T : E \longrightarrow F$ an operator. There is no concept of continuity for operators defined on Riesz spaces however, the notion of positivity, allows us to obtain interesting results. The operator T is said to be positive if $Tx \geq 0$ for every $x \in E^+$. An operator $T : E \longrightarrow F$ is a Riesz homomorphism if $T(x \vee y) = T(x) \vee T(y)$ for every $x, y \in E$. Note that since $x+y = x \wedge y + x \vee y$, $T(x \wedge y) = T(x) \wedge T(y)$ as well. Furthermore, every Riesz homomorphism is positive. A bijective Riesz homomorphism is a Riesz isomorphism and E and F are said to be Riesz isomorphic if there exists a Riesz isomorphism between them.

A Riesz subspace of E is a linear subspace closed for the operation of taking supremum and infimum. An ideal is a Riesz subspace I of E such that if $|x| \leq |y|$ for $x \in E$ and $y \in I$ then $x \in I$. The ideal generated by $y \in E$ is the subspace

$$\{x \in E : |x| \leq \lambda |y| \text{ for some } \lambda \geq 0\}.$$

Ideals of this form are called principal ideals.

We will need the following Theorem by Yosida (see for instance [JvR] Theorem 13.11):

Theorem 3.1.2. *Let E be a Riesz space and e be a unit in E. Then E is Riesz isomorphic to a subspace of $C(K)$ for some compact Hausdorff K.*

An element $e \in E^+$ is a (strong) unit if the principal ideal generated by e equals to E.

As in the Banach space case, a mapping between two Riesz spaces $P : E \longrightarrow F$ is said to be a n-homogeneous polynomial if there exists a n-linear form $A : E \times \ldots \times E \longrightarrow F$ such that $P(x) = A(x, \ldots, x)$. The correspondence between polynomials and symmetric n-linear forms is univocal.

Given $E_1, \ldots, E_n, F$ Riesz spaces, a multilinear form $A : E_1 \times \ldots \times E_n \longrightarrow F$ is said to be positive if $A(x_1, \ldots, x_n) \geq 0$ for every $x_1, \ldots, x_n$ positive elements. A polynomial is positive if its associated symmetric multilinear mapping is positive. A linear, multilinear mapping or a polynomial is regular if it can be decomposed as the difference of two positive multilinear mappings or two positive polynomials respectively.

As we have seen in Chapter 2, the theory of tensor products of Banach spaces is an important tool for the study of multilinear mappings and polynomials. In Riesz spaces, we have the notion of Fremlin tensor product which linearizes bimorphisms. We discuss now the main results on Fremlin tensor products (see for instance the paper by Fremlin [Fre] or the survey by Schaefer [Sch]). We will need directly or indirectly these results which are the key for the theory of squares and powers of Riesz spaces that will be used later.

We begin with the definition of bimorphism as given by Fremlin:

Definition 3.1.3 (Definition 3.1 [Fre]). *A bilinear mapping $B : E \times F \longrightarrow G$ between Riesz spaces is a (Riesz) bimorphism if the maps*

$$B_y : E \longrightarrow G,\ B_y(z) = B(z, y)$$
$${}_xB : F \longrightarrow G,\ {}_xB(z) = B(x, z)$$

are Riesz homomorphisms for every $x \in E^+$ and every $y \in F^+$.

The main result of Fremlin's paper reads as follows:

Theorem 3.1.4 (Theorem 4.2 [Fre]). *Let E and F be Archimedean Riesz spaces. Then there is an essentially unique Archimedean Riesz space G and Riesz bimorphism $\phi : E \times F \longrightarrow G$ such that:*

1. *Whenever H is an Archimedean Riesz space and $\psi : E \times F \longrightarrow H$ is a Riesz bimorphism, there is a unique Riesz homomorphism $T : G \longrightarrow H$ such that $T\phi = \psi$.*

2. *ϕ induces an embedding of $E \otimes F$ in G.*

3. *$E \otimes F$ is dense in G in the sense that for every $w \in G$ there exists $x_0 \in E$ and $y_0 \in F$ such that for every $\delta > 0$ there is a $v \in E \otimes F$ such that $|w - v| \leq \delta(x_0 \otimes y_0)$.*

4. *If $w > 0$ in G, then there exist $x \in E^+$ and $y \in F^+$ such that $0 < x \otimes y < w$.*

We will denote by $E\overline{\otimes}F$ the Archimedean Riesz space G.

We are not interested in bimorphisms but in positive mappings. Fremlin studied this case as well. As the bimorphisms are positive he introduced an additional hypothesis in the target space:

Theorem 3.1.5 (Theorem 5.3 [Fre]). *Let E and F be Archimedean Riesz spaces and H a relatively uniformly complete Archimedean Riesz space. Let $\psi : E \times F \longrightarrow H$ be a positive bilinear map. Then there exists a unique positive linear map $T : E\overline{\otimes}F \longrightarrow H$ such that $T\phi = \psi$.*

Where H is called relatively uniformly complete if, for every $x \in H$ and (y_n) sequence in H such that $|y_n - y_m| \leq 2^{-n}x$ for every $m \geq n$, then there exists a $y \in H$ such that $|y_n - y| \leq 2^{-n}x$ for every $n \in \mathbb{N}$.

3.2 Orthosymmetric Applications on Riesz spaces.

We begin with the definition of orthosymmetric bilinear mapping as it appears in [BvR1]:

Definition 3.2.1. *Let E and F be Riesz spaces, a bilinear application $A : E \times E \longrightarrow F$ is said to be orthosymmetric if for every $x, y \in E$ such that $x \wedge y = 0$, then $A(x, y) = 0$.*

It can be proved that this definition is equivalent to the fact that $A(x, y) = 0$ whenever $|x| \wedge |y| = 0$. Hence, a 2-homogeneous polynomial P is orthogonally additive if and only if its associated bilinear form is orthosymmetric.

As we said in the introduction, Buskes and van Rooij introduced the concept of orthosymmetry in the setting of f-algebras. These spaces are a particular type of Riesz algebras, that is a Riesz space together with a multiplication which makes the Riesz space an algebra that satisfies $xy \geq 0$ for every $x, y \geq 0$. Among the Riesz algebras, E is said to be an f-algebra whenever $x \wedge y = 0$ implies $(xz) \wedge y = (zx) \wedge y = 0$ for every $z \in E^+$.

It is known that every f-algebra is commutative. This result due to Amemiya and independently to Birkhoff and Pierce, can be found for instance in [AB], Theorem 2.56.

As in any f-algebra, two disjoint elements x and y verify that $xy = 0$, the bilinear mapping $B(x, y) = xy$ is orthosymmetric. That was the key point in the work by Buskes and van Rooij who proved that every positive orthosymmetric bilinear mapping is symmetric and in particular that every f-algebra is commutative. This result is obtained as a corollary of the following theorem:

Theorem 3.2.2 (Theorem 1 [BvR1]). *Let K be a compact Hausdorff space, E a Riesz subspace of $C(K)$ uniformly dense, F an Archimedean Riesz space and $A : E \times E \longrightarrow F$ orthosymmetric positive bilinear form. If we denote by E^2 the span of the set $\{xy : x, y \in E\}$ then there exists a positive linear application $B : E^2 \longrightarrow F$ such that $A(x, y) = B(xy)$ for $x, y \in C(K)$.*

Buskes and van Rooij introduced as well the square of a Riesz space:

Definition 3.2.3 (Definition 3 [BvR3]). *Let E be a Riesz space. $(E^{\odot}, \odot)$ is said to be a square of E if $E^{\odot}$ is a Riesz space and*

- *$\odot : E \times E \longrightarrow E^{\odot}$ is an orthosymmetric bimorphism.*

- *For every Riesz space F and an orthosymmetric bimorphism $T : E \times E \longrightarrow F$, there exists a unique Riesz homomorphism $T^{\odot} : E^{\odot} \longrightarrow F$ such that $T^{\odot} \circ \odot = T$.*

Using the Fremlin tensor product, it was proved that every Riesz space has a square and that it is unique up to Riesz isomorphism. Furthermore, it was shown that if E is a uniformly complete Riesz subspace of a semiprime f-algebra with multiplication $\cdot$, then $(E^2, \cdot)$ is a square. Finally, they used the functional calculus to prove that the 2-concavification of E is as well a square.

We study now the possible definitions of orthosymmetry for multilinear mappings that will generalize the definition of Buskes and van Rooij. Recall that for this purpose we need a definition equivalent to the orthogonally additivity for polynomials and we need as well that Theorem 3.2.2 can be generalized.

The first approximation, as it appears in Loane's Thesis, can be as follows, a multilinear mapping $A : E \times \ldots \times E \longrightarrow F$ would be called orthosymmetric if $A(x_1, \ldots, x_n) = 0$ for any $x_1, \ldots, x_n \in E$ pairwise disjoint. It is observed by Loane that this definition does not imply that the associated polynomial is orthogonally additive as it is shown by the following example:

Example 3.2.4 (Example 4.34 [L]). *Consider $\mathbb{R}^2$ with the order given by coordinates. The 3-linear application defined as*

$$A(x, y, z) = x_1 y_1 z_2 + x_1 y_2 z_2 + x_2 y_1 z_2$$

would be orthosymmetric with the previous definition (note that every 3-linear application would be) but the associated polynomial is not orthogonally additive.

Loane suggests the following definition:

Definition 3.2.5 (Definition 4.35 [L]). *Let E, F be Riesz spaces. A n-linear mapping $A : E \times \ldots \times E \longrightarrow F$ is said to be b-orthosymmetric if $A(x^k, y^{n-k}) = 0$ for $0 < k < n$ whenever $x, y \in E$ verify $x \wedge y = 0$.*

The notation $A(x^k, y^{n-k})$ stands for $A(x, \ldots, x, y, \ldots, y)$ where x appears k times and y appears $n - k$ times. Loane proves:

Proposition 3.2.6 (Proposition 4.38 [L]). *Let $P = \hat{A}$ be a n-homogeneous polynomial on a Riesz space E. Then P is orthogonally additive if and only if A is b-orthosymmetric.*

This definition is very useful when the multilinear mapping A is symmetric. However, it does not guarante that the positive applications will be symmetric as we will need:

Example 3.2.7. *Again in $\mathbb{R}^2$ with the order given by coordinates, the 3-linear form given by*

$$A(x,y,z) = x_1 y_1 z_1 + x_2 y_1 z_2$$

verifies for $x \wedge y = 0$ that $A(x,x,y) = A(x,y,y) = 0$ but it is not symmetric.

On the other hand, the definition of Boulabiar and Buskes appeared in his article [BB] reads as follows:

Definition 3.2.8. *A n-lineal application $A : E \times \ldots \times E \longrightarrow F$ is said to be b-orthosymmetric if it is positive and $A(x_1, \ldots, x_n) = 0$ if $x_1, \ldots, x_n \in E^+$ and whenever there is $i, j \in \{1, \ldots, n\}$ such that $x_i \wedge x_j = 0$.*

This definition can be reformulated equivalently as $A(x_1, \ldots, x_n) = 0$ if $x_1, \ldots, x_n \in E$ verifies $|x_i| \wedge |x_j| = 0$ for some $i, j \in \{1, \ldots, n\}$. The hypothesis of positivity is not essential in the definition but, as we will see, it is necessary if we want to obtain interesting conclusions about the b-orthosymmetric applications. The next result shows the relation between the b-orthosymmetry and the additive orthogonality.

Theorem 3.2.9. *Let E, F be Riesz spaces and $A : E \times \cdots \times E \longrightarrow F$ a symmetric positive multilinear form in E. A is b-orthosymmetric if and only if $P = \hat{A}$ is orthogonally additive.*

Proof. It is clear that if A is b-orthosymmetric, expanding

$$P(x+y) = A(x+y, \ldots, x+y)$$

for disjoint x and y, the unique nonzero terms are $A(x, \ldots, x)$ and $A(y, \ldots, y)$ so P is orthogonally additive.

Assume that P is orthogonally additive. Let $x_1, \ldots, x_n$ be elements of E with at least two disjoint. Consider $E_0 \subset E$ to be the ideal generated by $x_1, \ldots, x_n$ whose unit is $|x_1| + \cdots + |x_n|$. By Yosida's Representation Theorem (Theorem 3.1.2), E_0 is Riesz isomorphic to a dense subspace $\hat{E_0}$ of $C(K)$ for certain compact Hausdorff K. Let $\tilde{P}$ be the polynomial on $\hat{E_0}$ defined by the composition of the previous isomorphism with P. As the isomorphism preserves the order, $\tilde{P}$ is positive and orthogonally additive. Extend $\tilde{P}$ by denseness to a positive orthogonally additive n-homogeneous polynomial on $C(K)$. We denote the extension again by $\tilde{P}$. Note that this extension, being positive it will be also continuous (see [GR] Proposition 4.1).

Hence, by the representation theorem, there exists a measure μ such that

$$\tilde{P}(f) = \int f^n \mathrm{d}\mu,$$

and the symmetric n-linear form associated to $\tilde{P}$ is given by

$$(f_1, \ldots, f_n) \longmapsto \int f_1 \cdots f_n \mathrm{d}\mu,$$

then, this form as well as its restriction to $\hat{E}_0$ are b-orthosymmetric and by the Riesz isomorphism between E_0 and $\hat{E}_0$, A will be also orthosymmetric. □

As we have seen, in the proof of this theorem we have employed the representation theorem. Due to our objective of an independent representation theorem of polynomials on Riesz spaces, we cannot use this definition. That is the reason to introduce a new definition with the required properties and that allows us to formulate independent proofs. Our definition can be seen artificial at first sight but later our procedure will become clearer. To begin with, we introduce the concept of p-disjoint elements:

Definition 3.2.10. *We will say that $x_1, \ldots, x_n \in E$ are partitionally disjoint or p-disjoint it there exists a partition $I_1, \ldots, I_m$, $2 \leq m \leq n$ of the index set $I = \{1, \ldots, n\}$ such that the sets*

$$\{x_{i_k} : i_k \in I_k\}$$

are disjoint, that is $|x_{i_k}| \wedge |x_{i_l}| = 0$ whenever $k \neq l$.

Definition 3.2.11. *Given E, F Riesz spaces, a n-linear application $A : E \times \cdots \times E \longrightarrow F$ is said to be p-orthosymmetric if $A(x_1, \ldots, x_n) = 0$ for $x_1, \ldots, x_n$ p-disjoint.*

In order to prove that this new definition is coherent with the additive orthogonality of polynomials, we need a previous lemma:

Lemma 3.2.12. *Let E, F be Riesz spaces and $A : E \times \cdots \times E \longrightarrow F$ a symmetric n-linear form. The following assumptions are equivalent*

1. *A is p-orthosymmetric.*

2. $A(x^i, y^{n-i}) = 0$ *for x and y disjoint and* $1 < i < n$.

Proof. If A is p-orthosymmetric, it is obvious that $A(x^i, y^{n-i}) = 0$ for disjoint x and y with $1 < i < n$.

Reciprocally, let $\{x_1, \ldots, x_n\}$ be p-disjoint. Assume that the partition in disjoint subsets is given by two elements, that is there exists some $1 < i < n$ such that the sets $\{x_1, \ldots, x_i\}$ and $\{x_{i+1} \ldots, x_n\}$ are disjoints. The general case is analogous.

The notation $B = A_{x_{i+1},\ldots,x_n}$ will represent the multilinear form B defined by $B(y_1, \ldots, y_i) = A(y_1, \ldots, y_i, x_{i+1}, \ldots, x_n)$. By the polarization formula (Theorem 13).

$$A(x_1, \ldots, x_n) = B(x_1, \ldots, x_i) = \frac{1}{i!2^i} \sum_{\varepsilon_j = \pm 1} \varepsilon_1 \cdots \varepsilon_i B(\varepsilon_1 x_1 + \cdots + \varepsilon_i x_i)^i$$

where $B(x)^i = B(x, \overbrace{\ldots}^{i}, x)$.

We conclude showing that for each election of signs $\varepsilon_j = \pm 1$,

$$B(\varepsilon_1 x_1 + \cdots + \varepsilon_i x_i)^i = 0.$$

In order to simplify, let $\varepsilon \mathbf{x} = \varepsilon_1 x_1 + \cdots + \varepsilon_i x_i$, note that

$$\begin{aligned} B(\varepsilon \mathbf{x})^i &= B(\varepsilon \mathbf{x}, \ldots, \varepsilon \mathbf{x}) = A(\varepsilon \mathbf{x}, \ldots, \varepsilon \mathbf{x}, x_{i+1}, \ldots, x_n) = \\ & A_{\varepsilon \mathbf{x}, \ldots, \varepsilon \mathbf{x}}(x_{i+1}, \ldots, x_n) = C(x_{i+1}, \ldots, x_n) \end{aligned}$$

with the notations above.

Again by using the polarization formula, we get:

$$C(x_{i+1}, \ldots, x_n) = \frac{1}{(n-i)!2^{n-i}} \sum_{\delta_k = \pm 1} \delta_1 \cdots \delta_{n-i} C(\delta_1 x_{i+1} + \cdots + \delta_{n-i} x_n)^{n-i}$$

as $\varepsilon_1 x_1 + \cdots + \varepsilon_i x_i$ and $\delta_1 x_{i+1} + \cdots + \delta_{n-i} x_n$ are disjoint for every choice of signs $\varepsilon_j = \pm 1$ and $\delta_k = \pm 1$ our assumption allows us to conclude that the summands appearing in the previous expressions are zero and hence $A(x_1, \ldots, x_n) = 0$, as needed. □

Now we can prove:

Theorem 3.2.13. *Let E, F be Riesz spaces and $A : E \times \cdots \times E \longrightarrow F$ a symmetric multilinear form in E. A is p-orthosymmetric if and only if $P = \hat{A}$ is orthogonally additive.*

Proof. If A is p-orthosymmetric, as we saw before, expanding $P(x+y) = A(x+y,\dots,x+y)$ with x and y disjoint, the summands different from zero are just $A(x,\dots,x)$ and $A(y,\dots,y)$, hence P is orthogonally additive.

Reciprocally, if P is orthogonally additive and $|x| \wedge |y| = 0$, by Proposition 3.2.6 $A(x^i, y^{n-i}) = 0$ for $1 < i < n$ then, by the previous lemma A is p-orthosymmetric. □

Finally, we will see that the p-orthosymmetry and the positiveness guarantee the symmetry as in the bilinear case. The following lemma generalizes Theorem 1 in [BvR1]. To prove it we will follow a similar path from which it will be clear the reasons for our definition of orthosymmetry.

Lemma 3.2.14. *Let K be a compact Hausdorff space, E a uniformly dense Riesz subspace of $C(K)$, F an Archimedean Riesz space and $T : E \times \ldots \times E \longrightarrow F$ a positive p-orthosymmetric n-linear map. Let E^n the linear hull of the set*

$$\{f_1 \cdots f_n : f_1, \dots, f_n \in E\}.$$

Then there exists a positive linear map $A : E^n \longrightarrow F$ such that

$$T(f_1, \dots, f_n) = A(f_1 \cdots f_n) \text{ for every } f_1, \dots, f_n \in E.$$

Proof. As in [BvR1] there is no loss of generality if we consider $E = C(K)$. Define $A(h) = T(\mathbf{1}, \dots, \mathbf{1}, h)$ where $\mathbf{1}$ stands for the function identically 1. It is clear that A is linear and positive. Note that as T is positive by [GR] Proposition 4.1, it is continuous. We have to prove that if $f_1, \dots, f_n \in C(K)$ and $h = f_1 \cdots f_n$, $T(f_1, \dots, f_n) = T(\mathbf{1}, \dots, \mathbf{1}, h)$.

Given $\varepsilon > 0$, we will say, following [BvR1], that $X \subset K$ is small if for every $x, y \in X$

$$|f_1(x) - f_1(y)| < \varepsilon, \dots, |f_n(x) - f_n(y)| < \varepsilon \text{ and } |h(x) - h(y)| < \varepsilon.$$

Take $u_1, \dots, u_N \in C(K)^+$ verifying $\sum u_n = \mathbf{1}$ and such that for every j, the set $S_j = \{x \in K : u_j(x) \neq 0\}$ is small and nonempty, and take $s_j \in S_j$, for $j = 1, \dots, N$. Define for $i = 1, \dots, n$,

$$f_i' = \sum_{j=1}^{N} f(s_j)u_j \text{ and } h' = \sum_{j=1}^{N} h(s_j)u_j.$$

Because

$$\begin{aligned} |T(f_1,\ldots,f_n) - T(\mathbf{1},\ldots,\mathbf{1},h))| &\leq |T(f_1,\ldots,f_n) - T(f_1',\ldots,f_n')| \\ &+ |T(f_1',\ldots,f_n') - T(\mathbf{1},\ldots,\mathbf{1},h'))| \\ &+ |T(\mathbf{1},\ldots,\mathbf{1},h')) - T(\mathbf{1},\ldots,\mathbf{1},h))| \end{aligned}$$

we have to bound these three terms. The second is immediate:

$$|T(f_1',\ldots,f_n') - T(\mathbf{1},\ldots,\mathbf{1},h'))| \leq \|h-h'\|_\infty |T(\mathbf{1},\ldots,\mathbf{1})| \leq \varepsilon T(\mathbf{1},\ldots,\mathbf{1}).$$

For the first

$$\begin{aligned} &|T(f_1,\ldots,f_n) - T(f_1',\ldots,f_n')| \leq |T(f_1,\ldots,f_n) - T(f_1',f_2\ldots,f_n)| + \\ &|T(f_1',f_2,\ldots,f_n) - T(f_1',f_2',f_3\ldots,f_n)| + \cdots + |T(f_1',\ldots,f_{n-1}',f_n) \\ &-T(f_1',\ldots,f_n')| = \sum_{i=1}^{n} |T(f_1',\ldots,f_{i-1}',f_i-f_i',\ldots,f_n)| \leq \\ &\sum_{i=1}^{n} \|f_1'\|_\infty \cdots \|f_i-f_i'\|_\infty \cdots \|f_n\|_\infty T(\mathbf{1},\ldots,\mathbf{1}) \leq \\ &\varepsilon T(\mathbf{1},\ldots,\mathbf{1}) \sum_{i=1}^{n} \|f_1\|_\infty \cdot \overbrace{\cdots}^{[i]} \cdot \|f_n\|_\infty \end{aligned}$$

With the notation $\overbrace{\cdots}^{[i]}$ we mean that in the product, the factor i does not appear. Finally

$$|T(\mathbf{1},\dots,\mathbf{1},h')) - T(\mathbf{1},\dots,\mathbf{1},h))| \leq$$
$$\sum_{j_1,\dots,j_n} |f_1(s_{j_1})\cdots f_{n-1}(s_{j_{n-1}}) - f_1(s_{j_n})\cdots f_{n-1}(s_{j_n})||f_n(s_n)|T(u_{j_1},\dots,u_{j_n})$$
$$\leq \|f_n\|_\infty \sum_{j_1,\dots,j_n} |f_1(s_{j_1})\cdots f_{n-1}(s_{j_{n-1}}) - f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}})f_{n-1}(s_{j_n})|T_{j_1,\dots,j_n}$$
$$+\|f_n\|_\infty \sum_{j_1,\dots,j_n} |f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}}) - f_1(s_{j_n})\cdots f_{n-2}(s_{j_n})||f_{n-1}(s_{j_n})|T_{j_1,\dots,j_n}$$
$$\leq \|f_1\|_\infty \overbrace{\cdots}^{[n-1]} \|f_n\|_\infty \sum_{j_1,\dots,j_n} |f_{n-1}(s_{j_{n-1}}) - f_{n-1}(s_{j_n})|T_{j_1,\dots,j_n}$$
$$+\|f_{n-1}\|_\infty\|f_{n-1}\|_\infty \sum_{j_1,\dots,j_n} |f_1(s_{j_1})\cdots f_{n-2}(s_{j_{n-2}}) - f_1(s_{j_n})\cdots f_{n-2}(s_{i_n})|T_{j_1,\dots,j_n}$$

Where the notation $T_{j_1,\dots,j_n}$ stands for $T(u_{j_1},\dots,u_{j_n})$. If we repeat the process, we get:

$$|T(\mathbf{1},\dots,\mathbf{1},h')) - T(\mathbf{1},\dots,\mathbf{1},h))| \leq$$
$$\sum_{i=1}^{n} \|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty \sum_{j_1,\dots,j_n} |f_i(s_{j_i}) - f_{n-1}(s_{j_n})|T(u_{j_1},\dots,u_{j_n})$$

Now, there are two options, if $u_{j_1},\dots,u_{j_n}$ are p-disjoint, then

$$T(u_{j_1},\dots,u_{j_n}) = 0,$$

otherwise there is a path connecting the set S_{j_i} and the set S_{j_n}, hence

$$|f_i(s_{j_i}) - f_{n-1}(s_{j_n})| \leq n\varepsilon$$

We conclude that

$$|T(\mathbf{1},\dots,\mathbf{1},h')) - T(\mathbf{1},\dots,\mathbf{1},h))| \leq$$
$$\sum_{i=1}^{n} n\varepsilon\|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty \sum_{j_1,\dots,j_n} T(u_{j_1},\dots,u_{j_n})$$
$$\varepsilon T(\mathbf{1},\dots,\mathbf{1}) \sum_{i=1}^{n} n\|f_1\|_\infty \overbrace{\cdots}^{[i]} \|f_n\|_\infty$$

□

Then we obtain as a direct consequence of the previous lemma, the following result:

Theorem 3.2.15. *Let E and F be Archimedean Riesz spaces. Let $T : E \times \ldots \times E \longrightarrow F$ be an p-orthosymmetric positive n-linear map. Then T is symmetric.*

The proof of this fact is an easy application of Yosida's Theorem 3.1.2 which allows us to translate the problem to the $C(K)$ setting and using then Lemma 3.4.5. Notice that for symmetric positive multilinear forms the three notions of orthosymmetry are the same. In general the notion of b-orthosymmetry is weaker than p-orthosymmetry.

3.3 Powers of Riesz spaces.

We will continue with our objective of obtaining a representation theorem for n-homogeneous orthogonally additive polynomials on an appropriate class of Riesz spaces, then we will discuss the notion that extends the concavification of a Banach lattice to Riesz spaces.

Boulabiar and Buskes generalized in [BB] the results of Buskes and van Rooij about squares of Riesz spaces to the case of general powers. As we have said, the notion of b-orthosymmetry is weaker than p-orthosymmetry, however their proofs work for p-orthosymmetry without substantial changes and they are not worth repeating. The results of this section can be seen in the cited article. We begin with the general definition of the n-power of a Riesz space:

Definition 3.3.1 (Definition 3.1 [BB]). *Let E be an Archimedean Riesz space and $n \geq 2$. The pair $(\bigodot_n E, \odot_n)$ it is said to be an n-power of E if*

1. *$\bigodot_n E$ is a Riesz space,*
2. *$\odot_n : E \times \ldots \times E \longrightarrow \bigodot_n E$ is a b-orthosymmetric n-morphism of Riesz spaces and,*
3. *for every F Archimedean Riesz space and every b-orthosymmetric n-morphism $T : E \times \ldots \times E \longrightarrow F$, there exists a unique Riesz homomorphism $T^{\odot_n} : \bigodot_n E \longrightarrow F$ such that $T = T^{\odot_n} \circ \odot_n$*

The n-power is unique up to Riesz isomorphism:

Theorem 3.3.2 (Theorem 3.2 [BB]). *Let E be an Archimedean Riesz space and $n \geq 2$. Up to a Riesz isomorphism, E has a unique n-power $(\bigodot_n E, \odot_n)$.*

The n-power $(\bigodot_n E, \odot_n)$ is defined as follows. If we consider $\mathcal{I}$ to be the ideal of the n-fold Fremlin tensor product $\overline{\bigotimes}_n E$ generated by the elements $x_1 \otimes \ldots \otimes x_n$ with $x_1, \ldots, x_n$ p-disjoint, $\bigodot_n E$ is the quotient space of $\overline{\bigotimes}_n E$ with $\mathcal{I}$ and $\odot_n$ is the natural map

$$\bigodot_n : E \times \ldots \times E \longrightarrow \overline{\bigotimes}_n E/\mathcal{I}$$

composition of $\otimes$ with the canonical application $\pi : \overline{\bigotimes}_n E \longrightarrow \overline{\bigotimes}_n E/\mathcal{I}$.

We are not interested in n-morphism but in positive orthosymmetric mappings, to avoid working with n-morphisms, we have to assume the hypothesis of uniform completeness:

Theorem 3.3.3 (Theorem 3.3 [BB]). *Given $n \geq 2$ and E an Archimedean Riesz space. If G is an Archimedean Riesz space and $\Phi : E \times \ldots \times E \longrightarrow G$ is a b-orthosymmetric n-morphism of Riesz spaces, they are equivalent:*

1. *The pair (G, Φ) is the n-power of E*
2. *For every F uniformly complete Riesz space and every $T : E \times \ldots \times E \longrightarrow F$ positive orthosymmetric n-linear application, there exists a unique positive operator $\tilde{T} : G \longrightarrow F$ such that $T = \tilde{T} \circ \Phi$.*

Working with f-algebras, there is a simple characterization:

Theorem 3.3.4 (Theorem 4.1 [BB]). *Let E a uniformly complete Riesz subspace of an Archimedean semiprime f-algebra A. Define*

$$E^n = \{x_1 \cdots x_n : x_i \in E; i = 1, \ldots, n\}$$

and $*(x_1, \ldots, x_n) = x_1 \cdots x_n$ for $x_1, \ldots, x_n \in E$. Then the pair $(E^n, *)$ is the n-power of E.

It can also be given another characterization in an abstract setting (see [BB] Theorem 5.1) based on functional calculus but for our purposes the characterizations given in this section will be sufficient.

3.4 Representation of orthogonally additive polynomials.

Given E and F Riesz spaces, the space of linear applications from E to F is not in general a Riesz space. What is more, even the subspace $\mathcal{L}^r(E,F)$ of regular mappings (recall that a mapping is regular if it can be written as the difference of two positive mappings) is not necessarily a Riesz space (see for instance Example 1.17 in [AB]).

This problem can be overcome assuming that F is Dedekind complete. A Riesz space is said to be Dedekind complete if every nonempty bounded set from above has a supremum. In this case, F. Riesz for $F=\mathbb{R}$ and Kantorovich for the general case, proved:

Theorem 3.4.1 (F. Riesz-Kantorovich). *If E and F are Riesz spaces with F Dedekind complete, then $\mathcal{L}^r(E,F)$ is a Dedekind complete Riesz space.*

The proof of this result can be seen for instance in [AB], Theorem 1.18. In the multilinear setting, Loane obtains in his thesis an analogous result:

Proposition 3.4.2 (Lemma 2.12 [L]). *If $E_1,\dots,E_n,F$ are Riesz spaces and F is Dedekind complete,then $\mathcal{L}^r(E_1\times\dots\times E_n,F)$ is a Dedekind complete space Riesz. Furthermore, if $T\in\mathcal{L}^r(E_1\times\dots\times E_n,F)$ then $T=T^+-T^-$ where $T^\pm$ are positive and given by:*

$$T^+(x_1,\dots,x_n)=\sup\left\{\sum_{i_1,\dots,i_n}(T(u_{1,i_1},\dots,u_{n,i_n}))^+ : u_{k,i_k}\in\Pi_{x_k}, 1\le k\le n\right\}$$

$$T^-(x_1,\dots,x_n)=\sup\left\{\sum_{i_1,\dots,i_n}(T(u_{1,i_1},\dots,u_{n,i_n}))^- : u_{k,i_k}\in\Pi_{x_k}, 1\le k\le n\right\}$$

$$|T|(x_1,\dots,x_n)=\sup\left\{\sum_{i_1,\dots,i_n}|T(u_{1,i_1},\dots,u_{n,i_n})| : u_{k,i_k}\in\Pi_{x_k}, 1\le k\le n\right\}$$

Notation Π_x where $x\in E$ appears in [BvR2] and stands for the set of partition of the element x that is, the set of finite sequences of elements of E whose sum equals x.

The same author proves (Lemma 2.15 [L]) that the space of symmetric n-linear regular mappings $\mathcal{L}^r_s({}^nE,F)$ is a Riesz space and obtain for polynomials the following result:

Proposition 3.4.3 (Lemma 2.16 [L]). *Let E, F be Riesz spaces with F Dedekind complete, $P \in \mathcal{P}^r({}^nE, F)$ a regular n-homogeneous polynomial and A its n-linear associated form. Then, for $x \in E^+$*

$$P^+(x) = A^+(x, \dots, x),$$

$$P^-(x) = A^-(x, \dots, x),$$
$$|P|(x) = |A|(x, \dots, x).$$

In order to prove the representation theorem for orthogonally additive polynomials on Riesz spaces, we will make use of the theory of p-orthosymmetric n-linear forms presented in the previous sections. In our case, the role of the continuity of the Banach lattice setting is played by positivity. The following theorem shows that the space of positive orthogonally additive n-homogeneous polynomials is a Riesz space isomorphic to the space of positive n-linear p-orthosymmetric forms.

Theorem 3.4.4. *Let E, F be Riesz spaces with F Dedekind complete. The space of positive orthogonally additive n-homogeneous polynomials $\mathcal{P}_o^+({}^nE, F)$ a Dedekind complete Riesz space which is Riesz isomorphic to the space of positive n-linear orthosymmetric forms.*

Proof. By Proposition 3.4.3 and noting that if $x \in E$, $x^+ \wedge x^- = 0$ (see for instance [AB] Theorem 1.5), $P = \hat{A} \in \mathcal{P}_o^+({}^nE, F)$ verifies

$$\begin{aligned} P^+(x) &= A^+(x, \dots, x) \\ P^-(x) &= A^-(x, \dots, x) \\ |P|(x) &= |A|(x, \dots, x) \end{aligned} \tag{3.1}$$

for every $x \in E$. Hence, in order to prove that $\mathcal{P}_o^+({}^nE, F)$ has structure of a Riesz space it suffices to show that $P \wedge Q \in \mathcal{P}_o^+({}^nE, F)$ whenever $P, Q \in \mathcal{P}_o^+({}^nE, F)$.

Let $P = \hat{A}$ and $Q = \hat{B}$, with A, B positive and p-orthosymmetric. If $x_1 \dots, x_n$ are p-disjoint

$$0 \leq A \wedge B(x_1, \dots, x_n) \leq A(x_1, \dots, x_n) = 0$$

then $A \wedge B$ is n-orthosymmetric and so $P \wedge Q = \widehat{A \wedge B} \in \mathcal{P}_o^+({}^nE, F)$.

Equations 3.1 together with Theorems 3.2.13 and 3.2.15 guarantee that the application $A \longmapsto \hat{A}$ is the needed Riesz isomorphism. □

Our main tool to prove the representation theorem for polynomials will be the previous result together with a representation theorem for multilinear mappings:

Lemma 3.4.5. *Let E be an Archimedean Riesz space and F an uniformly complete Archimedean Riesz space. The space $\mathcal{L}_o^+({}^nE,F)$ of positive p-orthosymmetric n-linear applications from E to F is Riesz isomorphic to the space $\mathcal{L}^+(\bigodot_n E,F)$ of positive linear forms from $\bigodot_n E$ to F.*

Proof. In this result, we generalize the ideas of [BK] Theorem 3.1 to the n-linear case. If $T \in \mathcal{L}_o^+({}^nE,F)$ we will prove that there exists a unique $\Phi_T \in \mathcal{L}^+(\bigodot_n E,F)$ such that $\Phi_T \circ \bigodot_n = T$ and that the correspondence $T \longrightarrow \Phi_T$ is a Riesz isomorphism.

Given such T, let $\widetilde{T}$ be the unique positive operator $\widetilde{T} : \overline{\bigotimes}_n E \longrightarrow F$ verifying

$$T(x_1,\ldots,x_n) = \widetilde{T}(x_1 \otimes \ldots \otimes x_n).$$

If $\pi : \overline{\bigotimes}_n E \longrightarrow \bigodot_n E$ is the canonical projection, the operator $\Phi = \Phi_T$ that we need is the one satisfying $\widetilde{T} = \Phi \circ \pi$. The uniqueness of this operator is given by the uniqueness of $\widetilde{T}$. Moreover $\widetilde{T}$ it is positive since π is a Riesz homomorphism and $\widetilde{T}$ is positive. In particular, the correspondence $T \longmapsto \Phi$ respects the order.

To conclude the proof, we need to show that Φ is well defined. It is sufficient to show that the kernel of π is contained in the kernel of $\widetilde{T}$. We proceed as in [BvR3] Theorem 4. Let $f_1 \otimes \ldots \otimes f_n \in \ker \pi$ then $f_1,\ldots,f_n$ are p-disjoint hence,

$$\widetilde{T}(f_1 \otimes \ldots \otimes f_n) = T(f_1,\ldots,f_n) = 0$$

since T is p-orthosymmetric.

Finally, observe that the correspondence $T \longmapsto \Phi_T$ is a Riesz isomorphism since T and Φ are positive.

□

As an immediate consequence of the previous results, we obtain the following theorem:

Theorem 3.4.6. *Let E,F be Archimedean Riesz spacies with F uniformly complete and let $(\bigodot_n E, \odot_n)$ be the n-power of E. The space $\mathcal{P}_o^+({}^nE,F)$ of positive orthogonally additive n-homogeneous polynomials on E is Riesz isomorphic to the space $\mathcal{L}^+(\bigodot_n E,F)$ of positive linear applications from $\bigodot_n E$ to F.*

If we consider the case of f-algebras, the connection with the theorem of representation of orthogonally additive polynomials in Banach lattices becomes clearer:

Theorem 3.4.7. *Let E be an uniformly complete Archimedean Riesz subspace of a semiprime f-algebra A and F uniformly complete Archimedean. Then, for every positive orthogonally additive n-homogeneous polynomial $P \in \mathcal{P}_o^+({}^nE, F)$ there exists a unique positive linear application $L \in \mathcal{L}^+(E^n, F)$ such that $P(x) = L(x \cdots x)$ for every $x \in E$.*

Proof. By Theorem 3.3.4, E^n with its product is the n-power of E and by the uniqueness given by Theorem 3.3.2, there exists a Riesz isomporphism $i : E^n \longrightarrow \bigodot_n E$ such that $i(x_1 \cdots x_n) = x_1 \odot \ldots \odot x_n$. Define $L \in \mathcal{L}^+(E^n, F)$ as the linear application $L = \Phi_{\check{P}} \circ i$ (with the notation as in Lemma 3.4.5) then,

$$P(x) = \check{P}(x, \ldots, x) = \Phi_{\check{P}}(x \odot \ldots \odot x) = L(x \cdots x).$$

□

Chapter 4

Orthogonal and P-orthogonal Polynomials.

In the classical theory of orthogonal polynomials, the orthogonality is defined respect to a linear form. As we have seen in Chapter 2 an orthogonally additive polynomial P on a Banach lattice X, can be represented by a linear functional T_P defined on the concavification of X. It arises naturally the problem of study a theory of orthogonality with respect to an orthogonally additive polynomial instead of a linear form.

In this chapter we will use the correspondence between P and T_P and the classical theory of orthogonality respect to T_P to define a new orthogonality respect to P.

We begin with a brief introduction to the classical theory and then we study the new theory in the cases $C[0,1]$ and L^p. Finally we will focus our attention to the classical theory of orthogonal polynomials with respect to a bilinear form and we will prove a theorem inspired in a result by Durán appeared in [Dur2].

4.1 Introduction.

There are several classical books on the theory of orthogonal polynomials. The contents of this introduction can be seen in [Sze] or in [C] for instance. An interesting historical introduction can be found in [AN].

By algebraic polynomial we will mean a polynomial with complex coefficients in a real variable. We will say real algebraic polynomials when we consider real coefficients. The space of algebraic polynomials in a subset Ω of $\mathbb{R}$ will be denoted by $\mathbb{P}(\Omega)$.

Definition 4.1.1. *Given a sequence of complex numbers (μ_n), a linear form L in the space of algebraic polynomials such that $L[x^n] = \mu_n$ will be called moment functional associated to (μ_n).*

Definition 4.1.2. *A sequence of algebraic polynomials $(P_n(x))_n$ is said to be an orthogonal polynomial sequence (OPS) with respect to a moment functional L if:*

1. *$P_n(x)$ is a polynomial of degree n.*
2. *$L[P_n(x)P_m(x)] = 0$ if $n \neq m$.*
3. *$L[P_n^2(x)] \neq 0$.*

The following result provides a simple characterization to the previous definition

Theorem 4.1.3. *Let L be a linear form and $(P_n)_n$ a sequence of polynomials such that degree$(P_n) = n$. The following assumptions are equivalent:*

1. *$(P_n)_n$ is an OPS with respect to L.*
2. *$L[\pi P_n] = 0$ if degree$(\pi) < n$ and $L[\pi P_n] \neq 0$ if degree$(\pi) = n$.*
3. *$L[x^m P_n] = K_n \delta_{n,m}$ with $K_n \neq 0$, $m = 0, 1, \ldots, n$.*

The next theorem states that given an orthogonal polynomial sequence, every algebraic polynomial can be written as a (finite) linear combination of the sequence

Theorem 4.1.4. *Let $(P_n)_n$ be an OPS respect to L. Then, for every polynomial π of degree n, it is satisfied:*

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where} \quad c_k = \frac{L[\pi P_k]}{L[P_k^2]}, \quad k = 0, 1, \ldots, n.$$

Furthermore, every P_n in the OPS it is uniquely determined up to multiplicative constant.

One of the most useful properties of the OPS is that they satisfy a three term recurrence relation:

Theorem 4.1.5. *Given* (P_n) *an OPS with respect to* L*, the following relations hold:*

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x).$$

for some sequences of complex numbers $(\alpha_n)_n$,$(\beta_n)_n$ *and* $(\gamma_n)_n$.

If we consider $P_{-1}(x) = 0$ *and* $P_0(x) = 1$*, we have that the sequence is uniquely determined if the sequences* $(\alpha_n)_n$,$(\beta_n)_n$ *and* $(\gamma_n)_n$ *are given.*

It can be proved that the sequences $(\alpha_n)_n$,$(\beta_n)_n$ and $(\gamma_n)_n$ satisfy

$$\alpha_n = \frac{L[xP_nP_{n+1}]}{L[P_{n+1}^2]}, \qquad \beta_n = \frac{L[xP_n^2]}{L[P_n^2]}, \qquad \gamma_n = \frac{L[xP_nP_{n-1}]}{L[P_{n-1}^2]}.$$

Theorem 4.1.5 has a reciprocal due to Favard in his paper [F] appeared in 1935:

Theorem 4.1.6 (Favard's Theorem). *Let* $(\beta_n)_n$ *and* $(\gamma_n)_n$ *be two sequences of complex numbers. If the sequence of monic polynomials* $(P_n)_n$ *is defined by the recurrence formula*

$$P_n(x) = (x - \beta_{n-1})P_{n-1}(x) - \gamma_{n-1}P_{n-2}(x), \qquad n = 0, 1, \ldots,$$

where $P_{-1} = 0$ *and* $P_0(x) = 1$.

Then, there exists a unique moment functional L *verifying*

$$L[1] = \gamma_0, \qquad L[P_nP_m] = \delta_{n,m}K_n, \qquad K_n \neq 0.$$

4.2 P-orthogonal Polynomials.

Recall that the representation theorem of orthogonally additive polynomials states that given a m-homogeneous polynomial P of this type on a Banach lattice X, there exists a linear functional $L_P \in X_{(m)}^*$ such that

$$L_P(f) = P(f^{1/m}) \text{ for every } f \in X_{(m)}$$

This correspondence can be used to define a family of functions orthogonal with respect to P and translate known results of the theory of orthogonal polynomials to this new setting.

We need some restrictions in our lattice X, it is necessary that $X_{(m)}$ contains the algebraic polynomials. Hence, we present the theory firstly in the most simple case, $C[0,1]$ that can be generalized to any compact set in $\mathbb{R}$, and then in $L^p(\mathbb{R})$ where we have introduce a regularization function in order to fulfill our first restriction.

4.2.1 P-orthogonal Polynomials in $C[0,1]$.

Let P be a m-homogeneous orthogonally additive polynomial in $X = C[0,1]$. In this case $X = X_{(m)}$ and there is a linear functional $L \in C[0,1]^*$ associated to P. We will denote for a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and integers, $n \geq 0$, $m \geq 1$

$$f^{n/m}(s) = |f(s)|^{n/m} sign(f(s))$$

for $\frac{n}{m} \in \mathbb{Q} - \mathbb{Z}$ where $s \in \mathbb{R}$.

We begin with the definition of moment polynomial

Definition 4.2.1. *Given a sequence (μ_n), of complex numbers, we will say that $P \in \mathcal{P}_o(^mC[0,1])$ is the moment polynomial associated to (μ_n) if $L \in C[0,1]^*$ is the moment functional of the sequence (μ_n), that is, $L(x^n) = \mu_n$ (equivalently $P(x^{n/m}) = \mu_n$).*

The family of functions verifying an orthogonality condition with respect to P will be called m-root polynomials:

Definition 4.2.2. *It will be said that $f \in C[0,1]$ is an m-root polynomial of degree $r \in \frac{\mathbb{N}}{m} = \{0, \frac{1}{m}, \frac{2}{m}, \ldots, 1, \ldots\}$ if f^m is a polynomial of degree rm.*

Finally, we define a P-orthogonal polynomial sequence:

Definition 4.2.3. *Let (p_r) be a sequence of m-root polynomials with $r \in \frac{\mathbb{N}}{m}$. We will say that (p_r) is a P-Orthogonal Polynomial Sequence (P-OPS) with respect to P, m-homogeneous orthogonally additive polynomial if the polynomial sequence (q_n) given by $q_n = p^m_{\frac{n}{m}}$, $n \in \mathbb{N}$ is an OPS with respect to the linear functional associated L.*

Now, based on the linear theory of orthogonal polynomials, we can prove in a straightforward way the following results. We only give the proof of one of them.

Theorem 4.2.4. *The sequence of m-root polynomials (p_r) where the degree of p_r is $r \in \frac{\mathbb{N}}{m}$, is a P-OPS with respect to P if and only if it verifies:*

1. $P(p_r p_s) = 0$ *if* $s \neq r$.
2. $P(p_r^2) \neq 0$.

Theorem 4.2.5. *Let P be an orthogonally additive m-homogeneous polynomial and (p_r) a sequence of m-root polynomials. The following assumptions are equivalent:*

1. (p_r) *is an P-OPS with respect to* P.
2. $P(p_r p) = 0$ *for every p m-root polynomial of degree $s < r$.* $P(p_r p) \neq 0$ *if degree of p is equal to r.*
3. $P(x^s p_r) = K_r \delta_{r,s}$ *with* $K_r \neq 0$, $0 \leq s \leq r$

Proposition 4.2.6. *Let (p_r) be a P-OPS associated to certain orthogonally additive m-homogenous polynomial P. For each p, m-root polynomial of degree r,*

$$p = \left(\sum_{s=0}^{r} \alpha_s p_s^m \right)^{1/m} \qquad \text{where} \quad \alpha_s = \frac{P(p_s p)}{P(p_s^2)}$$

Proof. Consider the algebraic polynomial p^m of degree rm. If (q_n) is the OPS associated to (p_r), we can write

$$p^m = \sum_{i=0}^{rm} \mu_i q_i$$

and using the properties of $L = L_P$ the linear theory allows us to conclude that $\mu_i = \frac{L(p^m q_i)}{L(q_i^2)}$.

As $q_i = p_{i/m}^m$, we have

$$p^m = \sum_{s=0}^{r} \mu_{sm} p_s^m$$

as needed. $\square$

It can be proved as in Theorem 4.1.5 that there exists a three term recurrence relation for m-root polynomials:

$$xp_r^m(x) = \alpha_r p_{r+1/m}^m(x) + \beta_r p_r^m(x) + \gamma_r p_{r-1/m}^m(x)$$

And there is also a polynomial version of Favard's theorem:

Theorem 4.2.7 (Polynomial Favard's Theorem). *Let (p_r) a sequence of m-roots polynomials such that*

$$p_r^m = (x-\beta_r)p_{r-1/m}^m - \lambda_r p_{r-2/m}^m$$

for some sequences of complex numbers (β_r) and (λ_r), with $r \in \mathbb{N}/m$.

Then, there exists a m-homogeneous orthogonally additive polynomial $\mathcal{P}$ (not necessarily continuous) defined on X the span of the set

$$\{f : f^m \text{ algebraic polynomial }\} = \{m\text{-roots polynomials}\}$$

such that $P(p_s p_r) = K_r \delta_{r,s}$ with $K_r \neq 0$, $s = 0, 1/m, \dots, r$ and for every $r \in \mathbb{N}/m$.

Proof. The sequence (q_n) of algebraic polynomials associated to (p_r), verifies a recurrence relation and hence by Favard's theorem there is linear form L defined in $\mathbb{P}[0,1]$ (the space of algebraic polynomials in $[0,1]$) such that $L(q_i q_j) = K_i \delta_{i,j}$.

For $f \in X$, we define:

$$\hat{L}(f) = \begin{cases} L(f) & \text{if } f \in \mathbb{P}[0,1] \\ (L(f^m))^{1/m} & \text{if } f \notin \mathbb{P}[0,1] \end{cases}$$

and we extend linearly to the whole X.

Then $\hat{L}$ is a well defined linear function whose restriction to a $\mathbb{P}[0,1]$ coincides with L.

Now, the multilinear form given by $A(f_1, \dots, f_m) = \hat{L}(f_1 \cdots f_m)$ is symmetric in X. It is not difficult to check that it is well defined since if $f_i \in X$, their product will also be in X.

Finally, if $P(f) = A(f, \dots, f)$ is the m-homogeneous polynomial (not necessarily continuous) defined by A, $P(f) = \hat{L}(f^m)$. This relation guarantees that P is orthogonally additive and

$$\mathcal{P}(p_s p_r) = \hat{L}(q_i q_j) = L(q_i q_j) = K_i \delta_{i,j}.$$

□

4.2.2 P-orthogonal Polynomials in L^p.

We will work now with algebraic polynomials defined on the real line. Our lattice will be now $L^p = L^p(\mathbb{R})$. There is two options, the first one is to assume that this space is defined with respect to a measure such that it contains the algebraic polynomials and the second one is to work with the Lebesgue measure and use a regularizing function. We take the second approach. Our functions will be the functions in the Schwartz zero-class:

Definition 4.2.8. *A real function f is said to be in the Schwartz zero-class S_0 if it is a continuous function, bounded in $\mathbb{R}$ and verifyng*

$$\sup_{t\in\mathbb{R}} |t^k f(t)| < \infty$$

for every k.

Note that if $X = L^p$, $X_{(m)} = L^{p/m}$ and $\|\!| \cdot |\!\| = \| \cdot \|_{p/m}$. Hence, the representation theorem for L^p says:

Theorem 4.2.9. *Let $X = (L^p, \| \cdot \|_p)$ and $m \in \mathbb{N}$. Then, the application $T \mapsto P_T$ defined as $P_T(f) = T(f^m)$ is an isometric isommetry from $L^{p/p-m}$ to $\mathcal{P}_o(^m X)$.*

As the dual of the spaces L^p with $0 < p < 1$ is empty, we will only be interested in the case $m \leq p$.

We begin our study with a linear theory of orthogonal polynomials in L^p. As we will see, the results are analogous to those in Section 4.1 but we have to use a regularizing function. Then we will study the polynomial theory in L^p.

Linear theory in L^p spaces.

We begin with the definition of moment functional:

Definition 4.2.10. *Given a secuence of complex numbers (μ_n) and $\eta \in S_0$, a linear form L in $\mathbb{P}_\eta$ it will be called moment functional associated to the sequence (μ_n) if $L[\eta(x)x^n] = \mu_n$.*

Where the notation $\mathbb{P}_\eta = \mathbb{P}_\eta(\mathbb{R})$ stands for the set

$$\{\eta p : p \text{ algebraic polynomial}\}$$

The sequences of orthogonal polynomials are defined as follows:

Definition 4.2.11. *A sequence of algebraic polynomials $(P_n(x))_n$ it is said to be a orthogonal polynomial sequence (OPS) with respect to a moment functional L if it verifies:*

1. $P_n(x)$ *is an algebraic polynomial of degree* n.
2. $L[\eta(x)P_n(x)P_m(x)] = 0$ *if* $n \neq m$.
3. $L[\eta(x)P_n^2(x)] \neq 0$.

We enunciate now the basic results in this setting:

Theorem 4.2.12. *Let L be a linear form and $(P_n)_n$ a sequence of polynomial such that degree$(P_n) = n$. The following are equivalent:*

1. $(P_n)_n$ *is an OPS with respect to* L.
2. $L[\eta\pi P_n] = 0$ *if degree*$(\pi) < n$ *and* $L[\eta\pi P_n] \neq 0$ *if degree*$(\pi) = n$.
3. $L[x^m\eta(x)P_n(x)] = K_n\delta_{n,m}$ *con* $K_n \neq 0$, $m = 0, 1, \ldots, n$.

Theorem 4.2.13. *Let $(P_n)_n$ an OPS with respect to L. Then, for every polynomial π of degree n, we have:*

$$\pi(x) = \sum_{k=0}^{n} c_k P_k(x), \quad \text{where} \;\; c_k = \frac{L[\eta\pi P_k]}{L[\eta P_k^2]}, \quad k = 0, 1, \ldots, n.$$

Furethermore, each P_n of the OPS is determined uniquely up to a multiplicative constant.

There is a three term recurrence relation:

Theorem 4.2.14. *Given (P_n) an OPS with respect to L, the following recurrence relation is satisfied:*

$$x\eta(x)P_n(x) = \alpha_n\eta(x)P_{n+1}(x) + \beta_n\eta(x)P_n(x) + \gamma_n\eta(x)P_{n-1}(x).$$

for some sequences $(\alpha_n), (\beta_n), (\gamma_n)$ of complex numbers.

If we consider that $P_{-1}(x) = 0$ and $P_0(x) = 1$, we have that the sequence is uniquely determined by the sequences $(\alpha_n)_n$, $(\beta_n)_n$ and $(\gamma_n)_n$.

And finally, Favard's theorem:

Theorem 4.2.15 (Favard's theorem). *Let $(\beta_n)_n$ and $(\gamma_n)_n$ be two sequences of complex numbers such that $\gamma_{n-1} \neq 0$ for every $n \in \mathbb{N}$. If the sequence of monic polynomials $(P_n)_n$ is defined by the relation*

$$\eta(x)P_n(x) = (x - \beta_{n-1})\eta(x)P_{n-1}(x) - \gamma_{n-1}\eta(x)P_{n-2}(x), \qquad n = 0, 1, \ldots,$$

with $P_{-1} = 0$ y $P_0(x) = 1$.

Then, there exists a moment functional L such that

$$L[\eta] = \gamma_0, \qquad L[\eta P_n P_m] = \delta_{n,m} K_n, \qquad K_n \neq 0.$$

Remark 4.2.16. *The introduction of the regularization functions does not imply essential changes in the proofs of these results.*

Polynomial theory in L^p spaces.

We will study now the polynomial theory in L^p. The definitions and results are analogous to those in the case $C[0,1]$. We skip the definition of m-root polynomial and moment polynomial and we begin with the definition of P-Orthogonal Polynomial Sequence:

Definition 4.2.17. *Let (p_r) be a sequence of m-root polynomials with $r \in \frac{\mathbb{N}}{m}$. We will say that (p_r) is a P-Orthogonal Polynomial Sequence (P-OPS) with respect to P orthogonally additive m-homogeneous polynomial if the sequence of algebraic polynomials (q_n) given by $q_n = p_{\frac{n}{m}}^m$ $n \in \mathbb{N}$ is an OPS with respect to the linear functional L_P associated to P.*

Theorem 4.2.18. *The sequence of m-roots polynomials (p_r) where degree$(p_r) = r$ is a P-OPS with respect to P if and only if, there exists $\tilde{\eta}$ such that:*

1. *$P(\tilde{\eta} p_r p_s) = 0$ if $s \neq r$*
2. *$P(\tilde{\eta} p_r^2) \neq 0$*

Theorem 4.2.19. *Let P be an orthogonally additive polynomial and (p_r) a sequence of m-root polynomials suchc that degree$(p_r) = r$. They are equivalent:*

1. *(p_r) is a P-OPS with respect to P.*

2. *There exists* $\tilde{\eta}$ *such that* $P(\tilde{\eta}p_rp) = 0$ *for any* p *m-root polynomial of degree* $s < r$ *and* $P(\tilde{\eta}p_rp) \neq 0$ *if degree*$(p) = r$.

3. *There exists* $\tilde{\eta}$ *such that* $P(x^s\tilde{\eta}(x)p_r(x)) = K_r\delta_{r,s}$ *with* $K_r \neq 0$, $0 \leq s \leq r$.

Proposition 4.2.20. *Given* (p_r) *a P-OPS associated to certain* P *an orthogonally additive m-homogenous polynomial, for each* p *m-root polynomial of degree* r*, we have:*

$$\tilde{\eta}p = \left(\sum_{s=0}^{r} \alpha_s \eta p_s^m\right)^{1/m} \qquad \text{where} \quad \alpha_s = \frac{P(\tilde{\eta}p_sp)}{P(\tilde{\eta}p_s^2)}$$

We have as well the recurrence relation for P-OPS:

$$x\tilde{\eta}(x)p_r^m(x) = \alpha_r\tilde{\eta}(x)p_{r+1/m}^m(x) + \beta_r\tilde{\eta}(x)p_r^m(x) + \gamma_r\tilde{\eta}(x)p_{r-1/m}^m(x)$$

And Favard's theorem:

Theorem 4.2.21 (Polynomial Favard's Theorem). *Let* (p_r) *be a sequence of m-root polynomials such that*

$$\tilde{\eta}^m(x)p_r^m(x) = (x-\beta_r)\tilde{\eta}^m(x)p_{r-1/m}^m(x) - \lambda_r\tilde{\eta}^m(x)p_{r-2/m}^m(x)$$

for certain sequence of complex numbers (β_r) *and* (λ_r) *and some regularizing function* $\tilde{\eta}$ *with* $r = 0, 1/m, 2/m, \ldots, 1, \ldots$

Then, there exists P *orthogonally additive m-homogeneous polynomial (not necessarily continuous) defined in* X *the span of the set*

$$\{\tilde{\eta}f : f^m \text{ is an algebraic polynomial }\} = \{\text{ regularized m-root polynomials }\}$$

such that $P(\tilde{\eta}(x)p_sp_r(x)) = K_r\delta_{r,s}$ *with* $K_r \neq 0$, $s = 0, 1/m, \ldots, r$ *and for every* $r \in \mathbb{N}/m$.

Proof. We write $\eta = \tilde{\eta}^m$ and we use the linear version of the theorem to find some linear application L in lineal in $\mathbb{P}_\eta$ such that $L(\eta q_n q_m) = K_n\delta_{n,m}$.

With the same ideas as in the case $C[0,1]$, extend L to $\hat{L}$ defined in

$$Y = span\{\eta f : f^m \text{ is an algebraic polynomial }\}$$

Note that $\mathbb{P}_\eta \subset Y$.

Define, for any ηf with f^m algebraic polynomial:

$$\hat{L}(\eta f) = \begin{cases} L(\eta f) & \text{if } f \text{ is an algebraic polynomial} \\ (L(\eta f^m))^{1/m} & \text{otherwise} \end{cases}$$

and extend linearly to the whole Y.

By construction, the restriction of $\hat{L}$ to $\mathbb{P}_\eta$ coincides with L.

Let $A(f_1, \ldots, f_m) = \hat{L}(f_1 \cdots f_m)$ be a m-linear symmetric form in X. It is not difficult to check that it is well defined since if $f_i \in X$, then the product will be in Y.

Finally, if $P(f) = A(f, \ldots, f)$ is the m-homogeneous polynomial defined by A, we have that $P(\tilde{\eta} f) = \hat{L}(\eta f^m)$. Hence P is orthogonally additive and

$$P(\tilde{\eta} p_s p_r) = \hat{L}(\eta q_i q_j) = L(\eta q_i q_j) = K_i \delta_{i,j}.$$

□

4.2.3 The problem of existence

In the linear theory of orthogonal polynomials, the problem of existence of an orthogonal polynomial sequence with respect to L is intrinsically related to the moments (μ_n) defining the functional. It can be proved that if Δ_n denotes de determinant

$$\Delta_n = det(\mu_{i+j})_{i,j=0}^n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix}$$

then, a necessary and sufficient condition for the existence of (P_n) an OPS with respect to L is that $\Delta_n \neq 0$ for every $n \in \mathbb{N}$ (see for instance Theorem 3.1 in [C]).

We turn now to the problem of the existence of P-orthogonal polynomial sequences. We will find conditions in the moments defining the polynomial. Note that we need, not only the existence of the OPS associated but also that the continuity the linear functional of moments. Furthermore, in the case of L^p we will need to guarantee as well the existence of a regularizing function.

We begin with the case $C[0,1]$, our problem is to find necessary and sufficient condition in a sequence of moments (μ_n) such that the functional

L defined by $L(x^n) = \mu_n$ will be continuous. Note that then the the moment polynomial $P(f) = L(f^{1/m})$ has associated a P-OPS whenever there exists an OPS for L.

The first remark is that not every (μ_n) will be valid. The sequence has to be bounded since $|\mu_n| = |L(x^n)| \leq \|L\|\|x^n\|_\infty \leq \|L\|$. This problem can be solved using via the Hausdorff moment problem as it can be seen in [?]. A sequence (μ_n) is said to be a Hausdorff moment sequence if its moment functional is continuous.

We will sketch the necessary tools as we will deal with the multilinear generalization of this problem in Chapter 5.

We need some definitions

Definition 4.2.22.

$$\Delta^k \mu_n = \sum_{m=0}^{k} (-1)^m \binom{k}{m} \mu_{n+k-m} \qquad k = 0, 1, \dots$$

Definition 4.2.23.

$$\lambda_{k,m}(x) = \binom{k}{m} x^m (1-x)^{k-m} \qquad m \leq k = 0, 1, \dots$$

Definition 4.2.24.

$$\lambda_{k,m} = \binom{k}{m} (-1)^{k-m} \Delta^{k-m} \mu_m \qquad m \leq k = 0, 1, \dots$$

They are related as follows, if L is the moment functional of the sequence (μ_n), $L(\lambda_{k,m}(x)) = \lambda_{k,m}$.

The condition we are looking for is a condition of boundness of the sequence $\lambda_{k,m}$:

Definition 4.2.25. *We will say that the sequence of moments (μ_n) is bounded with constant C or C-bounded if*

$$\sum_{m=0}^{k} |\lambda_{k,m}| < C$$

for $m = 0, \dots, k$ and every $k \in \mathbb{N}$.

This condition is equivalent to the boundness of L:

Theorem 4.2.26 ([?]). *A sequence of moments (μ_n) is a Hausdorff moment sequence if and only if there exists a constant C such that (μ_n) is bounded.*

We study now the problem of existence when the lattice is L^p. Our main tool is the following Theorem by Durán:

Theorem 4.2.27 (Theorem 1 [Dur1]). *Let (μ_n) be a sequence of complex numbers, then there exists a function ξ such that*

1. *$\xi \in S$ where S is Schwartz's space with $\xi(x) = 0$ if $x < 0$,*
2. *$\mu_n = \int_{-\infty}^{\infty} x^n \xi(x)\mathrm{d}x$ for every $n \geq 0$.*

Recall that the space of Schwartz of rapidly decreasing functions is defined as

$$\begin{aligned} S &= \{\xi : \mathbb{R} \to \mathbb{R} : \xi \text{ is a function of class } C^\infty \text{ and} \\ &\quad \|\xi\|_{k,n} = \sup_{t\in\mathbb{R}} t^k |\xi^{(n)}(t)| < \infty \text{ for every } k, n \in \mathbb{N}\} \end{aligned}$$

This theorem solves the so called Stieltjes moment problem which can be solved for every sequence. Following Durán's notation, we will denote by $S \cap S^+$ the functions of Schwartz's space vanishing for negative reals.

Taking the function ξ as in Duran's theorem, we consider $\eta = \xi^{1/2}$. We denote as before $\mathbb{P}_\eta = \{\eta\pi : \pi \in \mathbb{P}\}$ and consider the measure ν given by $\mathrm{d}\nu = |\eta|\mathrm{d}x$ where $\mathrm{d}x$ is the Lebesgue measure. We will say that a function f belongs to $S_0 \cap S_0^+$ if it is in the zero class of Schwartz and it vanishes in the negative reals. We need two lemmas:

Lemma 4.2.28. *The functions of the class $S_0 \cap S_0^+$ are Lebesgue integrable.*

Proof. Consider $f \in S_0 \cap S_0^+$. We can find a constant C such that $x^2|f(x)| \leq C$ for every $x > 0$. Then

$$\begin{aligned} \int_{-\infty}^{\infty} |f(x)|\mathrm{d}x &= \int_0^\infty |f(x)|\mathrm{d}x = \int_0^1 |f(x)|\mathrm{d}x + \int_1^\infty |f(x)|\mathrm{d}x \leq \\ &\leq \|f\|_\infty + C\int_1^\infty \frac{1}{x^2}\mathrm{d}x < \infty \end{aligned}$$

□

As an is immediate corollary we have:

Corollary 4.2.29. *The measure ν is finite.*

We will also need that $\mathbb{P}_\eta$ is contained in L^q for every $q \geq 1$. Note that it is enough to check that $x^n\eta(x) \in L^q$ for every $n \in \mathbb{N}$. This is the content of the next lemma.

Lemma 4.2.30. *Given $q \geq 1$ the space $\mathbb{P}_\eta = \{\eta\pi : \pi\mathbb{P}\}$ is contained in L^q.*

Proof.

Case $q \geq 2$. We will work with the function ξ given by Duran's Theorem. It can be checked as before that $\int_0^\infty x^k|\xi(x)|\mathrm{d}x < \infty$ for every integer $k \geq 0$. Hence, for any integer q it is immediate that:

$$\int_{-\infty}^{\infty} |x^n\eta(x)|^q \mathrm{d}x = \int_0^\infty x^{nq}|\xi(x)|^{q/2}\mathrm{d}x \leq C\int_0^\infty x^{nq}|\xi(x)|\mathrm{d}x,$$

where $C = \|\xi\|_\infty^{q/2-1}$.

For general q, denoting by $[t]$ t,

$$\int_{-\infty}^{\infty} |x^n\eta(x)|^q \mathrm{d}x \leq \int_0^1 x^{[nq]}|\eta(x)|\mathrm{d}x + \int_1^\infty x^{[nq]+1}|\eta(x)|\mathrm{d}x \leq$$
$$\leq C\int_0^\infty x^{[nq]}|\xi(x)|\mathrm{d}x + C\int_0^\infty x^{[nq]+1}|\xi(x)|\mathrm{d}x < \infty.$$

Case $q = 1$. Note that for every positive integer k we can find a constant C_k such that $t^k|\eta(t)| \leq C_k$. Hence

$$\begin{aligned}\int_{-\infty}^{\infty} |x^n\eta(x)|\mathrm{d}x &= \int_0^1 x^n|\eta(x)|\mathrm{d}x + \int_1^\infty x^n|\eta(x)|\mathrm{d}x \\ &\leq C_n + C_{n+2}\int_1^\infty \frac{1}{x^2}\mathrm{d}x < \infty.\end{aligned}$$

Case $1 < q < 2$**.** The particularity of this case is that we have now integrals of the form:

$$\int_{-\infty}^{\infty} |x^n \eta(x)|^q \mathrm{d}x = \int_0^{\infty} x^{nq} |\xi(x)|^{q/2} \mathrm{d}x$$

with $q/2 < 1$ and we cannot use the same procedure as in the first case.

However if $\xi \in S \cap S^+$, then $\xi^{1/k} \in S_0 \cap S_0^+$ for real $k \geq 1$.

In fact, we just have to check that $\sup_{x>0} x^n |\xi(x)|^{1/k} < \infty$, or equivalently that $\sup_{x>0} x^{kn} |\xi(x)|$ is finite. This is immediate for when k is an integer. In the general case:

$$\sup_{x>0} x^{kn} |\xi(x)| \leq \max\{ \sup_{0<x<1} x^{[kn]} |\xi(x)|, \sup_{x \geq 1} x^{[kn]+1} |\xi(x)| \} < \infty.$$

Now the proof follows as in the previous cases.

□

We can prove now

Theorem 4.2.31. *Given a sequence of complex numbers* (μ_n) *and a real* $q \geq 1$*, there exists a regularizing function* η *and a functional* L *defined in* L^q *such that* $L(x^n \eta(x)) = \mu_n$*.*

Proof. Consider again the function ξ associated to the sequence (μ_n) as in Duran's Theorem and let $\eta = \xi^{1/2}$ and ν the measure defined above. Define the linear map

$$L : L^q \to \mathbb{R}; f \mapsto L(f) = \int_{-\infty}^{\infty} f(x)\eta(x)\mathrm{d}x.$$

By the Corollary, the measure ν is finite, using the inclusion $L^q(\nu) \hookrightarrow L^1(\nu)$ we get:

$$|\varphi(f)| \leq \int_{-\infty}^{\infty} |f| \mathrm{d}\nu \leq C \left(\int_{-\infty}^{\infty} |f(x)|^q \mathrm{d}\nu \right)^{1/q} \leq K \|f\|_q,$$

where $K = C\|\eta\|_{\infty}^{1/q}$ is a constant. We conclude that the L is continuous.

As $\mathbb{P}_\eta \in L^q$ we have

$$L(x^n \eta(x)) = \int_{-\infty}^{\infty} x^n \eta(x)\eta(x)\mathrm{d}x = \int_{-\infty}^{\infty} x^n \xi(x)\mathrm{d}x = \mu_n.$$

□

The functional L solves our problem, we can associate a polynomial as in the representation theorem that verifies

$$P(x^{n/m}\eta(x)^{1/m}) = L(x^n\eta(x)) = \mu_n.$$

Then the problem of existence of P-orthogonal polynomial sequences is reduced to the existence of Orthogonal Polynomial Sequences.

4.3 Orthogonal Polynomials respect to a bilinear form.

As a generalization of the classical theory of orthogonal polynomials there has been also studied a theory of orthogonality with respect to a bilinear form. We say that a sequence of algebraic polynomials (p_n) is orthogonal respect to B if $B(p_n, p_m) = K_n\delta_{nm}$ for every $n \in \mathbb{N}$ and all $m = 0, 1, \ldots, n$.

A general introduction can be found in [BM]. In particular, a special interest has been devoted to the theory of Sobolev orthogonal polynomials, those polynomials orthogonal with respect to the Sobolev inner product:

$$(f, g)_S = \sum_{j=0}^{k} \int f^{(j)} g^{(j)} \mathrm{d}\mu_j \tag{4.1}$$

where μ_j are positive measures and the derivatives are taken in the weak sense (see for instance [MAR], [MMB], [MF1], [MF2], [M] and the references therein).

The orthogonality with respect to a bilinear form reduces to the classical case of orthogonality with respect to a linear form if we have that $B(f, g) = L(fg)$. Obviously, this is not always general as the Sobolev inner product shows (see (2.4.1) in Chapter 2). Lets see that the Hadamard product (2.2) defined in Chapter 2 will always satisfy that for every inner product B, $B(f, g) = L(f \cdot g)$.

Recall that if H is a Hilbert separable and (e_n) is an orthonormal basis of H, we define the Hadamard product of $x, y \in H$ as $x \cdot y = \sum x_n y_n e_n$ (where x_n, y_n are the Fourier coefficients of x and y respect to the basis (e_n)). We state the notion of product-orthogonality:

Definition 4.3.1. *A sequence of algebraic polynomials (p_n) in a Hilbert space H of functions over some measure space is said to be product-orthogonal* *respect to some linear functional L defined in $H_1 := \{x \in H : (x_n)_n = ((x, e_n))_n \in \ell_1\}$ if $L(p_n \cdot p_m) = K_n \delta_{n,m}$ with $K_n \neq 0$ for every $n, m \in \mathbb{N}$.*

Proposition 4.3.2. *Let B be an inner product defined in the space $\mathbb{P}(\Omega)$ of algebraic polynomials in some interval $\Omega \subset \mathbb{R}$ and (p_n) be a sequence of polynomials orthogonal with respect to B. Then, (p_n) are product-orthogonal respect to some linear functional L.*

Proof. It suffices to complete the pre-Hilbert $(\mathbb{P}(\Omega), B)$ to obtain a Hilbert H which inner product will be denote as well B. This space is necessarily separable and taking an orthonormal basis (e_n), as we saw in Chapter 2, section 2.4, the polynomial $P(f) = B(f, f)$ is orthogonally additive with respect to the coordinate order induced by the basis. Let now L be the linear form, given by the representation theorem which verifies $L(f \cdot g) = B(f, g)$ then, the sequence (p_n) is product-orthogonal with respect to L. □

As we have seen, orthogonality respect to a bilinear form, can be reduced to this new orthogonality with respect to a linear form with the help of the representation theorem. We are going to explore further this ideas. In the paper [Dur2] Durán proves the next theorem:

Theorem 4.3.3. *Let B be an inner product defined in the space of real algebraic polynomials on $\mathbb{R}_+$. The following assertions are equivalent:*

- *The operator multiplication by t is symmetric with respect to B, that is $B(tf, g) = B(f, tg)$.*
- *There exists a non discrete measure μ such that*

$$B(f, g) = \int f(t)g(t)\mathrm{d}\mu(t).$$

This theorem applied to the theory of orthogonal polynomials shows when the bilinear-orthogonality can be reduced to the linear-orthogonality. The proof in [Dur2] is a direct consequence of the strong theorems of Boas ([B]) and Durán ([Dur1]) about the problem of moments.

On the other hand, following the classical book of Akhiezer (see chapter 4 in [Akh]), we can include the problem of moments in the spectral theory of operators. That is the reason to wonder whether there is a result related to previous theorem that can be proved using our technics of representation of orthogonally additive polynomials and its connections with the spectral theory of operators as we saw in Chapter 2. In fact we can extend this result to inner products defined on algebraic polynomials on $\mathbb{R}$ although we need the further assumption for the operator multiplication by t to be bounded.

Theorem 4.3.4. *Let B be an inner product defined on the space of real algebraic polynomials on $\mathbb{R}$. The following are equivalent:*

- *The operator multiplication by t is symmetric and bounded with respect to B.*
- *There exists a Borel regular measure on a compact of $\mathbb{R}$ such that*

$$B(p,q)=\int p(t)q(t)\mathrm{d}\mu(t).$$

Proof. Denote by $\mathbb{P}$ the space of real algebraic polynomials on $\mathbb{R}$ and complete the pre-Hilbert space $(\mathbb{P},B)$ with respect to the norm $\|p\|_B = B(p,p)$ to obtain H a Hilbert space. If we denote by $T:\mathbb{P}\longrightarrow\mathbb{P}$ the multiplication operator (that is $T(p(t)) = tp(t)$) the boundedness hypothesis implies that there exists a continuous self-adjoint extension to H denoted as well by T.

Let $K=\sigma(T)$, since $\|p(T)\| = \|p\|_\infty$ and $p(t)=p(T)1$ then

$$|B(p,q)| \leq \|p\|_B\|q\|_B \leq \|p(T)1\|_B\|q(T)1\|_B \leq \|1\|_B^2\|p\|_\infty\|q\|_\infty$$

therefore, B is continuous with respect to the uniform topology on $\mathbb{P}(K)$ and can be extended to $B: C(K)\times C(K)\longrightarrow\mathbb{R}$.

The proof now follows the steps of the proof of Theorem 2.5.1: the polynomial $P(f) = B(f,f)$ is orthogonally additive in $C(K)$ since $B(f,g) = \lim B(p_n,q_n)$ where (p_n) and (q_n) are sequences of polynomials on K converging uniformly to f and g. Since T is self adjoint $B(p_n,q_n) = B(1,p_nq_n)$ hence if f and g are disjoint elements, as (p_nq_n) converges to $fg=0$ in $C(K)$,

$$\begin{aligned} P(f+g) &= P(f)+2\lim B(p_n,q_n)+P(g) \\ &= P(f)+2\lim B(1,p_nq_n)+P(g) = P(f)+P(g). \end{aligned}$$

By the theorem of representation, there exists a Borel regular measure μ such that

$$B(f,g)=\int fg\mathrm{d}\mu$$

and the conclusion follows as in Theorem 2.5.1 since H is now $L^2(\mu)$.

The reciprocal is immediate. □

Chapter 5

The multilinear Hausdorff problem of moments.

5.1 Introduction.

The problem of moments in a modern setting can be stated as finding a regular Borel measure μ in a given interval $I \in \mathbb{R}$ such that

$$\mu_n = \int_I t^n \mathrm{d}\mu$$

where $(\mu_n) \subset \mathbb{R}$ is a given sequence.

According to Shohat and Tamarkin (see [ST]) the first mathematician who studied the problem of moments was Tchebichef. His main tool was the theory of continuous fractions but his concern was not centered in finding the measure but in its determinacy. More precisely, he asked if

$$\int_{-\infty}^{\infty} x^n p(x) \mathrm{d}x = \int_{-\infty}^{\infty} x^n e^{-x^2} \mathrm{d}x$$

is enough to guarantee that $p(x) = e^{-x^2}$. In other words, he wondered if the moment problem associated to the measure

$$\mathrm{d}\mu = e^{-x^2} \mathrm{d}x$$

is determined, that is, its solution is substantially unique.

An important milestone in the problem of moments are the papers by Stieltjes ([Sti]) appeared in 1894-1895 "Recherches sur les fractions continues". In this work, Stieltjes study the moment problem in relation with the continuous fractions but he also introduces its theory of integration to solve the following problem:

> "Trouver une distribution de masse positive sur une droite $(0, \infty)$, les moments d'ordre n étant donnés."

This moment problem where $I = (0, \infty)$ is called Stieltjes moment problem.

In a paper appeared in 1920 and some subsequent papers, H. Hamburger extends Stieltjes problem from $(0, \infty)$ to the whole real line. This problem is known as the Hamburger moment problem. The case of a finite interval was studied by Hausdorff in 1921. For a complete study on the problem of moments the reader is referred to the books [Akh] and [ST].

This chapter is devoted to the Hausdorff multilinear problem of moments, a generalization to the Hausdorff moment problem to a multilinear setting. We will consider three different cases of the multilinear problem, the strong moment problem, the weak moment problem and the classical moment problem.

Before stating those problems, we will review the classical Hausdorff (linear) moment problem.

As we saw in the previous chapter, this problem is concerned about looking for a Borel regular measure μ such that, given a sequence of moments (μ_n)

$$\mu_n = \int_0^1 t^n \mathrm{d}\mu.$$

For its study, we will use the moment functional L_μ associated to a sequence μ_k, $k \in \mathbb{N}_0$ of real numbers defined to be the element in the (algebraic) dual of the space of polynomials $\mathbb{R}[t]$ defined by:

$$L_\mu(p) = \sum_{k \geq 0} p_k \mu_k, \tag{5.1}$$

where $p(t) = \sum_{k \geq 0} p_k t^k \in \mathbb{R}[t]$ is an arbitrary polynomial.

It is clear that given an interval $I \subset \mathbb{R}$ the classical problem of moments for the sequence μ_k on the interval I amounts to the integrality of such linear operator, that is, under what conditions there exists a regular Borel measure μ on $I \subset \mathbb{R}$ such that $L_\mu(t^k) = \int_I t^k d\mu(t)$, $k = 0, 1 \ldots$

The well-known solution to the classical Hausdorff problem (see for instance [Wid] or [ST]) establishes that such a measure μ exists provided that there is a constant C such that:

$$\sum_{m=0}^{k} |\lambda_{(k;m)}| < C, \tag{5.2}$$

for all $k = 0, 1, \ldots$, where

$$\lambda_{(k;m)} = \binom{k}{m} L_\mu(t^m(1-t)^{k-m}).$$

A simple computation shows that:

$$\lambda_{(k,m)} = \binom{k}{m} \Delta^{k-m}\mu_m, \quad 0 \le m \le k,$$

where Δ denotes the difference operator:

$$\Delta\mu_n = \mu_n - \mu_{n+1},$$

that satisfies:

$$\Delta^r \mu_s = \sum_{l=0}^{r} (-1)^l \binom{r}{l} \mu_{s+l}.$$

The original solution to the Hausdorff problem of moments, based on compactness properties of the set of functions of bounded variation, was used to obtain various versions of Riesz' theorem (see [Hil2] for instance). The extension of this problem to the multivariate setting was solved by Hildebrant and Shoenberg, and Haviland in the thirties [Hil2, Hav1, Hav2] under the general assumption of positivity of the moment functional L_μ. Conversely, Riesz theorem was used to give an alternative proof for the Hausdorff problem of moments [Hil1]. The idea is that condition (5.2) guarantees the continuity of the moment functional associated.

It is also worth to point out the close relation between the problem of moments and the theory of orthogonal polynomials (see for instance [Akh], [Dei]) even though we will not pursue this approach further in this chapter.

Later on Morse and Transue [MT], continuing ideas of Fréchet [Fr3], studied integrality properties of bilinear functionals. They provide various characterizations of them that led to the notion of bimeasures. Again from the perspective of the theory of orthogonal polynomials, Durán [Dur2] has proved recently a generalization of Favard's theorem for polynomials on $[0,+\infty)$ satisfying a generalized recurrence relation by characterizing a class of positive bilinear functionals which are integral in a strong sense. More precisely, let B be a positive bilinear functional on the space of real polynomials on the real half-axis $[0,+\infty)$, then they are equivalent:

i) The operator multiplication by t is symmetric with respect to B, that is, $B(tp,q) = B(p,tq)$ for all polynomials p,q on $\mathbb{R}^+$

ii) there exists a nondiscrete positive measure μ on $\mathbb{R}^+$ such that $B(p,q) = \int p(t)q(t)d\mu(t)$.

The proof of this result relies on the solution of the Stieltjes problem of moments defined by a sequence of moments on the space of rapidly decreasing functions [Dur1].

These results suggest a natural extension of the moment problem to the multilinear case as follows. Let $\mathbf{k} = (k_1,\ldots,k_n)$ be a nonnegative multi-index of length n, that is, $k_1,\ldots,k_n = 0,1,\ldots$. A family $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^n$ of numbers will be called a multi-index moment sequence, a sequence of n-multimoments or just multimoments for short. We will denote by L_μ the n-linear moment functional defined by the multimoments $\mu_{\mathbf{k}}$ on the space of polynomials $\mathbb{R}[t]$ by means of the formula:

$$L_\mu(t^{k_1},\ldots,t^{k_n}) = \mu_{k_1,\ldots,k_n} \text{ for } k_1,\ldots,k_n = 0,1,\ldots.$$

Given a sequence of multimoments $\mu_{\mathbf{k}}$, the strong multilinear Hausdorff problem of moments consist in determining under which conditions there exists a signed regular Borel measure μ on $[0,1]$ such that

$$L_\mu(p_1(t),\ldots,p_n(t)) = \int_0^1 p_1(t)\cdots p_n(t)d\mu(t)$$

where $p_1(t),\ldots,p_n(t) \in \mathbb{R}[t]$.

Clearly this condition is too restrictive on the measure μ, hence the adjective "strong" in the definition. A weaker version of the multilinear Hausdorff moment problem can be stated by demanding that there will exist a signed measure μ on $[0,1]^n \subset \mathbb{R}^n$ such that:

$$L_\mu(p_1(t_1),\ldots,p_n(t_n)) = \int_{I^n} p_1(t_1)\cdots p_n(t_n)d\mu(t_1,\ldots,t_n).$$

Due to its similarities with the classical theory, we will call this problem the classical multilinear Hausdorff moment problem.

It has been recently proved [BV1] that the space of continuous n-linear functionals on spaces of continuous functions on compact sets are in one-to-one correspondence with the space of regular countably additive Borel polymeasures on the corresponding compact spaces.

This theorem (Theorem 7 in [BV1]) is a generalization of Riesz's theorem to the multilinear setting, taking into account the relation between Hausdorff moment problem and Riesz theorem in the linear case, we are naturally led to consider a weaker version of the multilinear Hausdorff problem of moments. Let $\mu_{\mathbf{k}}$ be a multimoment sequence and L_μ its associated n-linear moment functional, we will say that $\mu_{\mathbf{k}}$ satisfies the weak multilinear Hausdorff problem of moments if there exists a polymeasure γ on $\mathrm{Bo}[0,1] \times \cdots \times \mathrm{Bo}[0,1]$ such that:

$$\mu_{k_1,\ldots,k_n} = \int_{I^n} (t^{k_1}, \ldots, t^{k_n}) d\gamma$$

for all $k_1, \ldots, k_n = 0, 1, \ldots$.

As it turns out, the solution to the weak multilinear Hausdorff problem of moments is characterized by a condition that generalizes non-trivially the boundedness condition equation (5.2).

5.2 The weak multilinear Haussdorff moment problem.

5.2.1 Polymeasures and a multilinear Riesz representation theorem.

Before giving the proof of the weak problem, we will review briefly the notion of polymeasures and the theorem by Bombal and Villanueva that extends the classical Riesz's theorem to the multilinear setting. For more information about polymeasures, the reader is referred to the paper by Dobrakov [Dob] or to the Ph.D. Thesis by Villanueva [V].

Definition 5.2.1. *Let Σ_k be a σ-algebra of subsets of the set S_k, $k = 1, \ldots, n$. A polymeasure γ on the σ-algebras $\Sigma_1, \ldots, \Sigma_n$ is a separately valued σ-additive function on the cartesian product of $\Sigma_1, \ldots, \Sigma_n$*

We will consider only real valued polymeasures. According to the definition, if we fix $F_l \in \Sigma_l$, $l \neq k$, the polymeasure γ induces a measure on Σ_k by means of the formula $\gamma_k(\cdot) = \gamma(F_1, \dots, F_{k-1}, \cdot, F_{k+1}, \dots, F_n)$.

As in the linear case, we can define the variation of the polymeasure γ to be the set function $v(\gamma)\colon \Sigma_1 \times \cdots \times \Sigma_n \to [0, +\infty]$ given by:

$$v(\gamma)(A_1, \dots, A_n) = \sup \left\{ \sum_{k_1}^{r_1} \cdots \sum_{k_n}^{r_n} |\gamma(A_1^{k_1}, \dots, A_n^{k_n})| \right\},$$

where the supremum is taken over all finite partitions $\{A_l^{k_l}\}_{k_l=1}^{r_l}$ of the set $A_l \in \Sigma_l$. Following [BV1] we consider also the semivariation

$$\|\gamma\|\colon \Sigma_1 \times \cdots \times \Sigma_n \to [0, +\infty]$$

of the polymeasure γ defined as:

$$\|\gamma\|(A_1, \dots, A_n) = \sup \left\{ \Big| \sum_{k_1}^{r_1} \cdots \sum_{k_n}^{r_n} a_1^{k_1} \cdots a_n^{k_n} \gamma(A_1^{k_1}, \dots, A_n^{k_n}) \Big| \right\}, \quad (5.3)$$

where the supremum is taken over all finite partitions $\{A_l^{k_l}\}_{k_l=1}^{r_l}$ of the set $A_l \in \Sigma_l$ and all collections $\{a_l^{k_l}\}_{k_l=1}^{r_l}$ such that $|a_l^{k_l}| \leq 1$. In the linear case $n = 1$ the semivariation and variation of a measure coincides, however this is not the case for $n > 1$ [BV2].

If the polymeasure γ has finite semivariation an elementary integral denoted as:

$$\int (f_1, \dots, f_n) d\gamma,$$

can be constructed in the obvious way for families of bounded Σ_k-measurable scalar functions f_k, by taking the limits of the integrals of n-tuples of simple functions uniformly converging to the f_k's [Dob].

We will discuss now a n-linear version of Riesz representation theorem. We shall denote by $\mathrm{Bo}(K_l)$ the Borel σ-algebra on the compact space K_l. A polymeasure γ on the product of the σ-algebras $\mathrm{Bo}(K_1) \times \cdots \times \mathrm{Bo}(K_n)$ is said to be regular if for any Borel subsets $A_l \subset K_l$, $l \neq k$, the set function:

$$\gamma_k(A) = \gamma(A_1, \dots, A_{k-1}, A, A_{k+1}, \dots, A_n)$$

is a Radon measure on K_k, $k = 1, \dots, n$. The space of regular countably additive polymeasures on $\mathrm{Bo}(K_1) \times \cdots \times \mathrm{Bo}(K_n)$ will be denoted by rcapm(Bo(K_1), ..., Bo(K_n)) which is a Banach space equipped with the semivariation norm. On the other hand, we will denote the space of continuous scalar n-linear maps by $\mathcal{L}^n(C(K_1),\ldots,C(K_n);\mathbb{R})$. Then we have the following theorem:

Theorem 5.2.2 (Theorem 7 [BV1]). *Let $K_1,\ldots,K_n$ be Hausdorff compact topological spaces. There exists an isometric isomorphism between*

$$\mathcal{L}^n(C(K_1),\ldots,C(K_n);\mathbb{R})$$

and the space rcapm(Bo(K_1), ..., Bo(K_n)) *of regular countably additive Borel polymeasures defined on the product of the Borel σ-algebras of the Hausdorff compact spaces K_l equipped with the semivariation norm.*

The correspondence is easily constructed as follows: Let γ be a polymeasure, then we define the linear map $T_\gamma(f_1,\ldots,f_n) = \int(f_1,\ldots,f_n)d\gamma$. Conversely, given a linear map T, for any fixed Borel sets $A_1,\ldots,A_n$, we will define the set function $\gamma\colon \mathrm{Bo}(K_1)\times\ldots\times\mathrm{Bo}(K_n)\to\mathbb{R}$ as,

$$\gamma(A_1,\ldots,A_n) = T(\chi_{A_1},\ldots,\chi_{A_n}).$$

5.2.2 The weak Hausdorff multilinear moment problem.

Given a multimoment $\mu_{\mathbf{k}}$, $\mathbf{k}\in\mathbb{N}_0^n$, we will denote as above by $L_\mu(t_1^{k_1},\ldots,t_n^{k_n}) = \mu_{\mathbf{k}}$ the associated n-linear moment functional defined on the space of polynomials. We will say that a multimoment sequence $\mu_{\mathbf{k}}$, $\mathbf{k}\in\mathbb{N}_0^n$, is a Hausdorff multimoment family if there exists a polymeasure γ on $[0,1]^n$ such that:

$$\mu_{\mathbf{k}} = \int_{I^n}(t^{k_1},\ldots,t^{k_n})d\gamma, \quad \forall\mathbf{k}=(k_1,\ldots,k_n)\geq 0.$$

We will use throughout the rest of this chapter a consistent multi-index notation. If $\mathbf{r}=(r_1,\ldots,r_n)$ the usual notation stands for $|\mathbf{r}| = r_1+\cdots+r_n$,

$$\sum_{\mathbf{s}=0}^{\mathbf{r}} = \sum_{s_1}^{r_1}\cdots\sum_{s_n}^{r_n} \quad \text{and,} \quad \binom{\mathbf{r}}{\mathbf{s}} = \binom{r_1}{s_1}\cdots\binom{r_n}{s_n}.$$

We will say that $\mathbf{m}\leq\mathbf{k}$ if $m_i\leq k_i$ for all $i=1,\ldots,n$, and $\lim_{\mathbf{k}\to\infty} = \lim_{k_1\to\infty}\cdots\lim_{k_n\to\infty}$. Inspired by the linear case, we will introduce the symbols:

$$\Delta^{\mathbf{r}}\mu_{\mathbf{s}} = \Delta_1^{r_1}\Delta_2^{r_2}\cdots\Delta_n^{r_n}\mu_{s_1\ldots s_n},$$

where Δ_l denotes the difference operator on the lth component, $\Delta_l \mu_{\mathbf{k}} = \mu_{\mathbf{k}} - \mu_{\mathbf{k}+\mathbf{1}_l}$. Then we obtain easily:

$$\Delta^{\mathbf{r}} \mu_{\mathbf{s}} = \sum_{\mathbf{l}=0}^{\mathbf{r}} (-1)^{|\mathbf{l}|} \binom{\mathbf{r}}{\mathbf{l}} \mu_{\mathbf{s}+\mathbf{l}}$$

where $\mathbf{r}, \mathbf{s}, \mathbf{l}$ denote multindexes of length n.

Given a function f on $[0,1]$ we will define the N-Bernstein polynomial of f to be the Nth order polynomial:

$$B_N(f)(t) = \sum_{k=0}^{N} f(k/N) \binom{N}{k} t^k (1-t)^{N-k}.$$

If we denote by

$$\lambda_{k,m}(t) = \binom{k}{m} t^m (1-t)^{k-m},$$

we can write,

$$B_N(f)(t) = \sum_{k=0}^{N} f(k/N) \lambda_{N,k}(t).$$

The Bernstein polynomials $B_N(f)(t)$ converge uniformly to $f(t)$ on $[0,1]$. We will define the Bernstein coefficients $\lambda_{(\mathbf{k};\mathbf{m})}$ of a n-linear functional L, or of a multimoment sequence $\mu_{\mathbf{k}}$ whose associated moment functional is L, as:

$$L(\lambda_{k_1,m_1}(t), \ldots, \lambda_{k_n,m_n}(t)) = \lambda_{(\mathbf{k};\mathbf{m})}. \tag{5.4}$$

If the functional L is the moment functional defined by a multimomentum sequence $\mu_{\mathbf{k}}$, then a simple computation will give us the following relation between the Bernstein coefficients of the moment functional and the sequece of multimoments:

$$\lambda_{(\mathbf{k};\mathbf{m})} = \binom{\mathbf{k}}{\mathbf{m}} \Delta^{\mathbf{k}-\mathbf{m}} \mu_{\mathbf{m}}.$$

We will introduce now two notions of uniform boundedness for a multimoment sequence $\mu_{\mathbf{k}}$ that will characterize the solutions of the classical and weak multilinear Hausdorff problems. The first one is a natural n-linear extension of the boundedness condition equation (5.2).

Definition 5.2.3. *A sequence of multimoments $\mu_{\mathbf{k}}$ is said to be bounded with constant $C > 0$ if:*

$$\sum_{\mathbf{m}=0}^{\mathbf{k}} |\lambda_{(\mathbf{k};\mathbf{m})}| \leq C, \tag{5.5}$$

for all $\mathbf{k} \geq 0$.

As we will see (5.5) provides a sufficient condition for the solution of the weak Hausdorff problem however it is not necessary. The appropriate necessary condition for that, inspired on the definition of a polymeasure semivariation equation (5.3), is the following:

Definition 5.2.4. *A multimoment sequence $\mu_{\mathbf{k}}$ is said to be weakly bounded with constant $C > 0$ if:*

$$\left| \sum_{\mathbf{m}=0}^{\mathbf{k}} \mathbf{a}_{\mathbf{m}}^{\mathbf{k}} \lambda_{(\mathbf{k};\mathbf{m})} \right| \leq C, \tag{5.6}$$

where $\mathbf{a}_{\mathbf{m}}^{\mathbf{k}} = a_{m_1}^{k_1} \cdots a_{m_n}^{k_n}$, for all $a_{m_l}^{k_l}$ such that $|a_{m_l}^{k_l}| \leq 1$, $l = 1, \dots, n$, and for all $\mathbf{k} \geq 0$.

It is clear from the definitions that the boundedness condition (5.5) implies weak boundedness (5.6), however the converse is not true. Later on we will discuss the relation among them in more detail.

Theorem 5.2.5. *A sequence of multimoments $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}^n$ is a solution of the weak Hausdorff multilinear moment problem if and only if it is weakly bounded.*

Proof. If $\mu_{\mathbf{k}}$ is a solution of the weak Hausdorff multilinear moment problem, then there will exists a continuous multilinear functional L on $C[0,1]$ such that $L(\mathbf{t}^{\mathbf{k}}) = \mu_{\mathbf{k}}$. Let $\mathbf{a}_{\mathbf{m}}^{\mathbf{k}} = a_{m_1}^{k_1} \cdots a_{m_n}^{k_n}$, $\mathbf{k} \geq 0$, with $|a_{m_l}^{k_l}| \leq 1$, $l = 1, \dots, n$.

Then:

$$\begin{aligned}
\left|\sum_{\mathbf{m}=0}^{\mathbf{k}} \mathbf{a}_{\mathbf{m}}^{\mathbf{k}} \lambda_{(\mathbf{k};\mathbf{m})}\right| &= \left|\sum_{\mathbf{m}=0}^{\mathbf{k}} \mathbf{a}_{\mathbf{m}}^{\mathbf{k}} L(\lambda_{(k_1,m_1)}(t), \ldots, \lambda_{(k_n,m_n)}(t))\right| \\
&= \left|\sum_{\mathbf{m}=0}^{\mathbf{k}} L(a_{k_1}^{m_1} \lambda_{(k_1,m_1)}(t), \ldots, a_{k_n}^{m_n} \lambda_{(k_n,m_n)}(t))\right| \\
&= \left|L\left(\sum_{m_1=0}^{k_1} a_{k_1}^{m_1} \lambda_{(k_1,m_1)}(t), \ldots, \sum_{m_n=0}^{k_n} a_{k_n}^{m_n} \lambda_{(k_n,m_n)}(t)\right)\right| \\
&\leq \|L\| \prod_{l=1}^{n} \| \sum_{m_l=0}^{k_l} a_{k_l}^{m_l} \lambda_{(k_l,m_l)}(t)\|_\infty
\end{aligned}$$

Moreover:

$$\begin{aligned}
\left|\sum_{m_l=0}^{k_l} a_{k_l}^{m_l} \lambda_{(k_l,m_l)}(t)\right| &\leq \sum_{m_l=0}^{k_l} |a_{k_l}^{m_l}||\lambda_{(k_l,m_l)}(t)| \\
&\leq \sum_{m_l=0}^{k_l} \binom{k_l}{m_l} t^{k_l-m_l}(1-t)^{m_l} = 1.
\end{aligned}$$

and we conclude that

$$\left|\sum_{\mathbf{m}=0}^{\mathbf{k}} \mathbf{a}_{\mathbf{m}}^{\mathbf{k}} \lambda_{(\mathbf{k},\mathbf{m})}\right| \leq \|L\|,$$

for all $|a_{m_l}^{k_l}| \leq 1$, $\mathbf{k} \geq 0$.

Conversely, if we assume that the sequence of multimoments $\mu_{\mathbf{k}}$ is weakly bounded, let us prove that the multilinear functional L_μ defined on the space of polynomials on the interval $[0, 1]$ is continuous in the uniform topology. Let n be the length of the multi-index $\mathbf{k}$. We will show by induction on n that:

$$|L(p_1, \ldots, p_n)| \leq 2^n C \|p_1\|_\infty \cdots \|p_n\|_\infty,$$

for any family of polynomials $p_1, \ldots, p_n$, provided that the multimoment sequence $\mu_{\mathbf{k}}$ is weakly bounded.

For $n = 1$, the weakly bounded condition for multimomentum sequences because of Theorem 5.2 is equivalent to the bounded condition.

Now we will assume that if L' is a $(n-1)$–multilinear functional associated to the bounded $(n-1)$–multimoment sequence $\mu'_{\mathbf{k}'}$ with bounding constant C', then

$$|L'(q_1, \ldots, q_{n-1})| \leq 2^{n-1} C' \|q_1\|_\infty \cdots \|q_{n-1}\|_\infty,$$

for any family of polynomials $q_1, \ldots, q_{n-1}$.

Let us consider now the n-multimoment functional L associated to the bounded n-multimoment sequence $\mu_{\mathbf{k}}$. We shall denote by L_p the $(n-1)$–multilinear functional obtained by fixing the nth argument of L to be the polynomial p, that is,

$$L_p(q_1, \ldots, q_{n-1}) = L(q_1, \ldots, q_{n-1}, p).$$

Notice that if $p(t)$ is a polynomial of degree r then [ST]:

$$B_N(p)(t) = p(t) + S_N(t) = p(t) + \sum_{l=1}^{r-1} \frac{p_{r,l}(t)}{N^l},$$

where $p_{r,l}$ are polynomials of degree less or equal that r, not depending on N.

Now denoting by $a^{\mathbf{k}'}_{\mathbf{m}'} = a^{k_1}_{m_1} \cdots a^{k_{n-1}}_{m_{n-1}}$ with $|a^{k_l}_{m_l}| \leq 1$, it is clear that:

$$\left| \sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'} L_{B_N(p)}(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1})) \right| \leq$$
$$\leq \left| \sum_{\mathbf{m}'=0}^{\mathbf{k}'} \sum_{m_n=0}^{N} a^{\mathbf{k}'}_{\mathbf{m}'} p(m_n/N) L(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \lambda_{N,m_n}(t_n)) \right| \leq$$
$$\leq \|p\|_\infty \left| \sum_{\mathbf{m}=0}^{\mathbf{k}} a^{\mathbf{k}}_{\mathbf{m}} L(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \lambda_{k_n,m_n}(t_n)) \right|$$

with

$$\mathbf{k} = (k_1, \ldots, k_{n-1}, N), \quad \mathbf{m} = (m_1, \ldots, m_{n_1}, m_n),$$

and $a^{\mathbf{k}}_{\mathbf{m}} = a^{\mathbf{k}'}_{\mathbf{m}'}(p(m_n/N)/\|p\|_\infty)$. Hence,

$$\left|\sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'} L_{B_N(p)}(\lambda_{k_1,m_1}(t_1),\ldots,\lambda_{k_{n-1},m_{n-1}}(t_{n-1}))\right| \le$$
$$\le \|p\|_\infty \left|\sum_{\mathbf{m}=0}^{\mathbf{k}} a^{\mathbf{k}}_{\mathbf{m}} \lambda_{(\mathbf{k};\mathbf{m})}\right| \le C\|p\|_\infty$$

because $\mu_{\mathbf{k}}$ is weakly bounded with constant C. Then the $(n-1)$–multimoment sequence defined by $L_{B_N(p)}$ is weakly bounded with bound $C\|p\|_\infty$, and by induction we have:

$$|L_{B_N(p)}(p_1,\ldots,p_{n-1})| \le 2^{n-1}C\|p_1\|_\infty \cdots \|p_{n-1}\|_\infty \|p\|_\infty.$$

Similarly, we consider now $S_N(t) = \sum_{l=1}^{r-1} \frac{p_{r,l}(t)}{N^l}$. If the polynomials $p_{r,l}$ have the form, $p_{r,l}(t) = \sum_{j=0}^{r} a_{lj} t^j$, then by choosing:

$$a = \max\{|a_{lj}|\}, \quad N \ge \frac{a(r-1)(r+1)}{\|p\|_\infty}, \tag{5.7}$$

and using the notations above, we will get:

$$\begin{aligned}
&\left|\sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'} L_{S_N(p)}(\lambda_{k_1,m_1}(t_1),\ldots,\lambda_{k_{n-1},m_{n-1}}(t_{n-1}))\right| \le \\
&\le \left|\sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'} \sum_{l=1}^{r-1} L(\lambda_{k_1,m_1}(t_1),\ldots,\lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \frac{p_{r,l}(t_n)}{N^l})\right| = \\
&= \left|\sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'} \sum_{j=0}^{r}\sum_{l=1}^{r-1} \frac{a_{lj}}{N^l} L(\lambda_{k_1,m_1}(t_1),\ldots,\lambda_{k_{n-1},m_{n-1}}(t_{n-1}), t_n^j)\right| \le \\
&\le \frac{\|p\|_\infty}{r+1}\sum_{j=0}^{r}\left|\sum_{\mathbf{m}'=0}^{\mathbf{k}'} a^{\mathbf{k}'}_{\mathbf{m}'}\left(\frac{r+1}{\|p\|_\infty}\sum_{l=1}^{r-1}\frac{a_{lj}}{N^l}\right) L(\lambda_{k_1,m_1}(t_1),\ldots\right. \\
&\quad \left.\ldots,\lambda_{k_{n-1},m_{n-1}}(t_{n-1}),\lambda_{j,j}(t_n))\right|
\end{aligned} \tag{5.8}$$

If we denote by

$$a^j_j = \frac{r+1}{\|p\|_\infty}\sum_{l=1}^{r-1}\frac{a_{lj}}{N^l}$$

considering the conditions equation (5.7), we easily check that $|a_j^j| \leq 1$. Now we will define the collection of numbers $a_{m_n}^j$ in such a way that $a_{m_n}^j = 0$ for all $0 \leq m_n < j$. With these definitions, we get that last term in the sequence of inequalities (5.8), can be written as:

$$\frac{\|p\|_\infty}{r+1} \sum_{j=0}^{r} \left| \sum_{\mathbf{m}'=0}^{\mathbf{k}'} a_{\mathbf{m}'}^{\mathbf{k}'} a_j^j L(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \lambda_{j,j}(t_n)) \right| =$$

$$= \frac{\|p\|_\infty}{r+1} \sum_{j=0}^{r} \left| \sum_{\mathbf{m}'=0}^{\mathbf{k}'} \sum_{m_n=0}^{j} a_{\mathbf{m}'}^{\mathbf{k}'} a_{m_n}^j L(\lambda_{k_1,m_1}(t_1), \ldots \right.$$

$$\left. \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \lambda_{j,m_n}(t_n)) \right| =$$

$$= \frac{\|p\|_\infty}{r+1} \sum_{j=0}^{r} \left| \sum_{\mathbf{m}=0}^{\mathbf{k}} a_{\mathbf{m}}^{\mathbf{k}} L(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1}), \lambda_{j,m_n}(t_n)) \right|$$

with $\mathbf{k} = (k_1, \ldots, k_{n-1}, j)$ and $\mathbf{m} = (m_1, \ldots, m_n)$. Hence finally we obtain:

$$\left| \sum_{\mathbf{m}'=0}^{\mathbf{k}'} a_{\mathbf{m}'}^{\mathbf{k}'} L_{S_N(p)}(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}}(t_{n-1})) \right| \leq$$

$$\leq \frac{\|p\|_\infty}{r+1} \sum_{j=0}^{r} \left| \sum_{\mathbf{m}=0}^{\mathbf{k}} a_{\mathbf{m}}^{\mathbf{k}} \lambda_{(\mathbf{k};\mathbf{m})} \right| \leq \frac{\|p\|_\infty}{r+1} \sum_{j=0}^{r} C = C\|p\|_\infty$$

and the sequence of multimoments $\mu''_{\mathbf{k}''} = L_{S_N(p)}(\lambda_{k_1,m_1}(t_1), \ldots, \lambda_{k_{n-1},m_{n-1}})$ is weakly bounded with constant $C\|p\|_\infty$.

We conclude the argument by using the induction hypothesis and computing:

$$\begin{aligned} |L(p_1, \ldots, p_n)| &= |L_{p_n}(p_1, \ldots, p_{n-1})| \\ &\leq |L_{B_N(p_n)}(p_1, \ldots, p_{n-1})| + |L_{S_N(p_n)}(p_1, \ldots, p_{n-1})| \\ &\leq 2^n C \|p_1\|_\infty \cdots \|p_n\|_\infty \end{aligned}$$

□

5.3 The Hausdorff multilinear problem of moments and the classical multivariate Hausdorff problem of moments.

Hildebrant and Schoenberg [Hil2] obtained necessary and sufficient conditions to solve the Hausdorff moment problem in two variables. Later on Haviland [Hav1], [Hav2] extended these results to any number of variables and arbitrary domains under the assumption of positive moment sequences. We will revisit those results under the perspective of the multilinear problem discussed in this chapter.

Given a sequence of multimoments $\mu_{\mathbf{k}}$ we can also associate to it, besides the functional L_μ defined by equation (5.1), the linear functional $\hat{L}_\mu$ on the space of polynomials on n variables $\mathbb{R}[t_1, \ldots, t_n]$ given by the formula:

$$\hat{L}_\mu(\mathbf{t}^{\mathbf{k}}) = \hat{L}(t_1^{k_1} \cdots t_n^{k_n}) = \mu_{k_1,\ldots,k_n}, \quad \forall k_1, \ldots, k_n = 0, 1, \ldots.$$

Thus we can consider the solution for the multivariate Haussdorff problem of moments defined by $\mu_{\mathbf{k}}$: under what conditions there exists a signed measure μ on $[0,1]^n$ such that

$$\hat{L}_\mu(\mathbf{t}^{\mathbf{k}}) = \int_{[0,1]^n} \mathbf{t}^{\mathbf{k}} d\mu,$$

The well-known answer to this question is that the boundedness condition equation (5.5) for the multimoment sequence $\mu_{\mathbf{k}}$ solves the classical Hausdorff multivariate problem and we will give here a proof that is a refined version of the arguments in [Hil2].

Theorem 5.3.1. *Let $\mu_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{N}_0^n$ be a sequence of multimoments. Then there will exists a signed regular Borel measure μ on* $\mathrm{Bo}([0,1]^n)$ *solving the Haussdorff multilinear problem of moments defined by $\mu_{\mathbf{k}}$ if and only if the multimoment sequence $\mu_{\mathbf{k}}$ is bounded.*

Proof. Denote by $P_{\mathbf{N},\mathbf{k}}$, $0 \leq \mathbf{k} \leq \mathbf{N}$, the set of points in $[0,1]^n$ defined by:

$$P_{\mathbf{N},\mathbf{k}} = (k_1/N_1, \ldots, k_n/N_n).$$

Now let us define for each $\mathbf{N}$ the Dirac measure $\mu_{(\mathbf{N})}$ which are concentrated at the points $P_{\mathbf{N},\mathbf{k}} \in [0,1]^n$, $0 \leq \mathbf{k} \leq \mathbf{N}$, and with weights given by the Bernstein coefficients $\lambda_{(\mathbf{N},\mathbf{k})}$ of $\mu_{\mathbf{k}}$ respectively, this is

$$\mu_{(\mathbf{N})} = \sum_{\mathbf{k}=0}^{\mathbf{N}} \lambda_{(\mathbf{N},\mathbf{k})} \delta(P_{\mathbf{N},\mathbf{k}}).$$

Clearly the discrete measures $\mu_{(\mathbf{N})}$ have finite total variation given by:

$$\|\mu_{(\mathbf{N})}\| = \sum_{\mathbf{k}=0}^{\mathbf{N}} |\lambda_{(\mathbf{N},\mathbf{k})}|.$$

Hence, since the moment sequence $\mu_{\mathbf{k}}$ is bounded, there exists C such that:

$$\|\mu_{(\mathbf{N})}\| \leq C, \quad \forall \mathbf{N} \in \mathbb{N}_0^n.$$

Thus we conclude that the family of measures $\mu_{(\mathbf{N})}$ is contained in the ball of radius C on the space of regular Borel measures on $[0,1]^n \subset \mathbb{R}^n$. Because of Alaoglu-Bourbaki's theorem the unit ball on the space of regular Borel measures on $[0,1]^n$ is compact on the weak*-topology on $\mathrm{Bo}([0,1]^n) = C([0,1]^n)^*$, and consequently there is a subsequence, that will be denoted again by $\mu_{(\mathbf{N})}$, converging on the weak*-topology, that is, there exists a regular Borel measure μ on $[0,1]^n$ such that for all $f \in C([0,1]^n)$, we have:

$$\lim_{\mathbf{N}\to\infty} \int_{[0,1]^n} f(\mathbf{t}) d\mu_{(\mathbf{N})} = \int_{[0,1]^n} f(\mathbf{t}) d\mu.$$

We will conclude the proof by showing that $\int_{[0,1]^n} \mathbf{t}^{\mathbf{k}} d\mu = \mu_{\mathbf{k}}$. The Bernstein $B_N(f)(t)$ polynomials converge to f in the uniform topology as $N \to \infty$. Then because the sequence of moments $\mu_{\mathbf{k}}$ is bounded, its multilinear moment functional L_μ is continuous, and $L_\mu(B_{N_1}(t^{k_1}), \ldots, B_{N_n}(t^{k_n}))$ converges to $\mu_{\mathbf{k}}$ as $\mathbf{N} \to \infty$.

Now because

$$\lim_{\mathbf{N}\to\infty} \int_{[0,1]^n} \mathbf{t}^{\mathbf{k}} d\mu_{(\mathbf{N})} = \int_{[0,1]^n} \mathbf{t}^{\mathbf{k}} d\mu,$$

and

$$\int_{[0,1]^n} \mathbf{t}^{\mathbf{k}} d\mu_{(\mathbf{N})} = \sum_{\mathbf{j}=0}^{\mathbf{N}} P_{\mathbf{N},\mathbf{j}}^{\mathbf{k}} \lambda_{(\mathbf{N},\mathbf{j})},$$

since for $\mathbf{t} \in [0,1]^n$,

$$L_\mu(B_{N_1}(t^{k_1}), \ldots, B_{N_n}(t^{k_n})) = \sum_{\mathbf{j}=0}^{\mathbf{N}} P_{\mathbf{N},\mathbf{j}}^{\mathbf{k}} \lambda_{(\mathbf{N},\mathbf{j})},$$

the conclusion follows.

□

5.4 Positivity and the Hausdorff problem.

A different solution for the classical Hausdorff moment problem was obtained by using Riesz extension theorem for positive linear functionals (see for instance [Akh]). We shall explore in this section the relation between positivity and continuity of the two functionals L_μ and $\hat{L}_\mu$.

It is well-known that positivity plays a fundamental role in the analysis of the classical Hausdorff problem. In the one-dimensional case, we say that a given sequence of moments μ_k, $k = 0, 1, 2, ...$, is positive in the domain $\Omega \subset \mathbb{R}$ if the moment functional L_μ is positive on Ω, that is, if $L(p) \geq 0$ for all positive polynomials p on Ω (that is, polynomials such that $p(t) \geq 0$ for all $t \in \Omega$).

For a sequence of n-multimoments $\mu_{\mathbf{k}}$ with $n > 1$, there are two different notions of positivity depending on which one of the functionals L_μ or $\hat{L}_\mu$ we use to define it:

Definition 5.4.1. *We will say that the sequence of multimoments $\mu_{\mathbf{k}}$ is π-positive if for all polynomials $p_1, \ldots, p_n \in \mathbb{R}[t]$ positive on the interval $[0,1]$, $L_\mu(p_1, \ldots, p_n) \geq 0$.*

A sequence of multimoments $\mu_{\mathbf{k}}$ will be said to be ϵ-positive if $\hat{L}_\mu(p) \geq 0$, for all polynomials $p \in \mathbb{R}[t_1, \ldots, t_n]$ positive on the interval $[0,1]$.

It is clear that if $\mu_{\mathbf{k}}$ is ϵ-positive it is also π-positive. The converse of this however is not immediately obvious. The reason for that lies in the difference among the π (or projective) and the ϵ (or injective) topologies on the tensor products of the space $C[0,1]$.

The continuity of the n-linear functional L_μ is equivalent to the continuity of $\hat{L}_\mu$ with respect to the projective topology on $C[0,1] \otimes \cdots \otimes C[0,1]$, in other words, if the multilinear functional L_μ is continuous on the space of polynomials defined on the unit interval, then $\hat{L}_\mu$ can be extended to the space $C[0,1] \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi C[0,1]$.

In general, such operator cannot be extended to $C([0,1]^n)$. If such extension exists the operator is called integral and because Riesz theorem there will exists a positive Radon measure ν on $[0,1]^n$ representing it, that is

$$\hat{L}_\mu(f) = \int_{[0,1]^n} f d\nu.$$

It is well-known that the operator $\hat{L}_\mu$ is integral if and only if it is continuous with respect to the injective topology on the tensor product $C[0,1]\otimes\cdots\otimes C[0,1]$ because $C([0,1]^n) \cong C[0,1]\hat{\otimes}_\epsilon\cdots\hat{\otimes}_\epsilon C[0,1]$. Besides it is also well-known that the completion of the tensor product $C[0,1]\otimes\cdots\otimes C[0,1]$ with respect to the π-topology is strictly contained in its completion with respect to the ϵ-topology.

Furthermore the ϵ-positivity of $\mu_{\mathbf{k}}$ amounts to the positivity of $\hat{L}_\mu$ with respect to the cone of positive functions on $[0,1]^n$ restricted to the space of polynomials on $[0,1]^n$. In such case, because of Riesz extension theorem for positive functionals, the functional $\hat{L}_\mu$ can be extended to a positive functional on $C([0,1]^n)$, hence $\hat{L}_\mu$ will be continous with respect to the uniform topology. Now because of Riesz theorem there will exists a Radon measure associated to it and the sequence $\mu_{\mathbf{k}}$ will be a solution of the classical Hausdorff problem.

It is known that there are π-continuous n-linear functionals on $C[0,1]$ which are not integral and that cannot be extended to $C([0,1]^n)$. Hence, it could be suspected that the condition of π-positivity would be strictly weaker than ϵ-positivity. This is not the case as it is shown in the following theorem.

Theorem 5.4.2. *A weakly bounded multimomentum sequence $\mu_{\mathbf{k}}$ is π-positive if and only if $\mu_{\mathbf{k}}$ is ϵ-positive.*

Proof. If $\mu_{\mathbf{k}}$ is π-positive, then L_μ is positive. Hence because equation (5.4), Bernstein coefficients $\lambda_{(\mathbf{k};\mathbf{m})}$ are positive. Then, if the multimomentum sequence $\mu_{\mathbf{k}}$ is weakly bounded, then choosing all coefficients $\mathbf{a}_{\mathbf{m}}^{\mathbf{k}} = 1$ on (5.6) we obtain that $\mu_{\mathbf{k}}$ is bounded. Because of Theorem 5.3.1 the functional $\hat{L}_\mu$ is integral, hence it is continuous in the uniform topology on $[0,1]^n$. We will show that $\hat{L}_\mu$ is positive. Let $f \geq 0$ a positive function on $[0,1]^n$. We can approximate uniformly the function f by using the sequence of Bernstein polynomials $B_{\mathbf{k}}(f)(t_1,\ldots,t_n) = \sum_{\mathbf{m}=0}^{\mathbf{N}} f(\mathbf{m}/\mathbf{k})\lambda_{(\mathbf{k};\mathbf{m})}(t_1,\ldots,t_n)$. However, because $f \geq 0$, we have:

$$\hat{L}_\mu(B_{\mathbf{k}}(f)(t_1,\ldots,t_n)) = \sum_{\mathbf{m}=0}^{\mathbf{N}} f(\mathbf{m}/\mathbf{k})\lambda_{(\mathbf{k};\mathbf{m})} \geq 0.$$

Since $\hat{L}_\mu$ is continuous in the uniform topology and $B_{\mathbf{k}}(f)$ converges uniformly to f, the conclusion follows. □

Notice that the linear functional $\hat{L}_\mu$ is nothing but the linearization of the multilinear functional L_μ, that is there is a one-to-one correspondence between multilinear maps $L\colon E_1 \times \cdots \times E_n \to \mathbb{R}$ and linear maps $\hat{L}\colon E_1 \otimes \cdots \otimes E_n \to \mathbb{R}$ given by $\hat{L}(x_1 \otimes \cdots \otimes x_n) = L(x_1, \ldots, x_n)$ for all decomposable elements $x_1 \otimes \cdots \otimes x_n \in E_1 \otimes \cdots \otimes E_n$. In the particular instance we are considering here, all linear spaces E_k are copies of $\mathbb{R}[t]$ and the nth tensor product of $\mathbb{R}[t]$ with itself gives $\mathbb{R}[t] \otimes \cdots \otimes \mathbb{R}[t] \cong \mathbb{R}[t_1, \ldots, t_n]$.

As it was discussed before, the weak Hausdorff moment problem requires only the existence of a polymeasure on the product domain K^n. The fundamental point to consider here is that a polymeasure $\beta\colon \Sigma_1 \times \cdots \times \Sigma_n \to \mathbb{R}$ defines a finitely additive measure on the algebra generated by $\Sigma_1 \times \cdots \times \Sigma_n$ in the product σ-algebra $\Sigma_1 \otimes \cdots \otimes \Sigma_n$, but in general such measure cannot be extended to a measure on the product σ-algebra. In the particular case of the weak Hausdorff problem we can answer this question affirmatively by using the positivity condition above for its sequence of multimoments $\mu_{\mathbf{k}}$.

Theorem 5.4.3. *Let $\mu_{\mathbf{k}}$ be a weakly bounded multimoment sequence on the unit interval, then they are equivalent:*

i. The regular Borel polymeasure γ defined by $\mu_{\mathbf{k}}$ can be extended to a positive regular Borel measure in the Borel σ-algebra of $[0,1]^n$.

ii. $\lambda_{(\mathbf{k};\mathbf{m})} \geq 0$, for all $\mathbf{k} \geq 0$.

Proof. Suppose that there exists a positive measure μ on $[0,1]^n$ such that

$$\int (f_1, \ldots, f_n) d\gamma = \int f_1 \cdots f_n d\mu.$$

If $f_1, \ldots, f_n \geq 0$, because μ is positive, then:

$$\int (f_1, \ldots, f_n) d\gamma \geq 0.$$

In particular

$$\lambda_{(\mathbf{k},\mathbf{m})} = \int (t^{m_1}(1-t)^{k_1-m_1}, \ldots, t^{m_n}(1-t)^{k_n-m_n}) d\gamma \geq 0.$$

Conversely, because $\mu_{\mathbf{k}}$ is weakly bounded, there will exists a regular Borel polymeasure γ satisfying:

$$\mu_{\mathbf{k}} = \int (t^{k_1}, \dots, t^{k_n}) d\gamma.$$

We denote as usual by L the n-linear functional defined by

$$L(f_1, \dots, f_n) = \int (f_1, \dots, f_n) d\gamma \quad f_1, \dots, f_n \in C[0,1].$$

Now any positive polynomial on the unit interval $[0,1]$ can be uniformly approximate by a sequence of combinations of the polynomials λ_{k_l,m_l}, $l = 1, \dots, n$, with positive coefficients (by using Bernstein polynomials for instance), then the positivity condition (ii) implies that L_μ is positive and hence because of Theorem 5.4.2, $\hat{L}$ will be positive too on the space of polynomials on $[0,1]^n$. Moreover because of Riesz extension theorem there will exists a positive extension of $\hat{L}$ to $C([0,1]^n)$, hence a positive measure ν on $[0,1]^n$ such that:

$$\mu_{\mathbf{k}} = \int t_1^{k_1} \cdots t_n^{k_n} d\nu(t_1, \dots, t_n) = \int (t^{k_1}, \dots, t^{k_n}) d\gamma,$$

therefore, the polymeasure γ can be extended to the positive measure ν. □

The previous result can also be obtained by using the following observation:

Lemma 5.4.4. *Let $\mu_{\mathbf{k}}$ be a weakly bounded positive multimoment sequence, then the corresponding polymeasure γ has finite variation $v(\gamma)$.*

Proof. If the multimoment sequence $\mu_{\mathbf{k}}$ is positive the polymeasure γ will be positive, hence a simple computation shows that,

$$v(\gamma)([0,1]^n) = \mu_{0,\dots,0} < \infty.$$

□

The previous lemma together with the following characterization by Bombal and Villanueva [BV2] of Borel polymeasures that can be extended to a measure lead again to the previous result:

Theorem 5.4.5 (Theorem 3.3 [BV2]). *Let* $T: C(K_1) \times \cdots \times C(K_n) \to \mathbb{R}$ *be a continuous* n*-linear operator with representing polymeasure* γ*. Then the following are equivalent:*

(i) $v(\gamma) < \infty$.

(ii) The operator T *is integral.*

(iii) γ *can be extended to a Borel measure* μ *in* $\mathrm{Bo}(K_1 \times \cdots \times K_n)$.

5.5 The strong Hausdorff multilinear moment problem.

As it was discussed in the introduction, the results by Durán [Dur2] suggest a strong version of the multilinear problem of moments. Given a multimomentum sequence $\mu_{\mathbf{k}}$, $\mathbf{k} = (k_1, \cdots, k_n)$, $k_l = 0, 1, \ldots$, $l = 1, \ldots, n$, and a given subset $K \subset \mathbb{R}$, under what conditions there exists a finite measure μ on K such that the multilinear functional L defined on the space of real polynomials $\mathcal{P}$ on K by means of:

$$L(t_1^{k_1}, \cdots, t_n^{k_n}) = \mu_{k_1, \ldots, k_n}$$

has the simple expression:

$$L(f_1, \ldots, f_n) = \int_K f_1(t) \cdots f_n(t) d\mu(t)?$$

Obviously, a necessary condition for that is the existence of a sequence of moments μ_r such that $\mu_{k_1,\ldots,k_n} = \mu_{|\mathbf{k}|}$, for all $\mathbf{k} = (k_1, \ldots, k_n)$. As it turns out, if the multilinear sequence of moments $\mu_{\mathbf{k}}$ is bounded, the previous condition is also sufficient.

Definition 5.5.1. *We will say that a multimoment sequence* $\mu_{\mathbf{k}}$*,* $\mathbf{k} \in \mathbb{N}^n$*, is Hänkel if* $\mu_{\mathbf{k}+\mathbf{1}_l} = \mu_{\mathbf{k}+\mathbf{1}_{l+1}}$*, where* $(\mathbf{1}_l)_j = \delta_{lj}$*, for all* $l = 1, \ldots, n$.

Finally we present the proof of the strong Hausdorff moment problem which is based on the representation theorem of orthogonally additive polynomials by [BLLl] presented in Chapter 2.

Theorem 5.5.2. *Let K be compact subset of $\mathbb{R}$ and let $\mu_{\mathbf{k}}$ be a bounded multilinear sequence of moments. Then there exists a finite measure μ on K solving the strong problem of moments if and only if the multimomentum sequence $\mu_{\mathbf{k}}$ is Hänkel.*

Proof. Consider the n-linear moment functional defined on the space $\mathcal{P}$ of real polynomials on the subset K by the multimoment sequence $\mu_{\mathbf{k}}$. Because the multimomentum sequence $\mu_{\mathbf{k}}$ is bounded, then L can be extended to $C(K)$ (Theorem 5.2.5). We shall denote such extension with the same symbol L. We shall show now that the homogeneous polynomial P_L determined by L is orthogonally additive. For that we notice that $L(f_1, \dots, g \cdot f_l, f_{l+1}, \dots, f_n) = L(f_1, \dots, f_l, g \cdot f_{l+1}, \dots, f_n)$ for all $f_1, \dots, f_n, g \in C(K)$. In fact we can construct a sequence of polynomials p_{m_l}, q_m uniformly converging to f_l and g respectively ($l = 1, \dots, n$) on K, hence because the multimomentum sequence $\mu_{\mathbf{k}}$ is Hänkel, we clearly have:

$$L(p_{m_1}, \dots, q_m \cdot p_{m_l}, p_{m_{l+1}}, \dots, p_{m_n}) = L(p_{m_1}, \dots, p_{m_l}, q_m \cdot p_{m_{l+1}}, \dots, p_{m_n}),$$

and the conclusion follows because of the continuity of L.

Now suppose we have two disjoint positive functions f, g on $C(K)$, $f \wedge g = 0$. We compute:

$$\begin{aligned} P_L(f+g) &= L(f+g, \dots, f+g) = \sum_{r \geq 0} \binom{n}{r} L(f, \overset{n-r}{\dots}, f, g, \overset{r}{\dots}, g) = \\ &= L(f, \dots, f) + \sum_{r=1}^{n-1} \binom{n}{r} L(1, f, \overset{n-r-1}{\dots}, f, g, \overset{r-1}{\dots}, g, f \cdot g) \\ &+ L(g, \dots, g) = P_L(f) + P_L(g), \end{aligned}$$

because $f \cdot g = 0$.

Now we will apply Theorem 2.2.2. As the n-concavification of the Banach lattice $C(K)$ coincides with itself, $C(K)_{(n)} = C(K)$. Hence the polynomial P_L defines a bounded linear functional $T \colon C(K) \to \mathbb{R}$,

$$T(f^n) = P_L(f) = L(f, \cdots, f)$$

and then, by Riesz theorem, there will exists a Radon measure μ such that $T(f^n) = \int_K f(t)^n d\mu(t)$. Hence $L(t^{k_1}, \dots, t^{k_n}) = \int_K t^{k_1 + \dots + k_n} d\mu(t)$, and consequently we conclude that:

$$L(f_1, \dots, f_n) = \int_K f_1(t) \cdots f_n(t) d\mu(t),$$

and the measure μ solves the strong moment problem for $\mu_{\mathbf{k}}$.

$\square$

Chapter 6

The multilinear trigonometric problem of moments.

In this chapter we will provide a solution for the weak multilinear trigonometric problem of moments.

In the linear setting, the classical trigonometric problem of moments consists in determining necessary and sufficient conditions for the existence of a positive regular Borel measure μ on the circle $\mathbb{T}$ such that for a given sequence of complex numbers c_k, $k \in \mathbb{Z}$ we have:

$$c_k = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta), \quad k \in \mathbb{Z}. \tag{6.1}$$

Consider the Fourier-Stieltjes transform of the measure μ, that is the function $\hat{\mu}$ on $\mathbb{Z}$ (the dual group of $\mathbb{T}$) defined as

$$\hat{\mu}(k) = \int_{\mathbb{T}} e^{-ik\theta} \mu(\mathrm{d}\theta)$$

for all $k \in \mathbb{Z}$. The trigonometric moment problem is equivalent to find a function c defined on $\mathbb{Z}$ such that $c = \hat{\mu}$.

In 1911, Riesz and Herglotz [Rie, Her] showed independently that a necessary and sufficient condition for the existence of such a measure is that the function $c(k) = c_k$ is positive definite, that is, if for any positive integer N the quadratic form

$$\sum_{k,l=0}^{N} c(k-l)\bar{\xi}_k \xi_l$$

defined on $\mathbb{C}^{N+1}$ is positive.

The problem of moments can be extended to signed Borel measures. In such case, functions $c(k)$ for which a signed Borel measure exists satisfying equation (6.1) are finite linear combinations of positive definite functions (see for instance [Rud]).

This moment problem was also extended to the multivariate case without further difficulties as it can be seen in the papers [Hil2], [Hav1], [Hav2]. It consists in finding a positive regular Borel measure on μ on $\mathbb{T}^n$ such that

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} e^{-i\mathbf{k}\cdot\boldsymbol{\theta}}\mu(\mathrm{d}\boldsymbol{\theta}), \quad \mathbf{k} \in \mathbb{Z}^n, \quad \boldsymbol{\theta} = (\theta_1, \ldots, \theta_n) \in \mathbb{T}^n.$$

given a sequence of complex numbers $c_{\mathbf{k}}$, $\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$. Its solution can be settled in terms of an analogous positivity condition. The proof of this can be obtained by using M. Riesz extension principle for positive functionals (see for instance [Akh]).

Even in the more general situation of signed measures, the solution comes easily using the Jordan decomposition of signed measures.

In all cases, Riesz representation theorem plays a fundamental role. The situation is different if we consider the problem through a multilinear approach since that theorem, as we saw in the previous chapter, is no longer valid now.

Given a sequence of multimoments $c_{\mathbf{k}}$, we will define the multilinear functional L_c defined on the space of trigonometric polynomials as

$$L_c(p_1(\theta_{k_1}), \ldots, p_n(\theta_{k_n})) = \sum p_{k_1} \cdots p_{k_n} c_{-\mathbf{k}},$$

where for $j = 1, \ldots, n$, $p_j(\theta_{k_j}) = \sum p_{k_j} e^{ik_j\theta_{k_j}}$.

We will use again the representation theorem by Bombal and Villanueva, Theorem 5.2.2 that identifies bounded multilinear functionals with polymeasures. Then, the natural extension of the trigonometric problem of moments to the multilinear case, will study necessary and sufficient conditions to guarantee the existence of a regular polymeasure γ on the n-dimensional torus $\mathbb{T}^n$ such that:

$$c_{\mathbf{k}} = \int_{\mathbb{T}^n} (e^{-ik_1\theta_1}, \ldots, e^{-ik_n\theta_n})\,\gamma(\mathrm{d}\boldsymbol{\theta}), \quad \forall \mathbf{k} \in \mathbb{Z}^n,$$

where the family complex numbers $c_{\mathbf{k}}$ indexed by the multiindex $\mathbf{k} \in \mathbb{Z}^n$ is given.

According to our terminology in the previous chapter, this problem will be called the Weak Trigonometric Moment Problem. In order to solve it, we will introduce a norm, weaker than the supremum norm on the space of functions c on $\mathbb{Z}^n$.

6.1 The result.

Before proceeding with our proof, we define the norm $\|\cdot\|_w$:

Definition 6.1.1. *Given a function c on $\mathbb{Z}^n$ we define the norm $\|\cdot\|_w$ as*

$$\|c\|_w = \sup_{\substack{\|\varphi_l\|_\infty \leq 1,\, l=1,\ldots,n \\ \varphi_l(\theta)=\sum a_{k_l} e^{-ik_l\theta},\, N\in\mathbb{N}}} \left| \sum_{|\mathbf{k}|\leq N} a_{k_1}\cdots a_{k_n} c(k_1,\ldots,k_n) \right|,$$

with $|\mathbf{k}| = \sum_{l=1}^n |k_l|$.

We will denote by $BP_n(\mathbb{Z})$ the set of functions c on $\mathbb{Z}^n$ such that $\|c\|_w < \infty$ and we will say that $c \in BP_n(\mathbb{Z})$ is weakly bounded.

Moreover a given n-dimensional sequence of moments $c_{\mathbf{k}}$, $\mathbf{k} \in \mathbb{Z}^n$, will be said to be weakly bounded if $c(\mathbf{k}) = c_{\mathbf{k}} \in BP_n(\mathbb{Z})$.

We will also prove that this norm is related with the norm of the multilinear functional defined by the moment sequence $c_{\mathbf{k}} = c(\mathbf{k})$.

To see this relation, we introduce the Fourier-Stieltjes transform for a multilinear functional. If L denotes a continuous multilinear functional on $C(\mathbb{T})$ its Fourier-Stieltjes transform is simply defined in a similar manner to the linear case as the function on $\widehat{\mathbb{T}}^n \cong \mathbb{Z}^n$ given by

$$\widehat{L}(\chi_{k_1},\ldots,\chi_{k_n}) = L(\bar{\chi}_{k_1},\ldots,\bar{\chi}_{k_n}) = L(e^{-ik_1\theta_1},\cdots,e^{-ik_n\theta_n}).$$

where $\chi_k(\theta) = e^{ik\theta}$. The Fourier-Stieltjes transform of the polymeasure γ, will be denoted as well by $\hat{\gamma}$ and verifies that $c_{\mathbf{k}} = c(\mathbf{k}) = \hat{\gamma}(\mathbf{k})$.

Solving the weak multilinear trigonometric problem for the sequence of moments $c_{\mathbf{k}}$ amounts to determine the existence of a bounded multilinear functional L such that $c = \hat{L}$.

The set of functions solving the weak multilinear trigonometric problem can be equipped with a norm by importing directly the norm of the corresponding bounded multilinear functional, that is, we can define $\|c\|_w = \|L\|$ where $\hat{L} = c$.

We prefer to use our first definition for the norm $\|\cdot\|_w$ since it depends only on the multimoments and we will prove then that the previous relation holds.

The first theorem shows us that the function $\|\cdot\|_w$ defines in fact a norm and we will compare it with the norms $\|\cdot\|_{L_\infty}$, $\|\cdot\|_{L^1}$ and $\|\cdot\|_{F_n}$, the Fréchet norm defined as

$$\|c\|_{F_n} = \sup_{|a_k|\leq 1, l=1,\ldots,n} \left| \sum_{|\mathbf{k}|\leq N} a_{k_1}\cdots a_{k_n} c(k_1,\ldots,k_n) \right|.$$

Frechet norms are the natural norms generalizing the notion of semivariation norm on the space of polymeasures (see [Ble] for an exhaustive discussion on the subject).

Theorem 6.1.2. *The function $\|\cdot\|_w$ is a norm on $BP_n(\mathbb{Z})$ satisfying that $(BP_n(\mathbb{Z}), \|\cdot\|_w)$ is a closed subspace of $c_0(\mathbb{Z}^n)$. Moreover*

$$\|c\|_{L_\infty} \leq \|c\|_w \leq \|c\|_{F_n} \leq \|c\|_{L_1}$$

and the inequalities are strict.

Proof. Given $j_1,\ldots,j_m$, we choose the family of numbers $a_{k_l} = \delta_{j_l,k_l}$, $\mathbf{k} \in \mathbb{Z}^n$. Then the trigonometric polynomials

$$\varphi_l(\theta) = \sum_{k_l} a_{k_l} e^{-ik_l\theta}$$

are such that $\|\varphi_l\|_\infty = 1$ and

$$\sum a_{k_1}\cdots a_{k_n} c_{k_1\cdots k_n} = c_{j_1\cdots j_n}.$$

Thus

$$\|c\|_{L_\infty} = \sup_{\mathbf{j}} |c(\mathbf{j})| = \sup_{\mathbf{j}} |\sum a_{k_1}\cdots a_{k_n} c_{k_1\cdots k_n}| \leq \|c\|_w.$$

Assume now that $\|c\|_w = 0$, then $0 \leq \|c\|_{L_\infty} \leq \|c\|_w = 0$ which implies that $c = 0$. Moreover it is obvious that $\|\lambda c\|_w = |\lambda|\|c\|_w$ as well as the triangle inequality for $\|\cdot\|_w$.

To show that $BP_n(\mathbb{T}) \subset c_0(\mathbb{Z}^n)$ is a closed subspace is enough to prove it when $n = 1$. In that case the polynomials $\phi(\theta) = \frac{1}{2N+1} \sum_{|k|\leq N} e^{ik\theta}$ are such that $\|\varphi\|_\infty = 1$ and

$$\sum_{|k|\leq N} a_k c_k = \frac{1}{2N+1} \sum_{|k|\leq N} c_k$$

will not converge unless c_k goes to zero. Using once more the inequality $\|c\|_{L_\infty} \leq \|c\|_w$, any Cauchy sequence on the norm $\|\cdot\|_w$ will be convergent componentwise, hence the sequence converges in $BP_n(\mathbb{T})$.

To prove the inequality $\|c\|_w \leq \|c\|_{F_n}$ we will observe that if the trigonometric polynomial $\varphi(\theta) = \sum_{|k|\leq N} a_k e^{ik\theta}$ is such that $\|\varphi\|_\infty \leq 1$, then necessarily $|a_k| \leq 1$. Hence:

$$\|c\|_w \leq \sup_{|a_{k_l}|\leq 1, N\in\mathbb{N}} \left| \sum_{|\mathbf{k}|\leq N} a_{k_1} \cdots a_{k_n} c_{k_1\cdots k_n} \right| = \|c\|_{F_n}.$$

The last equality follows immediately from the definition of the Fréchet norm $\|\cdot\|_{F_n}$.

□

We prove now the theorem that solves the weak trigonometric multilinear problem.

Theorem 6.1.3. *A sequence of multimoments $c_\mathbf{k}$, $\mathbf{k} \in \mathbb{Z}^n$ is a solution of the weak trigonometric multilinear moment problem if and only if $\|c\|_w < \infty$. Moreover $\|c\|_w = \|L\|$ where L is a bounded multilinear functional such that $c = \hat{L}$.*

Proof. If $c = (c_\mathbf{k})$ is a solution of the weak multilinear trigonometric problem, then there will exist a polymeasure γ on $\mathrm{BP}(\mathbb{T}, \ldots, \mathbb{T})$ such that:

$$c_\mathbf{k} = \int_{\mathbb{T}^n} (e^{-ik_1\theta_1} \otimes \cdots \otimes e^{-ik_n\theta_n})\, \gamma(\mathrm{d}\boldsymbol{\theta}), \quad \mathbf{k} = (k_1, \ldots, k_n).$$

The functional $L_\gamma \colon C(\mathbb{T}) \times \cdots \times C(\mathbb{T}) \to \mathbb{C}$ associated to γ by Riesz multilinear theorem, Theorem 5.2.2, is bounded and $\|L_\gamma\| = \|\gamma\|$ where $\|\gamma\|$ denotes the semivariation norm of γ.

Now we have:

$$\begin{aligned}&\left|\sum_{|\mathbf{k}|\le N} a_{k_1}\cdots a_{k_n} c_{\mathbf{k}}\right| =\\ &= \left| L_\gamma\left(\sum_{k_1=-N}^{N} a_{k_1} e^{-ik_1\theta_1},\dots,\sum_{k_n=-N}^{N} a_{k_n} e^{-ik_n\theta_n}\right)\right|\\ &\le \|\gamma\|\,\|\varphi_{\mathbf{a}_1}\|_\infty\cdots\|\varphi_{\mathbf{a}_n}\|_\infty.\end{aligned}$$

with $\varphi_{\mathbf{a}_l} = \sum_{k_l=-N}^{N} a_{k_l} e^{-ik_l\theta_l}$ the corresponding trigonometric polynomials defined by the coefficients a_{k_l}.

As the polynomials $\varphi_{\mathbf{a}_l}$, $l = 1,\dots,n$ are such that $\|\varphi_{\mathbf{a}_l}\|_\infty \le 1$, $\|c\|_w \le \|\gamma\|$ and $c_{\mathbf{k}}$ is weakly bounded.

Conversely, if c is weakly bounded, then it is easy to check that if $\varphi_l = \sum_{k_l=-N_l}^{N_l} a_{k_l}^{N_l} e^{ik_l\theta}$, $l = 1,\cdots,n$ is a family of trigonometric polynomials, then:

$$\begin{aligned}|L_c(\varphi_1,\dots,\varphi_n)| &\le \left|\sum_{\mathbf{k}=-\mathbf{N}}^{\mathbf{N}} a_{k_1}^{N_1}\cdots a_{k_n}^{N_n} L(e^{ik_1\theta_1},\dots,e^{ik_n\theta_n})\right| =\\ &\le \left|\sum_{\mathbf{k}=-\mathbf{N}}^{\mathbf{N}} \frac{a_{k_1}^{N_1}}{\|\varphi_1\|_\infty}\cdots\frac{a_{k_n}^{N_n}}{\|\varphi_n\|_\infty} c_{-k_1\cdots-k_n}\right| \|\varphi_1\|_\infty\cdots\|\varphi_n\|_\infty \le\\ &\le \|c\|_w\|\varphi_1\|_\infty\cdots\|\varphi_n\|_\infty\end{aligned}$$

because the coefficients $c_{k_l}^{N_l} = a_{k_l}^{N_l}/\|\varphi_l\|_\infty$ are such that the associated trigonometric polynomials $\psi_l = \sum_{k_l=-N_l}^{N_l} c_{k_l}^{N_l} e^{-ik_l\theta}$, $l = 1,\dots,n$, satisfy $\|\psi_l\|_\infty \le 1$. □

6.2 Examples.

We conclude this chapter presenting two examples, one on the linear case and another on the bilinear, that show the differences between our norm and Fréchet norm.

Example 6.2.1 (The linear case.). *Let L be a complex bounded linear functional on $C(\mathbb{T})$, then we shall denote by $c(k)$ the Fourier-Stieltjes transform of L. Because of Riesz representation theorem, there exists a complex* *measure μ on $\mathbb{T}$ such that $L(f) = \int_{\mathbb{T}} f d\mu$. Any complex measure has finite total variation $\|\mu\|$, hence we have $\|L\| = \|\mu\| = \|c\|_w$. It is clear that in the linear case $\|c\|_{F_1} = \|c\|_{L^1}$ where $\|\cdot\|_{F_1}$ denotes the Fréchet norm with $n = 1$. Hence there are examples of bounded functionals such that $\|c\|_{F_1} = \infty$.*

Let us consider for instance the functional defined by $L(f) = \sum_{k\in S} \hat{f}(k)$, where $S \subset \mathbb{Z}$ is an infinite Sidon subset of $\mathbb{Z}$. If S is a Sidon set, then the restriction algebra $A_S(\mathbb{T})$ and $C_S(\mathbb{T})$ coincide and in consequence $|L_S(f)| = |\sum_{k\in S} \hat{f}(k)| \leq \sum_{k\in S} |\hat{f}(k)| \leq \|\hat{f}\|_{L^1}$ and the functional L will be bounded. However a simple check shows that $c(k) = \hat{L}(\chi_k) = 1$ if $k \in S$ and 0 otherwise. Thus for any infinite Sidon set S we have $\|c\|_{F_1} = \infty$.

Example 6.2.2 (The bilinear case.). *Let us consider the bilinear functional on $C(\mathbb{T})$, given by:*

$$L(f, g) = \sum_{k\in S} \hat{f}(k)\hat{g}(k),$$

where $S \subset \mathbb{Z}$ is an infinite subset of the integers. It is clear that L is bounded in the norm L^2, $|L(f, g)| \leq \|f\|_2\|g\|_2$, thus L is bounded with respect to the uniform topology. Hence, because of the multilinear Riesz theorem, Theorem 5.2.2, L defines a regular Borel bimeasure on $\mathbb{T}$ given explicitely by $\gamma(A, B) = \sum_{k\in S} \hat{\mathbf{1}}_A(k)\hat{\mathbf{1}}_B(k)$. Then a simple computation shows:

$$c_{mn} = \hat{L}(m, n) = \int_{\mathbb{T}^2} (e^{-im\theta} \otimes e^{-in\psi})\, \gamma(\mathrm{d}(\theta, \psi)) = \sum_{k\in S} \delta_{k+m}\delta_{k+n},$$

this is, $c_{mn} = 1$ if $m = n \in S$ and $c_{mn} = 0$ otherwise. Because of Theorem 6.1.3, the function c_{mn} on $\mathbb{Z}^2$ has finite weak norm $\|\cdot\|_w$. In fact it can be shown that $\|c\|_w = 1$.

However c_{mn} is unbounded in the Fréchet norm $\|\cdot\|_{F_2}$. Notice that if we select an increasing family of finite subsets $S_N \subset S$, with the property $S_N \subset S_{N+1}$, we can choose numbers $a_{mn} = \hat{\mathbf{1}}_{S_N}(m, n)$, then $\|c\|_{F_2} \geq N$ for all N.

Bibliography

[A] R. A. Adams, *Sobolev Spaces.* Academic Press, 1975.

[AMV] M. Adler, P. van Moerbecke, P. Vanhaecke. *Moment matrices and multi-component KP, with applications to random matrix theory.* (2006) ArXiv: math-ph/0612064.

[Akh] N.I. Akhiezer, *The Classical Moment Problem.* Oliver and Boyd Ltd, 1965.

[AB] C. D. Aliprantis and O. Burkinshaw, *Positive Operators.* Springer, 2006.

[AN] R. Álvarez-Nodarse, *Polinomios hipergeométricos y q-polinomios.* Monografías del seminario matemático García de Galdeano, n 26, 2003.

[AG] R. M. Aron and J. Globevnik, *Analytic Functions on* c_0. Rev. Mat. Univ. Complut. Madrid **2** (1989), suppl., 27–33.

[BLLl] Y. Benyamini, S. Lassalle and J.G. Llavona, *Homogeneous orthogonally-additive polynomials on Banach lattices.* Bull. London Math. Soc. **38** (2006), 459–469.

[Ber] M. Bertola *Bilinear semiclassical moment functionals and their integral representation.* J. Approx. Theory 121 (2003), no. 1, 71–99.

[BlK] P. Bleher, A. Kuijlaars. *Random matrices with external source and multiple orthogonal polynomials.* Int. Math. Res. Not. **3** (2004) 109–129.

[Ble] R. Blei. *Analysis in Integer and Fractional Dimensions.* Cambridge Studies in Advanced Mathematics **71**. Cambridge Univ. Press (2001).

[B] R. P. Boas, *The Stieltjes Moments Problem for functions of bounded vaiation.* Bull. Amer. Soc. **45** (1939), 399-404.

[BV1] F. Bombal, I. Villanueva, *Multilinear operators on spaces of continuous functions.* Funct. Approx. Comment. Math. **XXVI** (1998), 117-126.

[BV2] F. Bombal, I. Villanueva. *Integral operators on the product of $C(K)$ spaces.* J. Math. Anal. Appl., (2001) **264**, 107–121.

[BB] K. Boulabiar and G. Buskes, *Vector lattice powers: f-algebras and functional calculus.* Comm. Algebra **34** (2006), no. 4, 1435–1442.

[BM] M. I. Bueno, F. Marcellán, *Polynomial perturbations of bilinear functionals and Hessenberg matrices.* Linear Algebra Appl. **414** (2006), no. 1, 64–83.

[BK] G. Buskes and A. G. Kusraev, *Representation and extension of orthoregular bilinear operators.* Vladikavkaz Math. J. **9** (2007), no 1, 16-29.

[BvR1] G. Buskes and A. van Rooij, *Almost f-algebras: Commutativity and the Cauchy-Schwarz Inequality.* Positivity and its applications. Positivity **4** (2000), no. 3, 227-231.

[BvR2] G. Buskes and A. van Rooij, *Bounded Variation and Tensor Products of Banach Lattices.* Positivity **7** (2003), no. 1-2, 47-59.

[BvR3] G. Buskes and A. van Rooij, *Squares of Riesz spaces.* Rocky Mountain J. Math. **31** (2001), no. 1, 45-56.

[CLZ1] D. Carando, S. Lassalle and I. Zalduendo, *Orthogonally additive polynomials over $C(K)$ are measures—a short proof.* Integral Equations Operator Theory **56** (2006), no. 4, 597–602.

[CLZ2] D. Carando, S. Lassalle and I. Zalduendo, *Orthogonally additive holomorphic functions of bounded type over $C(K)$* . arXiv:0810.5352v1 [math.FA]

[CF] R. V. Chacon and N. Friedman, *Additive functionals.* Arch. Rational Mech. Anal. **18** 1965 230–240.

[C] T. S. Chihara, *An introduction to Orthogonal Polynomials.* Gordon and Breach Science Publishers, New York, 1978.

[Dei] P. Deift. *Orthogonal polynomials and random matrices: a Riemann-Hilbert approach.* Am. Math. Soc., Providence, Rhode Island (1998).

[D] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, S.M.M., Springer, 1999.

[Dob] I. Dobrakov. *On integration in Banach spaces VIII (polymeasures).* Czech. Math. J. (1987) **37**, 487–506.

[DO1] L. Drewnonwski and W. Orlicz, *On orthogonally additive functionals.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 16 1968 883–888.

[DO2] L. Drewnonwski and W. Orlicz, *On representation of orthogonally additive functionals.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **17** 1969 167–173.

[Dur1] A. J. Durán, *The Stieltjes Moments Problem for rapidly decreasing functions.* Proc. of the Amer. Math. Soc. vol 107, n3 (1989), 731-741.

[Dur2] A.J. Durán, *A Generalization of Favard's Theorem for Polynomials Satisfying a Recurrence Relation.* J. of Approx Theory **74**, 83-109 (1993).

[F] J. Favard, *Sur les polynomes de Tchebicheff*, C.R. Acad. Sci. Paris **200** (1935), 2052-2053.

[Fr1] M. Fréchet, *Une definition fonctionnelle des polynomes.* Nouv. Ann. Math., **9**, 1909, 145-162.

[Fr2] M. Fréchet, *Sur les fonctionnelles continues.* Ann. Sci. École Norm. Sup. Sér. 3, **27** (1910), 193-216.

[Fr3] M. Fréchet. *Sur les fonctionnelles bilineares.* Trans. Am. Math. Soc., (1915) **16**, 215–234.

[Fre] D. H. Fremlin, *Tensor products of Archimedean vector lattices.* Amer. J. Math. **94**, (1972) 777-798.

[FK1] N. Friedman and M. Katz, *A representation theorem for additive functionals.* Arch. Rational Mech. Anal. **21** 1966 49–57.

[FK2] N. Friedman and M. Katz, *Additive functionals of L_p spaces.* Canad. J. Math. **18** 1966 1264–1271.

[FK3] N. Friedman and M. Katz, *On additive functionals.* Proc. Amer. Math. Soc. **21** 1969 557–561.

[GR] B. Grecu and R. A. Ryan *Polynomials on Banach Spaces with Unconditional Bases.* Proc. Amer. Math. Soc. **133** (2005), no. 4, 1083–1091.

[Hav1] E.K. Haviland. *On the momentum problem for distribution functions in more than one dimension.* Am. J. Math., (1935) **57**, 562-568.

[Hav2] E.K. Haviland. *On the momentum problem for distribution functions in more than one dimension. II.* Am. J. Math., (1936) **58**, 164-168.

[Her] G. Herglotz. *Über Potenzreihen mit positivem reellen Teil im Einheitskreis.* Ber. Ver. Sächs. GEs. d. Wiss. Leipzig., **63** 501-511 (1911).

[Hil1] T.H. Hildebrandt. *On the moment problem for a finite interval*, Bull. Amer. Math. Soc., vol 38 (1932), 72-76.

[Hil2] T.H. Hildebrandt, I.J. Shoenberg. *On linear functional operations and the moment problem for a finite interval in one and several dimensions.* Ann. Maths. (1933) **34**, 317-328.

[ILL11] A. Ibort, P. Linares, J.G. Llavona. *On the multilinear Hausdorff problem of moments.* Preprint.

[ILL12] A. Ibort, P. Linares, J.G. Llavona. *On the multilinear trigonometric problem of moments.* To appear in Contemp Math.

[ILLl3] A. Ibort, P. Linares and J. G. Llavona *On the Representation of Orthogonally Additive Polynomials in ℓ_p* Publ. Res. Inst. Math. Sci. Volume 45, Number 2 (2009), 519-524.

[ILL14] A. Ibort, P. Linares, J.G. Llavona. *Spectral Theory and Orthogonally Additive Polynomials.*. Preprint.

[JvR] E. de Jonge and A. van Rooij, *Introduction to Riesz Spaces.* Mathematical Centre Tracts, No. **78**. Mathematische Centrum, Amsterdam, 1977.

[K] S. Kakutani, *Concrete Representation of Abstract (L)-Spaces and the Mean Ergodic Theorem.* The Annals of Mathematics, 2nd Ser., Vol. 42, No. 2. (Apr., 1941), pp. 523-537.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II.* Springer 1977.

[L] J. Loane, *Polynomials on Riesz Spaces.* Ph.D. Thesis, National University of Irleand, Galway. 2008.

[MAR] F. Marcellán, M. Alfaro, M. L. Rezola, *Orthogonal polynomials on Sobolev spaces: old and new directions.* Proceedings of the Seventh Spanish Symposium on Orthogonal Polynomials and Applications (VII SPOA) (Granada, 1991). J. Comput. Appl. Math. 48 (1993), no. 1-2, 113–131.

[MMB] F. Marcellán, J. J. Moreno Balcázar, *Asymptotics and zeros of Sobolev orthogonal polynomials on unbounded supports.* Acta Appl. Math. **94** (2006), no. 2, 163–192.

[MM] A. D. Martin and V. J. Mizel *A representation theorem for certain nonlinear functionals.* Arch. Rational Mech. Anal. **15** 1964 353–367.

[MF1] A. Martínez-Finkelshtein, *Asymptotic properties of Sobolev orthogonal polynomials.* Proceedings of the VIIIth Symposium on Orthogonal Polynomials and Their Applications (Seville, 1997). J. Comput. Appl. Math. **99** (1998), no. 1-2, 491–510.

[MF2] A. Martínez-Finkelshtein, *Analytic aspects of Sobolev orthogonal polynomials revisited.* Numerical analysis 2000, Vol. V, Quadrature and orthogonal polynomials. J. Comput. Appl. Math. **127** (2001), no. 1-2, 255–266.

[M] H. G. Meijer, *A short history of orthogonal polynomials in a Sobolev space. I. The non-discrete case.* 31st Dutch Mathematical Conference (Groningen, 1995). Nieuw Arch. Wisk. (4) 14 (1996), no. 1, 93–112.

[MN] P. Meyer-Nieberg, *Banach Lattices.* Springer-Verlag 1991.

[MT] M. Morse, W. Transue. *Integral representation of bilinear functionals.* Proc. Natl. Acad. Sci. USA (1949) **35**, 136-143.

[PPV] C. Palazuelos, A. M. Peralta and I. Villanueva *Orthogonally additive polynomials on C^*-Algebras*, Quart. J. Math. **59** (2008), 363-374.

[PGV] D. Pérez García and I. Villanueva, *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl. **306** (2005), 97-105.

[Rie] F. Riesz. *Sur certains systèmes singuliers d'équations intégrales.* Ann. Éc. Norm., **28**, 33-62 (1911).

[R1] R. A. Ryan, *Applications of topological tensor products to infinite dimensional holomorphy*, Ph.D. Thesis, Trinity College Dublin, 1980.

[R2] R. A. Ryan, *Introduction to Tensor Products of Banach spaces*, Springer 2002.

[Rud] W. Rudin. *Fourier Analysis on Groups.* Interscience Tracts in Pure and Applied Mathematics, vol 12. Inters. Publ. New York (1962).

[Sch] H. H. Schaefer, *Aspects of Banach lattices.* Studies in functional analysis, MAA Stud.Math., **21**, Math. Assoc. America, Washington, D.C., (1980), 158-221.

[ST] J. A. Shohat y J. D. Tamarkin, *The problem of moments*, Mathematical Surveys n1. American Mathematical Society (1970).

[Sti] T. J. Stieltjes, *Recherches sur les fractions continues*, Annales de la Faculté des Sciences de Toulouse, (1) 8 (1894), T 1-122, (1) 9 (1895) A 5-47.

[S1] K. Sundaresan, *Additive functionals on Orlicz spaces.* Studia Math. **32** 1969 269276.

[S2] K. Sundaresan, *Geometry of spaces of homogeneous polynomials on Banach lattices*, Applied geometry and discrete Mathematics 571-586, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Prov.,RI, 1991.

[Sze] G. Szegő, *Orhogonal Polinomials.* American Mathematical Society, colloquium publications ; 23. 1939.

[V] I. Villanueva. Ph.D. Thesis. Universidad Complutense de Madrid. 1999.

[W] J. Weidmann *Linear operators in Hilbert spaces* . Springer, 1980.

[Wid] D.V. Widder *The Laplace Transform* Princeton University Press, 1972.

[Z] I. Zalduendo, *An estimate for multilinear forms on ℓ_p spaces.* Proc. Royal Irish. Acad. 93A (1993) no 1, 137–142.